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SHARP INTEGRAL INEQUALITIES FOR PRODUCTS OF CONVEX FUNCTIONS

VILLŐ CSISZÁR AND TAMÁS F. MÓRI

DEPARTMENT OF PROBABILITY THEORY AND STATISTICS
LORÁND EÖTVÖS UNIVERSITY
PÁZMÁNY P. S. 1/C, H-1117 BUDAPEST, HUNGARY
villo@ludens.elte.hu

moritamas@ludens.elte.hu

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ABSTRACT. In this note we present exact lower and upper bounds for the integral of a product of nonnegative convex resp. concave functions in terms of the product of individual integrals. They are found by adapting the convexity method to the case of product sets.

Key words and phrases: Convexity, Chebyshev's integral inequality, Grüss inequality, Andersson inequality.

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1. INTRODUCTION

Let f and g be integrable functions defined on the interval $[a, b]$, such that fg is integrable. Let us introduce the quantities

$$\begin{aligned} A = A(f,g) &= \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx, \\ B = B(f,g) &= \frac{1}{b-a}\int_a^b f(x)g(x)\,dx. \end{aligned} \tag{1.1}$$

It is well known that $A \le B$ if both f and g are either increasing or decreasing. On the other hand, when f and g possess opposite monotonicity properties, $A \ge B$ holds. These are sometimes referred to as Chebyshev inequalities.

When f and g are supposed to be bounded, the classical Grüss inequality [5] provides an upper bound for the difference $B - A$.

For convex and increasing functions with $f(a) = g(a) = 0$ Andersson [1] showed that Chebyshev's inequality can be improved by a constant factor, namely, $B \ge \frac{4}{3}A$. The requirement of convexity can be somewhat relaxed, see Fink [4].

In the case where both f and g are nonnegative convex functions, Pachpatte [8] presented (and Cristescu [2] corrected) linear upper bounds for certain triple integrals in terms of $(b-a)^{-1}\int_a^b f(x)g(x)\,dx$ and $[f(a)+f(b)][g(a)+g(b)]$.

192-07

The aim of the present note is to analyse the exact connection between the quantities A and B in the case where both f and g are nonnegative and either convex or concave functions. We will compute exact upper and lower bounds by adapting the convexity method to our problem. That method is often applied to characterize the range of several integral-type functionals when the domain is a convex set of functions. A detailed description of the method and some examples of applications can be found in [3] or [7].

Notice that $a = 0$, $b = 1$ can be assumed without loss of generality. Indeed, let us introduce $\widetilde{f}(t) = f(a(1-t)+bt)$ and $\widetilde{g}(t) = g(a(1-t)+bt)$, $0 \le t \le 1$. Then $\widetilde{f}$ and $\widetilde{g}$ are convex (concave) functions, provided that f and g are, and

$$\frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx = \int_0^1 \widetilde{f}(t)\,dt \cdot \int_0^1 \widetilde{g}(t)\,dt,$$
$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx = \int_0^1 \widetilde{f}(t)\widetilde{g}(t)\,dt.$$

The paper is organized as follows.

Section 2 contains a description of a variant of the convexity method adapted to the case of product sets.

In Section 3 unimprovable upper and lower bounds are derived for B in terms of A and $[f(a)+f(b)][g(a)+g(b)]$, in the case of nonnegative convex continuous functions f and g, see Corollary 3.5.

In Section 4 the range of B is determined as a function of A, for nonnegative concave functions f and g.

In the last section we briefly deal with the more general case of multiple products.

2. The Convexity Method on Products of Convex Sets

Let $(\mathcal{X}, \mathcal{B}, \lambda)$ be a measure space and $\mathcal{F}$ a closed convex set of λ-integrable functions $f : \mathcal{X} \to \mathbb{R}$. Suppose $\mathcal{H} = \{h_\theta : \theta \in \Theta\} \subset \mathcal{F}$ is a generating subset, given in parametrized form, in the sense that for every $f \in \mathcal{F}$ one can find a probability measure μ defined on the Borel sets of the parameter space Θ such that

$$f(x) = \int_\Theta h_\theta(x)\,\mu(d\theta), \tag{2.1}$$

that is, every $f \in \mathcal{F}$ has a representation as a mixture of elements in $\mathcal{H}$. (Of course, the function $\theta \mapsto h_\theta(x)$ is supposed to be measurable, for λ-a. e. $x \in \mathcal{X}$.) Then all integrals of the form (2.1) belong to $\mathcal{F}$, and the set $\left\{\int_\mathcal{X} f\,d\lambda : f \in \mathcal{F}\right\}$ is equal to the closed convex hull of the set $\left\{\int_\mathcal{X} h_\theta\,d\lambda : \theta \in \Theta\right\}$.

Suppose we are given a pair of functions in the form

$$f(x) = \int_\Theta h_\theta(x)\,\mu(d\theta), \quad g(x) = \int_\Theta h_\theta(x)\,\nu(d\theta).$$

Then by interchanging the order of integration one can see that

$$\begin{aligned} B(f,g) = \int_\mathcal{X} fg\,d\lambda &= \int_\mathcal{X}\left(\int_\Theta h_\theta(x)\,\mu(d\theta)\int_\Theta h_\tau(x)\,\nu(d\tau)\right)\lambda(dx) \\ &= \int_\Theta\int_\Theta\left(\int_\mathcal{X} h_\theta(x)h_\tau(x)\,\lambda(dx)\right)\mu(d\theta)\,\nu(d\tau) \\ &= \int_\Theta\int_\Theta B(h_\theta, h_\tau)\,\mu(d\theta)\,\nu(d\tau), \end{aligned}$$

and similarly,

$$\begin{aligned}A(f,g) &= \int_{\mathcal{X}} f\,d\lambda \int_{\mathcal{X}} g\,d\lambda = \int_{\mathcal{X}}\int_{\Theta} h_\theta(x)\,\mu(d\theta)\,\lambda(dx)\ \int_{\mathcal{X}}\int_{\Theta} h_\tau(x)\,\nu(d\tau)\,\lambda(dx)\\ &= \int_{\Theta}\int_{\Theta}\left(\int_{\mathcal{X}} h_\theta(x)\,\lambda(dx)\int_{\mathcal{X}} h_\tau(x)\,\lambda(dx)\right)\mu(d\theta)\,\nu(d\tau)\\ &= \int_{\Theta}\int_{\Theta} A(h_\theta,h_\tau)\,\mu(d\theta)\,\nu(d\tau).\end{aligned}$$

(The order of integration can be interchanged by Fubini's theorem, under suitable conditions; for instance, when all functions in $\mathcal{F}$ are nonnegative.)

Thus, in this case we can say that the planar set

$$S(\mathcal{F}) = \left\{\big(A(f,g),\ B(f,g)\big) : f,g\in\mathcal{F}\right\}$$

is still a subset of the closed convex hull of

$$S(\mathcal{H}) = \left\{\big(A(h_\theta,h_\tau),\ B(h_\theta,h_\tau)\big) : \theta,\tau\in\Theta\right\},$$

but in general equality does not necessarily hold. However, if $S(\mathcal{H})$ entirely contains the boundary of its convex hull, we can conclude that

$$\begin{aligned}\text{(2.2)}\quad &\min/\max\{B(f,g) : f,g\in\mathcal{F},\ A(f,g)=A\}\\ &\qquad\qquad = \min/\max\{B(h_\theta,h_\tau) : \theta,\tau\in\Theta,\ A(h_\theta,h_\tau)=A\}.\end{aligned}$$

3. Exact Bounds in the Case of Convex Functions

When f and g are nonnegative and convex, we can suppose that $f(a)+f(b)=g(a)+g(b)=1$, because they appear as multiplicative factors in the integrals. If there is an upper or lower bound of the form

$$B(f,g) \le (\ge)\ F\big(A(f,g)\big)$$

in this particular case, it can be extended to the general case as

$$\text{(3.1)}\qquad B(f,g) \le (\ge)\ [f(a)+f(b)]\,[g(a)+g(b)]\,F\left(\frac{A(f,g)}{[f(a)+f(b)]\,[g(a)+g(b)]}\right).$$

So let

$$\text{(3.2)}\qquad \mathcal{F} = \{f : [0,1]\to\mathbb{R} :\ f \text{ is convex, continuous, } f\ge 0,\ f(0)+f(1)=1\}.$$

The following lemma describes the extremal points of $\mathcal{F}$.

Lemma 3.1 ([7, Theorem 2.1])**.** *The set of extremal points of $\mathcal{F}$ is equal to*

$$\mathcal{H} = \{h_\theta,\ k_\theta : 0<\theta\le 1\},$$

where $h_\theta(x) = \left(1-\frac{x}{\theta}\right)^+$, and $k_\theta(x) = h_\theta(1-x) = \left(1-\frac{1-x}{\theta}\right)^+$.

We are going to find the set $S(\mathcal{F})$ by using the method described in Section 2.

Theorem 3.2.

$$S(\mathcal{F}) = \left\{(A,B) :\ 0<A\le\frac14,\ \max\left(0,\ \frac{(4\sqrt{A}-1)^3}{24A}\right)\le B\le\frac23\sqrt{A}\right\}.$$

Proof. By (2.2), the first thing we do is characterize $S(\mathcal{H})$. It is the union of the following four sets.

$$\begin{aligned} S_{11} &= \left\{ \big(A(h_\theta, h_\tau),\ B(h_\theta, h_\tau)\big) : \theta, \tau \in \Theta \right\}, \\ S_{12} &= \left\{ \big(A(h_\theta, k_\tau),\ B(h_\theta, k_\tau)\big) : \theta, \tau \in \Theta \right\}, \\ S_{21} &= \left\{ \big(A(k_\theta, h_\tau),\ B(k_\theta, h_\tau)\big) : \theta, \tau \in \Theta \right\}, \\ S_{22} &= \left\{ \big(A(k_\theta, k_\tau),\ B(k_\theta, k_\tau)\big) : \theta, \tau \in \Theta \right\}. \end{aligned}$$

Since A and B are symmetric functions, S_{12} and S_{21} are obviously identical. In addition, $S_{11} \equiv S_{22}$, because transformation $t \leftrightarrow 1-t$ does not alter the integrals but it maps h_θ into k_θ. Thus, it suffices to deal with S_{11} and S_{12}.

Let us start with S_{11}. By symmetry we can assume that $\theta \le \tau$. Then clearly, $A(h_\theta, h_\tau) = \frac{\theta\tau}{4}$, and $B(h_\theta, h_\tau) = \frac{\theta(3\tau-\theta)}{6\tau}$. Let us fix $A(h_\theta, h_\tau) = A$, then $\theta \le 2\sqrt{A} \le \tau$, and $B(h_\theta, h_\tau) = \frac{\theta(12A-\theta^2)}{24A}$ is maximal if $\theta = \tau = 2\sqrt{A}$, with a maximum equal to $\frac{2}{3}\sqrt{A}$.

Turning to S_{12} we find that $A(h_\theta, k_\tau) = \dfrac{\theta\tau}{4}$ again, and $B(h_\theta, k_\tau) = \frac{(\theta+\tau-1)^3}{6\theta\tau}$ if $\theta+\tau > 1$, and 0 otherwise. Hence B is minimal if, and only if $\theta+\tau$ is minimal; that is, $\theta = \tau = 2\sqrt{A}$. The minimum is equal to $\frac{(4\sqrt{A}-1)^3}{24A}$, if $A > 1/16$, and 0 otherwise. Finally, by Chebyshev's inequality cited in the Introduction we have that

$$B(h_\theta, k_\tau) \le A(h_\theta, k_\tau) = A(h_\theta, h_\tau) \le B(h_\theta, h_\tau),$$

thus the upper boundary of $S_{11} \cup S_{12}$ is that of S_{11}, and the lower boundary is that of S_{12} (see Figure 3.1 after Remark 3.3).

If we show that the lower boundary of $S(\mathcal{H})$ is convex and the upper one is concave, (2.2) will imply that $S(\mathcal{F})$ has the same lower and upper boundaries. It is obvious for the upper boundary, and it follows for the lower boundary by the positivity of the second derivative

$$\frac{d^2B}{dA^2} = -\frac{2}{3}\,A^{-3/2} + \frac{3}{8}\,A^{-5/2} - \frac{1}{12}\,A^{-3} = \frac{(4\sqrt{A}-1)(2-\sqrt{A}-4A)}{24A^3}$$

for $1/16 < A \le 1/4$.

Finally, we show that every point of the convex hull of $S(\mathcal{H})$ is an element of $S(\mathcal{F})$. Let $0 < A \le 1/4$, and $B(h_\theta, k_\theta) < B < B(h_\theta, h_\theta)$, where $\theta = 2\sqrt{A}$. Then $B = \alpha B(h_\theta, k_\theta) + (1-\alpha)B(h_\theta, h_\theta)$ for some α, $0 < \alpha < 1$. Suppose first that $\alpha > 1/2$ and look for f and g in the form $f = ph_\theta + (1-p)k_\theta$, $g = (1-p)h_\theta + pk_\theta$, with a suitable $p \in (0,1)$. By the bilinearity of B we have that

$$\begin{aligned} B(f,g) &= p(1-p)B(h_\theta, h_\theta) + p^2 B(h_\theta, k_\theta) + (1-p)^2 B(k_\theta, h_\theta) + (1-p)pB(k_\theta, k_\theta) \\ &= 2p(1-p)B(h_\theta, h_\theta) + [p^2 + (1-p)^2]B(h_\theta, k_\theta), \end{aligned}$$

thus we obtain the equation $2p(1-p) = 1-\alpha$. It is satisfied by $p = \frac{1}{2}\big(1 \pm \sqrt{2\alpha - 1}\,\big)$.

Next, suppose that $\alpha \le 1/2$. This time let $f = g = ph_\theta + (1-p)k_\theta$. Then

$$\begin{aligned} B(f,g) &= p^2 B(h_\theta, h_\theta) + p(1-p)B(h_\theta, k_\theta) + (1-p)pB(k_\theta, h_\theta) + (1-p)^2 B(k_\theta, k_\theta) \\ &= 2p(1-p)B(h_\theta, k_\theta) + [p^2 + (1-p)^2]B(h_\theta, h_\theta), \end{aligned}$$

therefore $2p(1-p) = \alpha$, and the solution is $p = \frac{1}{2}\big(1 \pm \sqrt{1-2\alpha}\,\big)$. □

Remark 3.3. Linear upper and lower bounds can be obtained by drawing the tangent lines to the upper resp. lower boundaries at the points $(1/4,\ 1/3)$, resp. $(1/4,\ 1/6)$. They are as follows.

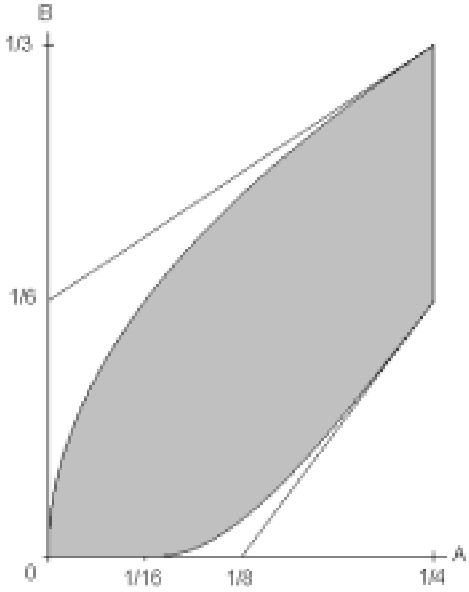

Figure 3.1: $S(\mathcal{F})$ with the linear bounds of (3.3).

$$\frac{4}{3}A - \frac{1}{6} \leq B \leq \frac{2}{3}A + \frac{1}{6}. \tag{3.3}$$

Remark 3.4. Based solely on A, that is, without involving another quantity like $[f(a) + f(b)]\,[g(a) + g(b)]$, we cannot expect any useful bound for B. Indeed, let A be fixed, and $f = 4Ah_\theta/\theta$ with a small θ. Then choosing $g = h_\theta$ gives $A(f,g) = A$ and $B(f,g) = \frac{4}{3}A/\theta$, thus B can be arbitrarily large. On the other hand, with $g = k_\theta$ we have $B = 0$.

At the end of this section we repeat our main result in the original setting. Theorem 3.2 combined with (3.1) yields the following exact bounds. With the notations of (1.1) and $C = [f(a) + f(b)]\,[g(a) + g(b)]$ we have

Corollary 3.5.

(1) *Upper bound.*

$$B \leq \frac{2}{3}\sqrt{AC}.$$

(2) *Lower bound.*

If $A < C/16$, there is no lower estimate better than the trivial one $B \geq 0$.
On the other hand, if $A \geq C/16$, then

$$B \geq \frac{\sqrt{C}\left(4\sqrt{A} - \sqrt{C}\right)^3}{24A}.$$

If one prefers linear lower and upper bounds of Cristescu style [2] at the expense of accuracy, (3.3) transforms into

$$\frac{4}{3}A - \frac{1}{6}C \leq B \leq \frac{2}{3}A + \frac{1}{6}C. \tag{3.4}$$

4. EXACT BOUNDS IN THE CASE OF CONCAVE FUNCTIONS

Let f and g be nonnegative concave functions. We shall suppose that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b g(x)\,dx = 1.$$

We fix $A = 1$, and by computing the range of B we obtain exact lower and upper bounds for the ratio $B(f,g)/A(f,g)$ in the general case.

Thus, the set of functions in consideration is

$$\mathcal{F} = \left\{ f : [0,1] \to \mathbb{R} :\ f \text{ is concave, } f \ge 0,\ \int_0^1 f(x)\,dx = 1 \right\}.$$

The extremal points of $\mathcal{F}$ are the triangle functions.

Lemma 4.1 ([3, Example 5 in Section 1])**.** *The set of extremal points of $\mathcal{F}$ is equal to*

$$\mathcal{H} = \{h_\theta : 0 \le \theta \le 1\},$$

where $h_0(x) = 2(1-x),\ h_1(x) = 2x$*, and*

$$h_\theta(x) = \begin{cases} 2\,\frac{x}{\theta}, & \textit{if} \quad 0 \le x < \theta, \\ 2\,\frac{1-x}{1-\theta}, & \textit{if} \quad \theta \le x \le 1, \end{cases}$$

for $0 < \theta < 1$.

Theorem 4.2. $\{B(f,g) : f, g \in \mathcal{F}\} = [2/3,\ 4/3]$.

Proof. By the reasoning of Section 2 we can see that

$$\{B(f,g) : f,g \in \mathcal{F}\} \subset \left[\min_{\theta,\tau} B(h_\theta, h_\tau),\ \max_{\theta,\tau} B(h_\theta, h_\tau)\right]. \tag{4.1}$$

While computing the right-hand side we can assume that $\theta \le \tau$. Thus,

$$\begin{aligned}\int_0^1 h_\theta(x)h_\tau(x)\,dx &= \int_0^\theta \frac{4x^2}{\theta\tau}\,dx + \int_\theta^\tau \frac{4(1-x)x}{(1-\theta)\tau}\,dx + \int_\tau^1 \frac{4(1-x)^2}{(1-\theta)(1-\tau)}\,dx \\ &= \frac{4\theta^2}{3\tau} + \frac{6(\tau^2-\theta^2) - 4(\tau^3-\theta^3)}{3(1-\theta)\tau} + \frac{4(1-\tau)^2}{3(1-\theta)} \\ &= \frac{4\tau - 2\theta^2 - 2\tau^2}{3(1-\theta)\tau}.\end{aligned}$$

This is a decreasing function of τ for every fixed θ, hence the maximum is attained when $\tau = \theta$, and the minimum, when $\tau = 1$. In the former case $B = 4/3$, independently of θ. In the latter case $B = \frac{2}{3}(1+\theta)$, which is minimal for $\theta = 0$.

On the other hand, since the range of $B(h_0, h_\tau)$, as τ runs from 0 to 1, is equal to the closed interval $[2/3,\ 4/3]$, we get that (4.1) holds with equality. □

Corollary 4.3. *Let f and g be nonnegative concave functions defined on $[a,b]$. Then*

$$\begin{aligned}&\frac{2}{3}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx \\ &\qquad \le \frac{1}{b-a}\int_a^b f(x)g(x)\,dx \le \frac{4}{3}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx.\end{aligned}$$

5. MULTIPLE PRODUCTS

A natural generalization of the problem is the case of multiple products, that is, where $f_1, \dots, f_n$ all belong to some class $\mathcal{F}$, and

$$A = \prod_{i=1}^{n} \int_0^1 f_i(x)\,dx, \quad B = \int_0^1 \prod_{i=1}^{n} f_i(x)\,dx.$$

The aim is to find lower and upper estimates for B in terms of A.

The reasoning of Section 2 can easily be extended to this case. Convex lower and concave upper estimates derived in the particular case where all functions are taken from a generating set $\mathcal{H} \subset \mathcal{F}$ remain valid even if the functions can come from $\mathcal{F}$.

The easiest to repeat among the results of Sections 3 and 4 is the upper estimate for convex functions. Let $\mathcal{F}$ be the set defined in (3.2), and $\mathcal{H}$ the set of extremals characterized by Lemma 3.1. Then we have the following sharp upper bound.

Theorem 5.1.

$$B \le \frac{2}{n+1}\,A^{1/n}. \tag{5.1}$$

(Compare this with Andersson's result $B \ge \dfrac{2^n}{n+1}\,A$, which is valid for increasing convex functions with $f(0) = 0$.)

Proof. Let us divide $S(\mathcal{H})$ into $n+1$ parts, $S(\mathcal{H}) = \cup_{i=0}^{n} S_i$, according to the number of functions h_θ among the n arguments (the other functions are of the form k_θ). Clearly, $S_i \equiv S_{n-i}$. When dealing with $\max B$ for fixed A, we may focus on S_0, because A does not change if every k_θ is substituted with the corresponding h_θ, while B increases by Chebyshev's inequality. Thus, let our convex functions be $f_i = h_{\theta_i}$, $1 \le i \le n$, with $0 \le \theta_1 \le \cdots \le \theta_n \le 1$, and suppose that

$$\prod_{i=1}^{n} \theta_i = 2^n A$$

is fixed. Maximize

$$B = \int_0^{\theta_1} \prod_{i=1}^{n} \Big(1 - \frac{x}{\theta_i}\Big)\,dx.$$

We are going to show that the the integrand is pointwise maximal if $\theta_1 = \cdots = \theta_n$. Then by increasing θ_1 we also increase the domain of integration, hence

$$\max B = \int_0^{\theta} \Big(1 - \frac{x}{\theta}\Big)^n dx = \frac{\theta}{n+1}\,,$$

where $\theta^n = 2^n A$.

Let $z_i = -\log \theta_i$, then $(z_1 + \cdots + z_n)/n = -\log \theta$. We have to show that

$$\prod_{i=1}^{n} \Big(1 - \frac{x}{\theta_1}\Big) \le \Big(1 - \frac{x}{\theta}\Big)^n,$$

or equivalently,

$$\frac{1}{n} \sum_{i=1}^{n} \varphi(z_i) \le \varphi\Big(\frac{z_1 + \cdots + z_n}{n}\Big), \tag{5.2}$$

where $\varphi(t) = 1 - x\,e^t$. Here φ is concave, for its second derivative

$$\varphi''(t) = -\frac{x\,e^t}{\varphi(t)^2} \le 0.$$

Thus, (5.2) is implied by the Jensen inequality.

Now the proof can be completed by noting that the upper bound in (5.1) is a concave function of A. □

Theorem 5.1 immediately implies the following sharp inequality.

Corollary 5.2. *Let $f_1, \dots, f_n$ be nonnegative convex continuous functions defined on the interval $[a, b]$. Then*

$$\int_a^b \prod_{i=1}^n f_i(x)\,dx \le \frac{2}{n+1} \left(\prod_{i=1}^n \int_a^b f_i(x)\,dx \right)^{\frac{1}{n}} \left(\prod_{i=1}^n [f_i(a) + f_i(b)] \right)^{1-\frac{1}{n}}.$$

Remark 5.3. The continuity of the functions f_i can be left out from the set of conditions. Being convex, they are continuous on the open interval (a, b), but can have jumps at a or b. If we redefine them at the endpoints so that they become continuous, the integrals do not change, but the sums $f_i(a) + f_i(b)$ decrease. Therefore the upper bound obtained for continuous functions remains valid in the general case.

REFERENCES

[1] B.X. ANDERSON, An inequality for convex functions, *Nordisk Mat. Tidsk*, **6** (1958), 25–26.

[2] G. CRISTESCU, Improved integral inequalities for products of convex functions, *J. Inequal. Pure and Appl. Math.*, **6**(2) (2005), Art. 35. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=504`].

[3] V. CSISZÁR AND T.F. MÓRI, The convexity method of proving moment-type inequalities, *Statist. Probab. Lett.*, **66** (2004), 303–313.

[4] A.M. FINK, Andersson's inequality, *Math. Inequal. Appl.*, **6** (2003), 241–245.

[5] G. GRÜSS, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt$, *Math. Z.*, **39** (1935), 215–226.

[6] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer, Dordrecht, 1993.

[7] T.F. MÓRI, Exact integral inequalities for convex functions, *J. Math. Inequal.*, **1** (2007), 105–116.

[8] B.G. PACHPATTE, On some inequalities for convex functions, *RGMIA Res. Rep. Coll.*, **6**(E) (2004). [ONLINE: `http://rgmia.vu.edu.au/v6(E).html`].