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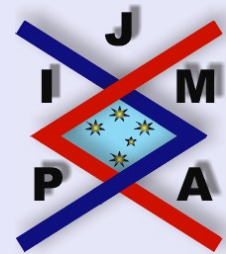
ON GRÜSS LIKE INTEGRAL INEQUALITIES VIA POMPEIU'S MEAN VALUE THEOREM

B.G. PACHPATTE

57 Shri Niketan Colony
Near Abhinay Talkies
Aurangabad 431 001 (Maharashtra) India.

EMail: bgpachpatte@hotmail.com

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Abstract

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Abstract

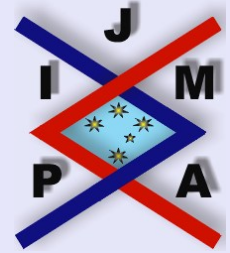
In the present note we establish two new integral inequalities similar to that of the Grüss integral inequality via Pompeiu's mean value theorem.

2000 Mathematics Subject Classification: 26D15, 26D20.

Key words: Grüss like integral inequalities, Pompeiu's mean value theorem, Lagrange's mean value theorem, Differentiable, Properties of modulus.

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1. Introduction

In 1935 G. Grüss [4] proved the following integral inequality (see also [5, p. 296]):

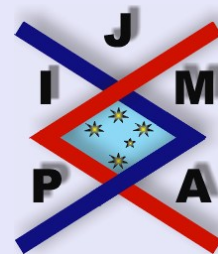
$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq \frac{1}{4} (P-p)(Q-q),$$

provided that f and g are two integrable functions on $[a, b]$ such that

$$p \leq f(x) \leq P, \quad q \leq g(x) \leq Q,$$

for all $x \in [a, b]$, where p, P, q, Q are constants.

The inequality (1.1) has evoked the interest of many researchers and numerous generalizations, variants and extensions have appeared in the literature, see [1], [3], [5] – [10] and the references cited therein. The main aim of this note is to establish two new integral inequalities similar to the inequality (1.1) by using a variant of Lagrange's mean value theorem, now known as the Pompeiu's mean value theorem [11] (see also [12, p. 83] and [2]).



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2. Statement of Results

In what follows, \mathbb{R} and $'$ denote the set of real numbers and derivative of a function respectively. For continuous functions $p, q : [a, b] \rightarrow \mathbb{R}$ which are differentiable on (a, b) , we use the notations

$$G[p, q] = \int_a^b p(x) q(x) dx - \frac{1}{b^2 - a^2} \left[\left(\int_a^b p(x) dx \right) \left(\int_a^b xq(x) dx \right) + \left(\int_a^b q(x) dx \right) \left(\int_a^b xp(x) dx \right) \right],$$

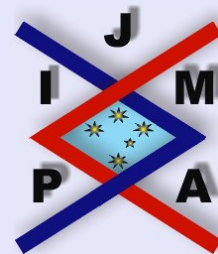
$$H[p, q] = \int_a^b p(x) q(x) dx - \frac{3}{b^3 - a^3} \left(\int_a^b xp(x) dx \right) \left(\int_a^b xq(x) dx \right),$$

to simplify the details of presentation and define $\|p\|_\infty = \sup_{t \in [a, b]} |p(t)|$.

In the proofs of our results we make use of the following theorem, which is a variant of the well known Lagrange's mean value theorem given by Pompeiu in [11] (see also [2, 12]).

Theorem 2.1 (Pompeiu). *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$ there exists a point c in (x_1, x_2) such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(c) - cf'(c).$$



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Our main result is given in the following theorem.

Theorem 2.2. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $[a, b]$ not containing 0. Then*

$$(2.1) \quad |G[f, g]| \leq \|f - lf'\|_\infty \int_a^b |g(x)| \left| \frac{1}{2} - \frac{x}{a+b} \right| dx \\ + \|g - lg'\|_\infty \int_a^b |f(x)| \left| \frac{1}{2} - \frac{x}{a+b} \right| dx,$$

where $l(t) = t, t \in [a, b]$.

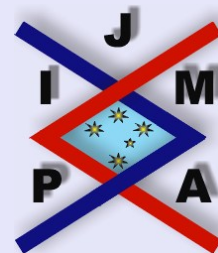
A slight variant of Theorem 2.2 is embodied in the following theorem.

Theorem 2.3. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $[a, b]$ not containing 0. Then*

$$(2.2) \quad |H[f, g]| \leq \|f - lf'\|_\infty \|g - lg'\|_\infty |M|,$$

where $l(t) = t, t \in [a, b]$ and

$$(2.3) \quad M = (b - a) \left\{ 1 - \frac{3}{4} \cdot \frac{(a + b)^2}{a^2 + ab + b^2} \right\}.$$



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3. Proofs of Theorems 2.2 and 2.3

From the hypotheses of Theorems 2.2 and 2.3 and using Theorem 2.1 for $t \neq x$, $x, t \in [a, b]$, there exist points c and d between x and t such that

$$(3.1) \quad t f(x) - x f(t) = [f(c) - cf'(c)](t - x),$$

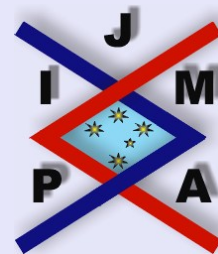
$$(3.2) \quad t g(x) - x g(t) = [g(d) - dg'(d)](t - x).$$

Multiplying (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we have

$$(3.3) \quad 2t f(x)g(x) - xg(x)f(t) - xf(x)g(t) \\ = [f(c) - cf'(c)](t - x)g(x) + [g(d) - dg'(d)](t - x)f(x).$$

Integrating both sides of (3.3) with respect to t over $[a, b]$ we have

$$(3.4) \quad (b^2 - a^2) f(x)g(x) - xg(x) \int_a^b f(t) dt - xf(x) \int_a^b g(t) dt \\ = [f(c) - cf'(c)] \left\{ \frac{b^2 - a^2}{2} g(x) - xg(x)(b - a) \right\} \\ + [g(d) - dg'(d)] \left\{ \frac{b^2 - a^2}{2} f(x) - xf(x)(b - a) \right\}.$$



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Now, integrating both sides of (3.4) with respect to x over $[a, b]$ we have

$$\begin{aligned}
 (3.5) \quad & (b^2 - a^2) \int_a^b f(x) g(x) dx \\
 & - \left(\int_a^b f(t) dt \right) \left(\int_a^b xg(x) dx \right) - \left(\int_a^b g(t) dt \right) \left(\int_a^b xf(x) dx \right) \\
 & = [f(c) - cf'(c)] \left\{ \frac{(b^2 - a^2)}{2} \int_a^b g(x) dx - (b - a) \int_a^b xg(x) dx \right\} \\
 & + [g(d) - dg'(d)] \left\{ \frac{(b^2 - a^2)}{2} \int_a^b f(x) dx - (b - a) \int_a^b xf(x) dx \right\}.
 \end{aligned}$$

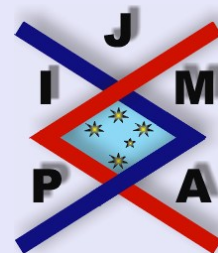
Rewriting (3.5) we have

$$\begin{aligned}
 (3.6) \quad G[f, g] &= [f(c) - cf'(c)] \int_a^b g(x) \left\{ \frac{1}{2} - \frac{x}{a+b} \right\} dx \\
 &+ [g(d) - dg'(d)] \int_a^b f(x) \left\{ \frac{1}{2} - \frac{x}{a+b} \right\} dx.
 \end{aligned}$$

Using the properties of modulus, from (3.6) we have

$$\begin{aligned}
 |G[f, g]| &\leq \|f - lf'\|_\infty \int_a^b |g(x)| \left| \frac{1}{2} - \frac{x}{a+b} \right| dx \\
 &+ \|g - lg'\|_\infty \int_a^b |f(x)| \left| \frac{1}{2} - \frac{x}{a+b} \right| dx.
 \end{aligned}$$

This completes the proof of Theorem 2.2.



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Multiplying the left sides and right sides of (3.1) and (3.2) we get

$$(3.7) \quad t^2 f(x) g(x) - (x f(x)) (t g(t)) - (x g(x)) (t f(t)) + x^2 f(t) g(t) \\ = [f(c) - c f'(c)] [g(d) - d g'(d)] (t - x)^2.$$

Integrating both sides of (3.7) with respect to t over $[a, b]$ we have

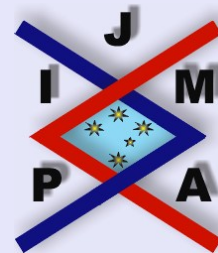
$$(3.8) \quad \frac{(b^3 - a^3)}{3} f(x) g(x) \\ - x f(x) \int_a^b t g(t) dt - x g(x) \int_a^b t f(t) dt + x^2 \int_a^b f(t) g(t) dt \\ = [f(c) - c f'(c)] [g(d) - d g'(d)] \left\{ \frac{(b^3 - a^3)}{3} - x(b^2 - a^2) + x^2(b - a) \right\}.$$

Now, integrating both sides of (3.8) with respect to x over $[a, b]$ we have

$$(3.9) \quad \frac{(b^3 - a^3)}{3} \int_a^b f(x) g(x) dx - \left(\int_a^b x f(x) dx \right) \left(\int_a^b t g(t) dt \right) \\ - \left(\int_a^b x g(x) dx \right) \left(\int_a^b t f(t) dt \right) + \frac{(b^3 - a^3)}{3} \int_a^b f(t) g(t) dt \\ = [f(c) - c f'(c)] [g(d) - d g'(d)] \\ \times \left\{ \frac{(b^3 - a^3)}{3} (b - a) - (b^2 - a^2) \frac{(b^2 - a^2)}{2} + (b - a) \frac{(b^3 - a^3)}{3} \right\}.$$

Rewriting (3.9) we have

$$(3.10) \quad H[f, g] = [f(c) - c f'(c)] [g(d) - d g'(d)] M.$$



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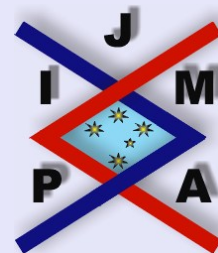
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Using the properties of modulus, from (3.10) we have

$$|H[f, g]| \leq \|f - lf'\|_\infty \|g - lg'\|_\infty |M|.$$

The proof of Theorem 2.3 is complete.



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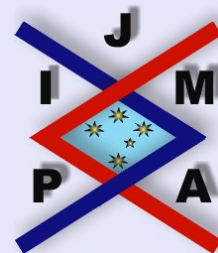
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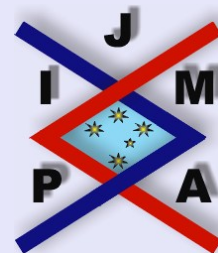
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