INEQUALITIES ON WELL-DISTRIBUTED POINT SETS ON CIRCLES

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Abstract: The setting is a finite set P of points on the circumference of a circle, where all points are assigned non-negative real weights $w(p)$. Let P_i be all subsets of P with i points and no two distinct points within a fixed distance d. We prove that $W_k^2 \geq W_{k+1}W_{k-1}$ where $W_k = \sum_{A \in P_i} \prod_{p \in A} w(p)$. This is done by first extending a theorem by Chudnovsky and Seymour on roots of stable set polynomials of claw-free graphs.

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1. Introduction

In this note a weighted type extension of a theorem by Chudnovsky and Seymour is proved, and then used to derive some inequalities about well-distributed points on the circumference of circles. Some basic graph theory will be used: A *stable set* in a graph, is a subset of its vertex set with no adjacent vertices. For a graph G, its *stable set polynomial* is

$$p_G(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n,$$

where p_i counts the stable sets in G with i vertices, and there are n vertices in the largest stable sets. It was conjectured by Stanley [8] and Hamidoune [5] that the roots of stable set polynomials of claw-free graphs are real. In a *claw-free* graph there are no four distinct vertices a, b, c, and d, with a adjacent to b, c, and d, but none of b, c, and d are adjacent. The conjecture was proved by Chudnovsky and Seymour [2]. For some subclasses of claw-free graphs, weighted versions of the theorem exist, and they are used in mathematical physics [6]. If w is a real valued function on the vertex set of a graph G, then the *weighted stable set polynomial* is

$$p_{G,w}(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n,$$

where

$$p_i = \sum_{S \text{ stable in } G \text{ and } \#S = i} \; \prod_{v \in S} w(v)$$

for $i > 0$ and $p_0 = 1$. Theorem 2.5 states that if w is non-negative, and G is claw-free then $p_{G,w}$ is real rooted. The proof is in three steps, first for integer weights, then rational, and finally for real weights.

In the last section, points on circles are described by claw-free graphs, and Newton's inequalities are used to derive information on well-distributed point sets of them.

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2. A Weighted Version of Chudnovsky and Seymour's Theorem

Some graph notation is needed. The neighborhood of a vertex v in G, denoted $N_G(v)$, is the set of vertices adjacent to v, and $N_G[v] = N_G(v) \cup \{v\}$. The vertex set of a graph G is $V(G)$ and the edge set is $E(G)$. The induced subgraph of G on $S \subseteq V(G)$, denoted by $G[S]$, is the maximal subgraph of G with vertex set S.

Lemma 2.1. *Let G be a claw-free graph with non-negative integer vertex weights $w(v)$. Then there is an unweighted claw-free graph H with $p_{G,w}(x) = p_H(x)$.*

Proof. If there are any vertices in G with weight zero they can be discarded and we assume further on that the weights are positive.

Let H be the graph with vertex set

$$\bigcup_{v\in V(G)} \{v\} \times \{1, 2, \ldots, w(v)\}$$

and edge set

$$\{\{(u, i), (v, j)\} \subseteq V(H) \mid \{u, v\} \in E(G), \text{ or } u = v \text{ and } i \neq j\}.$$

We will later use that if $v \in V(G)$ and $1 \leq i, j \leq w(v)$ then $N_H[(v, i)] = N_H[(v, j)]$.

First we check that H is claw-free. Let $(v_1, i_1), \ldots, (v_4, i_4)$ be four distinct vertices of H and assume that the subgraph they induce is a claw. If all of v_1, v_2, v_3, v_4 are distinct, then their induced subgraph of G is a claw, which contradicts that G is claw-free. The other case is that not all of v_1, v_2, v_3, v_4 are distinct; we can assume without loss of generaliy that $v_1 = v_2$. But $N_{H[(v_1,i_1),\ldots,(v_4,i_4)]}[(v_1, i_1)] = N_{H[(v_2,i_2),\ldots,(v_4,i_4)]}[(v_1, i_1)]$ and this is never the case for the neighborhoods of two distinct vertices in a claw. Thus H is claw-free.

The surjective map $\phi : \{S \text{ is stable in } H\} \to \{S \text{ is stable in } G\}$ defined by $\{(v_1, i_1), (v_2, i_2), \ldots, (v_t, i_t)\} \mapsto \{v_1, v_2, \ldots, v_t\}$ satisfy $\#\phi^{-1}(S) = \prod_{v\in S} w(v)$, which shows that $p_{G,w}(x) = p_H(x)$. $\square$

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Theorem 2.2 ([2]). *The roots of the stable set polynomial of a claw free graph are real.*

Lemma 2.3. *Let G be a claw-free finite graph with non-negative real vertex weights $w(v)$, and $\varepsilon > 0$ a real number. Then there is a polynomial $f(x) = f_0 + f_1 x + \cdots + f_d x^d$ of the same degree as $p_{G,w}(x) = p_0 + p_1 x + \cdots + p_d x^d$ satisfying $0 \le p_i - f_i \le \varepsilon$ for all i, and all of its roots are real and negative. In addition, $f_0 = 1$.*

Proof. We can assume that $\varepsilon < 1$. Let $\tilde{w}$ be the largest weight of a vertex in G, and let $\tilde{w} = 1$ if no weight is larger than 1. Set $n = (4\tilde{w})^{\#V(G)}\varepsilon^{-1}$. Note that $n, \tilde{w} \ge 1$. Let $w'(v) = \lfloor n w(v) \rfloor$ be non-negative integer weights of G. By Lemma 2.1, there is a graph H with $p_H(x) = p_{G,w'}(x)$, and by Theorem 2.2 all roots of $p_H(x)$ are real. They are negative since all coefficients are non-negative. The roots of

$$f(x) = p_{G,w'}(x/n) = f_0 + f_1 x^1 + f_2 x^2 + \cdots + f_d x^d$$

are then also real and negative.

$$\begin{aligned}
0 &\le p_i - f_i \\
&= \sum_{S \text{ stable in } G \text{ and } \#S = i} \left(\prod_{v \in S} w(v) - n^{-i} \prod_{v \in S} w'(v) \right) \\
&= \sum_{S \text{ stable in } G \text{ and } \#S = i} \left(\prod_{v \in S} w(v) - \prod_{v \in S} \frac{\lfloor n w(v) \rfloor}{n} \right) \\
&\le \sum_{S \text{ stable in } G \text{ and } \#S = i} \left(\prod_{v \in S} w(v) - \prod_{v \in S} \left(w(v) - \frac{1}{n} \right) \right)
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{S \text{ stable in } G \text{ and } \#S=i} \sum_{U \subsetneqq S} -(-\frac{1}{n})^{\#S-\#U} \prod_{v\in U} w(v) \\
&\leq \frac{1}{n} \sum_{S \text{ stable in } G \text{ and } \#S=i} \sum_{U \subsetneqq S} n^{1+\#U-\#S} \tilde{w}^{\#U} \\
&\leq \frac{1}{n} 2^{\#V(G)} 2^{\#V(G)} 1 \tilde{w}^{\#V(G)} \\
&= \varepsilon.
\end{aligned}
$$

We have that $f_0 = 1$ since $p_{G,w'}$, hence it is a stable set polynomial. $\square$

This is a nice way to state the old fact that the roots and coefficients of complex polynomials move continuously with each other.

Theorem 2.4 ([3]). *The space $\mathcal{P}_n$ of all monic complex polynomials of degree n with the distance function $d_{\mathcal{P}_n}(f,g) = \max\{|f_0 - g_0|, \dots, |f_{n-1} - g_{n-1}|\}$ for $f(z) = f_0 + f_1 z + \cdots + f_{n-1} z^{n-1} + z^n$ and $g(z) = g_0 + g_1 z + \cdots + g_{n-1} z^{n-1} + z^n$ is a metric space.*

The set $\mathcal{L}_n$ of all multisets of complex numbers with n elements with distance function

$$d_{\mathcal{L}_n}(U,V) = \min_{\pi \in \Pi_n} \max_{1 \leq j \leq n} \left| u_j - v_{\pi(j)} \right|$$

for $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ is a metric space.

The map $\{z_1, z_2, \dots, z_n\} \mapsto (z - z_1)(z - z_2) \cdots (z - z_n)$ from $\mathcal{L}_n$ to $\mathcal{P}_n$ is a homeomorphism.

Theorem 2.5. *If G is a claw-free graph with real non-negative vertex weights w then all roots of $p_{G,w}(z)$ are real and negative.*

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Proof. Assume that the the statement is false since there is a graph G with weights w such that $p_{G,w}(a+bi)=0$, where a and b are real numbers and $b\neq 0$. Assume that $p_{G,w}(z)=p_0+p_1z+p_2z^2+\cdots+p_dz^d$, where $p_d\neq 0$. Since p_0 and p_d are non-zero the map $r\mapsto 1/r$ is a bijection between the multiset of roots of $p_{G,w}(z)$ and the multiset of roots of the monic polynomial $\tilde{p}(z)=p_d+p_{d-1}z+p_{d-2}z^2+\cdots+p_0z^d$. The distance in $\mathcal{L}_d$, as defined in Theorem 2.4, between the multiset of roots of $\tilde{p}(z)$ and the multiset of roots of any real rooted polynomial is at least $|b|/(a^2+b^2)$ since

$$\left|r-\frac{1}{a+bi}\right|=\left|\left(r-\frac{a}{a^2+b^2}\right)+\frac{b}{a^2+b^2}i\right|\geq\frac{|b|}{a^2+b^2}$$

for any real r. Now we will find a contradiction to the homeomorphism statement in Theorem 2.4 by constructing polynomials which are arbitrary close to $\tilde{p}(z)$ in $\mathcal{P}_d$, but on distance at least $|b|/(a^2+b^2)$ in $\mathcal{L}_d$. Let $\varepsilon>0$ be arbitrarily small, at least smaller than $p_d/2$. By Lemma 2.3 there is a real rooted polynomial $f(x)=f_0+f_1x+\cdots+f_dx^d$ such that $0\leq p_i-f_i\leq\varepsilon$ and $f_0=1$. We assumed that $\varepsilon<p_d/2$ so that both f_0 and f_d are non-zero. All roots of the monic polynomial $\tilde{f}(z)=f_d+f_{d-1}z+f_{d-2}z^2+\cdots+f_0z^d$ are real, since they are the inverses of the roots of $f(z)$, which are real. Hence the distance between the roots of $\tilde{p}(z)$ and $\tilde{f}(z)$ in $\mathcal{L}_d$ is at least $|b|/(a^2+b^2)$. But since $|p_i-f_i|\leq\varepsilon$, the distance between $\tilde{p}(z)$ and $\tilde{f}(z)$ in $\mathcal{P}_d$ is at most ε.

The roots are negative since all coefficients of $p_{G,w}(z)$ are non-negative, and $p_{G,w}(0)=1$. $\square$

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3. Weighted Points on a Circle

The circumference of the circle is parametrized by $C = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$, and the distance between two points is the ordinary euclidean metric. To a set $P \subseteq C$ of points on the circle and a distance d, we associate a graph $G(P;d)$ with P as a vertex set, and two distinct vertices a and b are adjacent if their distance is not more than d.

Lemma 3.1. *The graph $G(P;d)$ is claw-free.*

Proof. Assume that the points p_1, p_2, p_3, p_4 lie clockwise on the circle and form a claw in the graph with p_1 adjacent to the other ones. Not both p_2 and p_4 can be further away from p_1 than p_3 is from p_1, since they are on clockwise order on the circle. But the distance from p_2 and p_4 to p_1 is larger than d, and the distance between p_1 and p_3 is at most d since they are in a claw. We have a contradiction and thus $G(P;d)$ is claw-free. □

If the points are equally distributed on the circle, we get a class of graphs which was studied in a topological setting by Engström [4] and used in the proof of Lovász's conjecture by Babson and Kozlov [1].

Now we can use the extension of Chudnovsky and Seymour's theorem.

Theorem 3.2. *Let P be a finite set of points on the circumference of a circle, where all points are assigned non-negative real weights $w(p)$. And let P_k be the set of all subsets of P with k points and no two points within a fixed distance d. Then the roots of*

$$f(x) = W_0 + W_1 x + W_2 x^2 + \cdots$$

are real and negative if

$$W_k = \sum_{A \in P_k} \prod_{p \in A} w(p)$$

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and $W_0 = 1$.

Proof. By Lemma 3.1 the graph $G(P;d)$ is claw-free. The sums of products of weights is W_k, and by Theorem 2.5 the roots of the polynomial $f(x) = p_{G(P;d),w}(z)$ are real and negative. $\square$

Newton's inequalities used for coefficients of polynomials with real and non-positive roots as described in [7] gives the following corollary.

Corollary 3.3. *Using the notation of Theorem 3.2, with n the largest integer such that $W_n \neq 0$, we have*

$$\frac{W_k^2}{\binom{n}{k}^2} \geq \frac{W_{k-1}}{\binom{n}{k-1}} \frac{W_{k+1}}{\binom{n}{k+1}}$$

and

$$\frac{W_k^{1/k}}{\binom{n}{k}^{1/k}} \geq \frac{W_{k+1}^{1/(k+1)}}{\binom{n}{k+1}^{1/(k+1)}}$$

for $0 < k < d$.

There is an easily stated slightly weaker version, $W_k^2 \geq W_{k-1}W_{k+1}$.

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