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SHARP GRÜSS-TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE OF BOUNDED VARIATION

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ABSTRACT. Sharp Grüss-type inequalities for functions whose derivatives are of bounded variation (Lipschitzian or monotonic) are given. Applications in relation with the well-known Čebyšev, Grüss, Ostrowski and Lupaş inequalities are provided as well.

Key words and phrases: Riemann-Stieltjes integral, Functions of bounded variation, Lipschitzian functions, Integral inequalities, Čebyšev, Grüss, Ostrowski and Lupaş type inequalities.

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1. INTRODUCTION

In 1998, S.S. Dragomir and I. Fedotov [10] introduced the following *Grüss type error functional*

$$D(f;u) := \int_a^b f(t)\,du(t) - [u(a)-u(b)] \cdot \frac{1}{b-a}\int_a^b f(t)\,dt$$

in order to approximate the *Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ by the simpler quantity

$$[u(a)-u(b)] \cdot \frac{1}{b-a}\int_a^b f(t)\,dt.$$

In the same paper the authors have shown that

$$|D(f;u)| \le \frac{1}{2} \cdot L(M-m)(b-a), \tag{1.1}$$

provided that u is $L-$*Lipschitzian*, i.e., $|u(t)-u(s)| \le L|t-x|$ for any $t,s \in [a,b]$ and f is *Riemann integrable* and satisfies the condition

$$-\infty < m \le f(t) \le M < \infty \quad \text{for any } t \in [a,b].$$

The constant $\frac{1}{2}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

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In [11], the same authors established another result for $D\left(f;u\right)$, namely

$$|D\left(f;u\right)| \leq \frac{1}{2}K\left(b-a\right)\bigvee_{a}^{b}\left(u\right), \tag{1.2}$$

provided that u is of *bounded variation* on $[a,b]$ with the *total variation* $\bigvee_{a}^{b}\left(u\right)$ and f is $K-$Lipschitzian. Here $\frac{1}{2}$ is also best possible.

In [8], by introducing the *kernel* $\Phi_u : [a,b] \to \mathbb{R}$ given by

$$\Phi_u\left(t\right) := \frac{1}{b-a}\left[\left(t-a\right)u\left(b\right)+\left(b-t\right)u\left(a\right)\right]-u\left(t\right), \quad t \in [a,b], \tag{1.3}$$

the author has obtained the following *integral representation*

$$D\left(f;u\right) = \int_a^b \Phi_u\left(t\right)df\left(t\right), \tag{1.4}$$

where $u, f : [a,b] \to \mathbb{R}$ are bounded functions such that the Riemann-Stieltjes integral $\int_a^b f\left(t\right)du\left(t\right)$ and the Riemann integral $\int_a^b f\left(t\right)dt$ exist. By the use of this representation he also obtained the following bounds for $D\left(f;u\right)$,

$$|D\left(f;u\right)| \leq \begin{cases} \sup\limits_{t\in[a,b]}|\Phi_u\left(t\right)|\cdot\bigvee_a^b\left(f\right) & \text{if } u \text{ is continuous and } f \text{ is of bounded variation;} \\ L\int_a^b|\Phi_u\left(t\right)|dt & \text{if } u \text{ is Riemann integrable and } f \text{ is } L\text{-Lipschitzian;} \\ \int_a^b|\Phi_u\left(t\right)|dt & \text{if } u \text{ is continuous and } f \text{ is monotonic nondecreasing.} \end{cases} \tag{1.5}$$

If u is *monotonic nondecreasing* and $K\left(u\right)$ is defined by

$$K\left(u\right) := \frac{4}{\left(b-a\right)^2}\int_a^b\left(t-\frac{a+b}{2}\right)u\left(t\right)dt\left(\geq 0\right),$$

then

$$|D\left(f;u\right)| \leq \frac{1}{2}L\left(b-a\right)\left[u\left(b\right)-u\left(a\right)-K\left(u\right)\right] \leq \frac{1}{2}L\left(b-a\right)\left[u\left(b\right)-u\left(a\right)\right], \tag{1.6}$$

provided that f is $L-$Lipschitzian on $[a,b]$.

Here $\frac{1}{2}$ is best possible in both inequalities.

Also, for u monotonic nondecreasing on $[a,b]$ and by defining $Q\left(u\right)$ as

$$Q\left(u\right) := \frac{1}{b-a}\int_a^b u\left(t\right)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\left(\geq 0\right),$$

we have

$$|D\left(f;u\right)| \leq \left[u\left(b\right)-u\left(a\right)-Q\left(u\right)\right]\cdot\bigvee_a^b\left(f\right) \leq \left[u\left(b\right)-u\left(a\right)\right]\cdot\bigvee_a^b\left(f\right), \tag{1.7}$$

provided that f is of bounded variation on $[a,b]$. The first inequality in (1.7) is sharp.

Finally, the case when u is convex and f is of bounded variation produces the bound

$$|D\left(f;u\right)| \leq \frac{1}{4}\left[u'_-\left(b\right)-u'_+\left(a\right)\right]\left(b-a\right)\bigvee_a^b\left(f\right), \tag{1.8}$$

with $\frac{1}{4}$ the best constant (when $u'_{-}(b)$ and $u'_{+}(a)$ are finite) and if f is monotonic nodecreasing and u is convex on $[a,b]$, then

$$0 \leq D(f;u) \tag{1.9}$$
$$\leq 2 \cdot \frac{u'_{-}(b) - u'_{+}(a)}{b-a} \cdot \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\, dt$$
$$\leq \begin{cases} \frac{1}{2}\left[u'_{-}(b) - u'_{+}(a)\right] \max\{|f(a)|, |f(b)|\}(b-a) \\ \frac{1}{(q+1)^{1/q}}\left[u'_{-}(b) - u'_{+}(a)\right] \|f\|_p (b-a)^{1/q} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[u'_{-}(b) - u'_{+}(a)\right] \|f\|_1, \end{cases}$$

where 2 and $\frac{1}{2}$ are sharp constants (when $u'_{-}(b)$ and $u'_{+}(a)$ are finite) and $\|\cdot\|_p$ are the usual Lebesgue norms, i.e., $\|f\|_p := \left(\int_a^b |f(t)|^p\, dt\right)^{\frac{1}{p}}$, $p \geq 1$.

The main aim of the present paper is to provide sharp upper bounds for the absolute value of $D(f;u)$ under various conditions for u', the derivative of an absolutely continuous function u, and f of bounded variation (Lipschitzian or monotonic). Natural applications for the Čebyšev functional that complement the classical results due to Čebyšev, Grüss, Ostrowski and Lupaş are also given.

2. PRELIMINARY RESULTS

We have the following integral representation of Φ_u.

Lemma 2.1. *Assume that* $u : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$ *and such that the derivative* u' *exists on* $[a,b]$ *(eventually except at a finite number of points). If* u' *is Riemann integrable on* $[a,b]$*, then*

$$\Phi_u(t) := \frac{1}{b-a}\int_a^b K(t,s)\, du'(s), \quad t \in [a,b], \tag{2.1}$$

where the kernel $K : [a,b]^2 \to \mathbb{R}$ *is given by*

$$K(t,s) := \begin{cases} (b-t)(s-a) & \text{if } s \in [a,t], \\ (t-a)(b-s) & \text{if } s \in (t,b]. \end{cases} \tag{2.2}$$

Proof. We give, for simplicity, a proof only in the case when u' is defined on the entire interval, and for which we have used the usual convention that $u'(a) := u'_{+}(a)$, $u'(b) := u'_{-}(b)$ and the lateral derivatives are finite.

Since u' is assumed to be Riemann integrable on $[a,b]$, it follows that the Riemann-Stieltjes integrals $\int_a^t (s-a)\, du'(s)$ and $\int_t^b (b-s)\, du'(s)$ exist for each $t \in [a,b]$. Now, integrating by parts in the Riemann-Stieltjes integral, we have succesively

$$\int_a^b K(t,s)\, du'(s) = (b-t)\int_a^t (s-a)\, du'(s) + (t-a)\int_t^b (b-s)\, du'(s)$$
$$= (b-t)\left[(s-a)u'(s)\Big|_a^t - \int_a^t u'(s)\, ds\right]$$
$$+ (t-a)\left[(b-s)u'(s)\Big|_t^b - \int_t^b u'(s)\, ds\right]$$

$$
\begin{aligned}
&= (b-t)\left[(t-a)\,u'(t) - (u(t)-u(a))\right] \\
&\qquad + (t-a)\left[-(b-t)\,u'(t) + u(b) - u(t)\right] \\
&= (t-a)\left[u(b)-u(t)\right] - (b-t)\left[u(t)-u(a)\right] \\
&= (b-a)\,\Phi_u(t)\,,
\end{aligned}
$$

for any $t \in [a,b]$, and the representation (2.1) is proved. □

The following result provides a sharp bound for $|\Phi_u|$ in the case when u' is of bounded variation.

Theorem 2.2. *Assume that $u : [a,b] \to \mathbb{R}$ is as in Lemma 2.1. If u' is of bounded variation on $[a,b]$, then*

$$
|\Phi_u(t)| \le \frac{(t-a)(b-t)}{b-a} \bigvee_a^b (u') \le \frac{1}{4}(b-a) \bigvee_a^b (u')\,, \tag{2.3}
$$

where $\bigvee_a^b (u')$ denotes the total variation of u' on $[a,b]$.

The inequalities are sharp and the constant $\frac{1}{4}$ is best possible.

Proof. It is well known that, if $p : [\alpha,\beta] \to \mathbb{R}$ is continuous and $v : [\alpha,\beta] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\,dv(s)$ exists and

$$
\left| \int_\alpha^\beta p(s)\,dv(s) \right| \le \sup_{s\in[\alpha,\beta]} |p(s)| \bigvee_\alpha^\beta (v)\,.
$$

Now, utilising the representation (2.1) we have successively:

$$
\begin{aligned}
&|\Phi_u(t)| \\
&\le \frac{1}{b-a}\left[(b-t)\left|\int_a^t (s-a)\,du'(s)\right| + (t-a)\left|\int_t^b (b-s)\,du'(s)\right|\right] \\
&\le \frac{1}{b-a}\left[(b-t)\sup_{s\in[a,t]}(s-a)\cdot\bigvee_a^t (u') + (t-a)\sup_{s\in[t,b]}(b-s)\cdot\bigvee_t^b (u')\right] \\
&= \frac{(t-a)(b-t)}{b-a}\left[\bigvee_a^t (u') + \bigvee_t^b (u')\right] = \frac{(t-a)(b-t)}{b-a}\bigvee_a^b (u')\,.
\end{aligned} \tag{2.4}
$$

The second inequality is obvious by the fact that $(t-a)(b-t) \le \frac{1}{4}(b-a)^2$, $t \in [a,b]$.

For the sharpness of the inequalities, assume that there exist $A, B > 0$ so that

$$
|\Phi_u(t)| \le A \cdot \frac{(t-a)(b-t)}{b-a}\bigvee_a^b (u') \le B(b-a)\bigvee_a^b (u')\,, \tag{2.5}
$$

with u as in the assumption of the theorem. Then, for $t = \frac{a+b}{2}$, we get from (2.5) that

$$
\left| \frac{u(a)+u(b)}{2} - u\left(\frac{a+b}{2}\right) \right| \le \frac{1}{4}A(b-a)\bigvee_a^b (u') \le B(b-a)\bigvee_a^b (u')\,. \tag{2.6}
$$

Consider the function $u : [a,b] \to \mathbb{R}$, $u(t) = \left|t - \frac{a+b}{2}\right|$. This function is absolutely continuous, $u'(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a,b] \setminus \left\{\frac{a+b}{2}\right\}$ and $\bigvee_a^b (u') = 2$. Then (2.6) becomes $\frac{b-a}{2} \le \frac{1}{2}A(b-a) \le 2B(b-a)$, which implies that $A \ge 1$ and $B \ge \frac{1}{4}$. □

Corollary 2.3. *With the assumptions of Theorem 2.2, we have*

$$\left|\frac{u(a)+u(b)}{2}-u\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{4}(b-a)\bigvee_{a}^{b}(u'). \tag{2.7}$$

The constant $\frac{1}{4}$ is best possible.

The Lipschitzian case is incorporated in the following result.

Theorem 2.4. *Assume that $u:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ with the property that u' is $K-$Lipschitzian on (a,b). Then*

$$|\Phi_u(t)|\leq\frac{1}{2}(t-a)(b-t)K\leq\frac{1}{8}(b-a)^2K. \tag{2.8}$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible.

Proof. We utilise the fact that, for an $L-$Lipschitzian function $p:[\alpha,\beta]\to\mathbb{R}$ and a Riemann integrable function $v:[\alpha,\beta]\to\mathbb{R}$, the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\,dv(s)$ exists and

$$\left|\int_\alpha^\beta p(s)\,dv(s)\right|\leq L\int_\alpha^\beta|p(s)|\,ds.$$

Then, by (2.1), we have that

$$\begin{aligned}|\Phi_u(t)|&\leq\frac{1}{b-a}\left[(b-t)\left|\int_a^t(s-a)\,du'(s)\right|+(t-a)\left|\int_t^b(b-s)\,du'(s)\right|\right]\\&\leq\frac{1}{b-a}\left[\frac{1}{2}K(b-t)(t-a)^2+\frac{1}{2}K(t-a)(b-t)^2\right]\\&=\frac{1}{2}(t-a)(b-t)K,\end{aligned} \tag{2.9}$$

which proves the first part of (2.8). The second part is obvious.

Now, for the sharpness of the constants, assume that there exist the constants $C,D>0$ such that

$$|\Phi_u(t)|\leq C(b-t)(t-a)K\leq D(b-a)^2K, \tag{2.10}$$

provided that u is as in the hypothesis of the theorem. For $t=\frac{a+b}{2}$, we get from (2.10) that

$$\left|\frac{u(a)+u(b)}{2}-u\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{4}CK(b-a)^2\leq D(b-a)^2K. \tag{2.11}$$

Consider $u:[a,b]\to\mathbb{R}$, $u(t)=\frac{1}{2}\left|t-\frac{a+b}{2}\right|^2$. Then $u'(t)=t-\frac{a+b}{2}$ is Lipschitzian with the constant $K=1$ and (2.11) becomes

$$\frac{1}{8}(b-a)^2\leq\frac{1}{4}C(b-a)^2\leq D(b-a)^2,$$

which implies that $C\geq\frac{1}{2}$ and $D\geq\frac{1}{8}$. □

Corollary 2.5. *With the assumptions of Theorem 2.4, we have*

$$\left|\frac{u(a)+u(b)}{2}-u\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{8}(b-a)^2K. \tag{2.12}$$

The constant $\frac{1}{8}$ is best possible.

Remark 2.6. If u' is absolutely continuous and $\|u''\|_\infty := ess\sup_{t\in[a,b]} |u''(t)| < \infty$, then we can take $K = \|u''\|_\infty$, and we have from (2.8) that

$$|\Phi_u(t)| \leq \frac{1}{2}(t-a)(b-t)\|u''\|_\infty \leq \frac{1}{8}(b-a)^2\|u''\|_\infty . \tag{2.13}$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible in (2.13).

From (2.12) we also get

$$\left|\frac{u(a)+u(b)}{2} - u\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{8}(b-a)^2\|u''\|_\infty , \tag{2.14}$$

in which $\frac{1}{8}$ is the best possible constant.

3. Bounds in the Case when u' is of Bounded Variation

We can start with the following result:

Theorem 3.1. *Assume that $u : [a,b] \to \mathbb{R}$ is as in Lemma 2.1. If u' and f are of bounded variation on $[a,b]$, then*

$$|D(f;u)| \leq \frac{1}{4}(b-a)\bigvee_a^b(u')\cdot\bigvee_a^b(f), \tag{3.1}$$

and the constant $\frac{1}{4}$ is best possible in (3.1).

Proof. We use the following representation of the functional $D(f;u)$ obtained in [8] (see also [9] or [6]):

$$D(f;u) = \int_a^b \Phi_u(t)\,df(t). \tag{3.2}$$

Then we have the bound

$$\begin{aligned} |D(f;u)| &= \left|\int_a^b \Phi_u(t)\,df(t)\right| \leq \sup_{t\in[a,b]}|\Phi_u(t)|\bigvee_a^b(f) \\ &\leq \frac{1}{b-a}\bigvee_a^b(u')\sup_{t\in[a,b]}[(t-a)(b-t)]\cdot\bigvee_a^b(f) \\ &= \frac{1}{4}(b-a)\bigvee_a^b(u')\cdot\bigvee_a^b(f), \end{aligned}$$

where, for the last inequality we have used (2.3).

To prove the sharpness of the constant $\frac{1}{4}$, assume that there is a constant $E > 0$ such that

$$|D(f;u)| \leq E(b-a)\bigvee_a^b(u')\cdot\bigvee_a^b(f). \tag{3.3}$$

Consider $u : [a,b] \to \mathbb{R}$, $u(t) = \left|t - \frac{a+b}{2}\right|$. Then $u'(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t \in [a,b]\setminus\left\{\frac{a+b}{2}\right\}$. The total variation on $[a,b]$ is 2 and

$$D(f;u) = -\int_a^{\frac{a+b}{2}} f(t)\,dt + \int_{\frac{a+b}{2}}^b f(t)\,dt = \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)f(t)\,dt.$$

Now, if we choose $f(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, then we obtain from (3.1) $b-a \leq 4E(b-a)$, which implies that $E \geq \frac{1}{4}$. □

The following result can be stated as well:

Theorem 3.2. *Assume that* $u:[a,b]\to\mathbb{R}$ *is as in Lemma 2.1. If the derivative* u' *is of bounded variation on* $[a,b]$ *while* f *is* $L-$*Lipschitzian on* $[a,b]$, *then*

$$|D(f;u)|\le\frac{1}{6}L(b-a)^2\bigvee_a^b(u'). \tag{3.4}$$

Proof. We have

$$\begin{aligned}|D(f;u)|&=\left|\int_a^b\Phi_u(t)\,df(t)\right|\le L\int_a^b|\Phi_u(t)|\,dt\\&\le\frac{L}{b-a}\bigvee_a^b(u')\int_a^b(t-a)(b-t)\,dt\\&=\frac{1}{6}L(b-a)^2\bigvee_a^b(u'),\end{aligned}$$

where for the second inequality we have used the inequality (2.3). □

Remark 3.3. It is an open problem whether or not the constant $\frac{1}{6}$ is the best possible constant in (3.4).

When the integrand f is monotonic, we can state the following result as well:

Theorem 3.4. *Assume that* u *is as in Theorem 3.1. If* f *is monotonic nondecreasing on* $[a,b]$, *then*

$$\begin{aligned}|D(f;u)|&\le2\cdot\frac{\bigvee_a^b(u')}{b-a}\cdot\int_a^b\left|t-\frac{a+b}{2}\right|f(t)\,dt\\&\le\begin{cases}\frac{1}{2}\bigvee_a^b(u')\max\{|f(a)|,|f(b)|\}(b-a);\\[2ex]\frac{1}{(q+1)^{1/q}}\bigvee_a^b(u')\|f\|_p(b-a)^{1/q}\quad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\bigvee_a^b(u')\|f\|_1,\end{cases}\end{aligned} \tag{3.5}$$

where $\|f\|_p:=\left(\int_a^b|f(t)|^p\,dt\right)^{\frac{1}{p}}$, $p\ge1$ *are the Lebesgue norms. The constants* 2 *and* $\frac{1}{2}$ *are best possible in (3.5).*

Proof. It is well known that, if $p:[\alpha,\beta]\to\mathbb{R}$ is continuous and $v:[\alpha,\beta]\to\mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and $\left|\int_\alpha^\beta p(t)\,dv(t)\right|\le\int_\alpha^\beta|p(t)|\,dv(t)$. Then, on applying this property for the integral $\int_a^b\Phi_u(t)\,df(t)$, we have

$$\begin{aligned}|D(f;u)|&=\left|\int_a^b\Phi_u(t)\,df(t)\right|\le\int_a^b|\Phi_u(t)|\,df(t)\\&\le\frac{\bigvee_a^b(u')}{b-a}\cdot\int_a^b(t-a)(b-t)\,df(t),\end{aligned} \tag{3.6}$$

where for the last inequality we used (2.3).

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\int_a^b (t-a)(b-t)\,df(t) = f(t)(b-t)(t-a)\Big|_a^b - \int_a^b [-2t+(a+b)]\,f(t)\,dt$$
$$= 2\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt,$$

which together with (3.6) produces the first part of (3.5).

The second part is obvious by the Hölder inequality applied for the integral $\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt$ and the details are omitted.

For the sharpness of the constants we use as examples $u(t) = \left|t-\frac{a+b}{2}\right|$ and $f(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t \in [a,b]$. The details are omitted. □

4. BOUNDS IN THE CASE WHEN u' IS LIPSCHITZIAN

The following result can be stated as well:

Theorem 4.1. *Let $u : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ with the property that u' is $K-$Lipschitzian on (a,b). If f is of bounded variation, then*

$$|D(f;u)| \le \frac{1}{8}(b-a)^2 K \bigvee_a^b (f). \tag{4.1}$$

The constant $\frac{1}{8}$ is best possible in (4.1).

Proof. Utilising (2.8), we have successively:

$$|D(f;u)| = \left|\int_a^b \Phi_u(t)\,df(t)\right| \le \sup_{t\in[a,b]} |\Phi_u(t)| \bigvee_a^b (f)$$
$$\le \frac{1}{2}K \sup_{t\in[a,b]} [(b-t)(t-a)] \bigvee_a^b (f)$$
$$= \frac{1}{8}(b-a)^2 K \bigvee_a^b (f),$$

and the inequality (4.1) is proved.

Now, for the sharpness of the constant, assume that the inequality holds with a constant $G > 0$, i.e.,

$$|D(f;u)| \le G(b-a)^2 K \bigvee_a^b (f). \tag{4.2}$$

for u and f as in the statement of the theorem.

Consider $u(t) := \frac{1}{2}\left(t-\frac{a+b}{2}\right)^2$ and $f(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t \in [a,b]$. Then $u'(t) = t-\frac{a+b}{2}$ is $K-$Lipschitzian with the constant $K = 1$ and

$$D(f;u) = \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\cdot\left(t-\frac{a+b}{2}\right) dt = \frac{(b-a)^2}{4}.$$

Since $\bigvee_a^b (f) = 2$, hence from (4.2) we get $\frac{(b-a)^2}{4} \le 2G(b-a)^2$, which implies that $G \ge \frac{1}{8}$. □

The following result may be stated as well:

Theorem 4.2. *Let $v:[a,b]\to\mathbb{R}$ be as in Theorem 4.1. If f is $L-$Lipschitzian on $[a,b]$, then*

$$|D(f;u)| \le \frac{1}{12}(b-a)^3 KL. \tag{4.3}$$

The constant $\frac{1}{12}$ is best possible in (4.3).

Proof. We have by (2.8), that:

$$\begin{aligned} |D(f;u)| &= \left|\int_a^b \Phi_u(t)\,df(t)\right| \\ &\le L\int_a^b |\Phi_u(t)|\,dt \\ &\le \frac{1}{2}LK\int_a^b (b-t)(t-a)\,dt = \frac{1}{12}KL(b-a)^3, \end{aligned}$$

and the inequality is proved.

For the sharpness, assume that (4.3) holds with a constant $F>0$. Then

$$|D(f;u)| \le F(b-a)^3 KL, \tag{4.4}$$

provided f and u are as in the hypothesis of the theorem.

Consider $f(t)=t-\frac{a+b}{2}$ and $u(t)=\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2$. Then u' is Lipschitzian with the constant $K=1$ and f is Lipschitzian with the constant $L=1$. Also,

$$D(f;u) = \int_a^b \left(t-\frac{a+b}{2}\right)^2 dt = \frac{(b-a)^3}{12},$$

and by (4.4) we get $\frac{(b-a)^3}{12} \le F(b-a)^3$ which implies that $F \ge \frac{1}{2}$. □

Finally, the case of monotonic integrands is enclosed in the following result.

Theorem 4.3. *Let $u:[a,b]\to\mathbb{R}$ be as in Theorem 4.1. If f is monotonic nondecreasing, then*

$$\begin{aligned} |D(f;u)| &\le K\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt \\ &\le \begin{cases} \frac{1}{4}K\max\{|f(a)|,|f(b)|\}(b-a)^2; \\ \frac{1}{2(q+1)^{1/q}}K\|f\|_p (b-a)^{1+1/q} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(b-a)K\|f\|_1. \end{cases} \end{aligned} \tag{4.5}$$

The first inequality is sharp. The constant $\frac{1}{4}$ is best possible.

Proof. We have

$$\begin{aligned} |D(f;u)| &\le \int_a^b |\Phi_u(t)|\,df(t) \\ &\le \frac{1}{2}K\int_a^b (b-t)(t-a)\,df(t) \\ &= K\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt \end{aligned}$$

and the first inequality is proved. The second part follows by the Hölder inequality.

The sharpness of the first inequality and of the constant $\frac{1}{4}$ follows by choosing $u(t) = \left|t-\frac{a+b}{2}\right|$ and $f(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$. The details are omitted. □

5. APPLICATIONS FOR THE ČEBYŠEV FUNCTIONAL

The above result can naturally be applied in obtaining various sharp upper bounds for the absolute value of the Čebyšev functional $C(f,g)$ defined by

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt, \tag{5.1}$$

where $f,g : [a,b] \to \mathbb{R}$ are Lebesgue integrable functions such that fg is also Lebesgue integrable.

There are various sharp upper bounds for $|C(f,g)|$ and in the following we will recall just a few of them.

In 1934, Grüss [13] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n) \tag{5.2}$$

under the assumptions that f and g satisfy the bounds

$$-\infty < m \le f(t) \le M < \infty \quad \text{and} \quad -\infty < n \le g(t) \le N < \infty \tag{5.3}$$

for almost every $t \in [a,b]$, where m, M, n, N are real numbers. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Another less known result, even though it was established by Čebyšev in 1882 [1], states that

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{5.4}$$

provided that f', g' exist and are continuous in $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be replaced by a smaller quantity. The Čebyšev inequality also holds if f,g are absolutely continuous on $[a,b]$, $f', g' \in L_\infty[a,b]$ and $\|\cdot\|_\infty$ is replaced by the $ess\sup$ norm $\|f'\|_\infty = ess\sup_{t\in[a,b]} |f'(t)|$.

In 1970, A. Ostrowski [16] considered a mixture between Grüss and Čebyšev inequalities by proving that

$$|C(f,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{5.5}$$

provided that f satisfies (5.3) and g is absolutely continuous and $g' \in L_\infty[a,b]$.

Three years after Ostrowski, A. Lupaş [14] obtained another bound for $C(f,g)$ in terms of the Euclidean norms of the derivatives. Namely, he proved that

$$|C(f,g)| \le \frac{1}{\pi^2}(b-a)\|f'\|_2 \|g'\|_2, \tag{5.6}$$

provided that f and g are absolutely continuous and $f', g' \in L_2[a,b]$. Here $\frac{1}{\pi^2}$ is also best possible.

Recently, Cerone and Dragomir [2], proved the following result:

$$|C(f,g)| \le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \tag{5.7}$$

provided $f \in L[a,b]$ and $g \in C[a,b]$.

As particular cases of (5.7), we can state the results:

$$|C(f,g)| \le \|g\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{5.8}$$

if $g \in C[a,b]$ and $f \in L[a,b]$ and

$$|C(f,g)| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \tag{5.9}$$

where $m \leq g(x) \leq M$ for $x \in [a,b]$. The constants 1 in (5.8) and $\frac{1}{2}$ in (5.9) are best possible. The inequality (5.9) has been obtained before in a different way in [5].

For generalisations in abstract Lebesgue spaces, best constants and discrete versions, see [3]. For other results on the Čebyšev functional, see [6], [7] and [12].

Now, assume that $g : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$. Then the function $u(t) := \int_a^t g(s)\,ds$ is absolutely continuous on $[a,b]$ and we can consider the function

$$\tilde{\Phi}_g(t) := \Phi_u(t) = \int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds, \quad t \in [a,b]. \tag{5.10}$$

Utilising Lemma 2.1, we can state the following representation result.

Lemma 5.1. *If g is absolutely continuous, then*

$$\tilde{\Phi}_g(t) = \frac{1}{b-a}\int_a^b K(t,s)\,dg(s), \quad t \in [a,b], \tag{5.11}$$

where K is given by (2.2).

As a consequence of Theorems 2.2 and 2.4, we also have the inequalities:

Proposition 5.2. *Assume that g is Lebesgue integrable on $[a,b]$.*

(i) *If g is of bounded variation on $[a,b]$, then*

$$\left|\tilde{\Phi}_g(t)\right| \leq \frac{(t-a)(b-t)}{b-a}\bigvee_a^b(g) \leq \frac{1}{4}(b-a)\bigvee_a^b(g). \tag{5.12}$$

The inequalities are sharp and $\frac{1}{4}$ is best possible.

(ii) *If g is $K-$Lipschitzian on $[a,b]$, then*

$$\left|\tilde{\Phi}_g(t)\right| \leq \frac{1}{2}(b-t)(t-a)K \leq \frac{1}{8}(b-a)^2 K. \tag{5.13}$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible.

We notice that the functions $g_1 : [a,b] \to \mathbb{R}$, $g_1(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ and $g_2 : [a,b] \to \mathbb{R}$, $g(t) = \left(t - \frac{a+b}{2}\right)$ realise equality in (5.12) and (5.13), respectively.

Now, we observe that for $u(t) = \int_a^t g(s)\,ds$, $s \in [a,b]$, we have the identity:

$$D(f,u) = (b-a)\,C(f,g). \tag{5.14}$$

Utilising this identity and Theorems 3.1 and 3.4, we can state the following result.

Proposition 5.3. *Assume that g is of bounded variation on $[a,b]$.*

(i) *If f is of bounded variation on $[a,b]$, then*

$$|C(f,g)| \leq \frac{1}{4}\bigvee_a^b(g)\cdot\bigvee_a^b(f). \tag{5.15}$$

The constant $\frac{1}{4}$ is best possible in (5.15).

(ii) *If f is monotonic nondecreasing, then*

$$|C(f,g)| \leq 2\bigvee_a^b (g) \cdot \frac{1}{(b-a)^2} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\, dt \tag{5.16}$$

$$\leq \begin{cases} \frac{1}{2} \cdot \bigvee_a^b (g) \max\{|f(a)|, |f(b)|\}; \\ \\ \frac{1}{(q+1)^{1/q}} \bigvee_a^b (g) \|f\|_p (b-a)^{-1/p} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{b-a} \bigvee_a^b (g) \|f\|_1 . \end{cases}$$

The multiplicative constants 2 and $\frac{1}{2}$ are best possible in (5.16).

Finally, by Theorems 4.1 – 4.3 we also have the following sharp bounds for the Čebyšev functional $C(f,g)$.

Proposition 5.4. *Assume that g is $K-$Lipschitzian on $[a,b]$.*

(i) *If f is of bounded variation, then*

$$|C(f,g)| \leq \frac{1}{8} \cdot (b-a) K \bigvee_a^b (f). \tag{5.17}$$

The constant $\frac{1}{8}$ is best possible.

(ii) *If f is $L-$Lipschitzian, then*

$$|C(f,g)| \leq \frac{1}{12} (b-a)^2 KL. \tag{5.18}$$

The constant $\frac{1}{12}$ is best possible in (5.18).

(iii) *If f is monotonic nondecreasing, then*

$$|C(f,g)| \leq K \cdot \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\, dt \tag{5.19}$$

$$\leq \begin{cases} \frac{1}{4} K (b-a) \max\{|f(a)|, |f(b)|\}; \\ \\ \frac{1}{2(q+1)^{1/q}} K (b-a)^{1/q} \|f\|_p \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2} K \|f\|_1 . \end{cases}$$

The first inequality is sharp. The constant $\frac{1}{4}$ is best possible.

Remark 5.5. The inequalities (5.15) and (5.17) were obtained by P. Cerone and S.S. Dragomir in [4, Corollary 3.5]. However, the sharpnes of the constants $\frac{1}{4}$ and $\frac{1}{8}$ were not discussed there. Inequality (5.18) is similar to the Čebyšev inequality (5.4).

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