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# ON GRÜSS LIKE INTEGRAL INEQUALITIES VIA POMPEIU'S MEAN VALUE THEOREM

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**Abstract**

In the present note we establish two new integral inequalities similar to that of the Grüss integral inequality via Pompeiu's mean value theorem.

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# Contents

# 1. Introduction

In 1935 G. Grüss [4] proved the following integral inequality (see also [5, p. 296]):

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \le \frac{1}{4}(P-p)(Q-q), \tag{1.1}$$

provided that $f$ and $g$ are two integrable functions on $[a,b]$ such that

$$p \le f(x) \le P, \qquad q \le g(x) \le Q,$$

for all $x \in [a,b]$ , where $p, P, q, Q$ are constants.

The inequality (1.1) has evoked the interest of many researchers and numerous generalizations, variants and extensions have appeared in the literature, see [1], [3], [5] – [10] and the references cited therein. The main aim of this note is to establish two new integral inequalities similar to the inequality (1.1) by using a variant of Lagrange's mean value theorem, now known as the Pompeiu's mean value theorem [11] (see also [12, p. 83] and [2]).

## 2. Statement of Results

In what follows, $\mathbb{R}$ and $'$ denote the set of real numbers and derivative of a function respectively. For continuous functions $p, q : [a,b] \to \mathbb{R}$ which are differentiable on $(a,b)$, we use the notations

$$G[p,q] = \int_a^b p(x)\,q(x)\,dx - \frac{1}{b^2-a^2}\left[\left(\int_a^b p(x)\,dx\right)\left(\int_a^b xq(x)\,dx\right) + \left(\int_a^b q(x)\,dx\right)\left(\int_a^b xp(x)\,dx\right)\right],$$

$$H[p,q] = \int_a^b p(x)\,q(x)\,dx - \frac{3}{b^3-a^3}\left(\int_a^b xp(x)\,dx\right)\left(\int_a^b xq(x)\,dx\right),$$

to simplify the details of presentation and define $\|p\|_\infty = \sup_{t\in[a,b]} |p(t)|$.

In the proofs of our results we make use of the following theorem, which is a variant of the well known Lagrange's mean value theorem given by Pompeiu in [11] (see also [2, 12]).

**Theorem 2.1 (Pompeiu).** *For every real valued function $f$ differentiable on an interval $[a,b]$ not containing $0$ and for all pairs $x_1 \neq x_2$ in $[a,b]$ there exists a point $c$ in $(x_1, x_2)$ such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(c) - cf'(c).$$

Our main result is given in the following theorem.

**Theorem 2.2.** *Let* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* $(a, b)$ *with* $[a, b]$ *not containing* $0$*. Then*

$$|G[f,g]| \leq \|f - lf'\|_\infty \int_a^b |g(x)| \left|\frac{1}{2} - \frac{x}{a+b}\right| dx + \|g - lg'\|_\infty \int_a^b |f(x)| \left|\frac{1}{2} - \frac{x}{a+b}\right| dx, \tag{2.1}$$

*where* $l(t) = t$*,* $t \in [a, b]$*.*

A slight variant of Theorem 2.2 is embodied in the following theorem.

**Theorem 2.3.** *Let* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* $(a, b)$ *with* $[a, b]$ *not containing* $0$*. Then*

$$|H[f,g]| \leq \|f - lf'\|_\infty \|g - lg'\|_\infty |M|, \tag{2.2}$$

*where* $l(t) = t$ *,* $t \in [a, b]$ *and*

$$M = (b-a)\left\{1 - \frac{3}{4} \cdot \frac{(a+b)^2}{a^2 + ab + b^2}\right\}. \tag{2.3}$$

## 3. Proofs of Theorems 2.2 and 2.3

From the hypotheses of Theorems 2.2 and 2.3 and using Theorem 2.1 for $t \neq x$, $x, t \in [a, b]$, there exist points $c$ and $d$ between $x$ and $t$ such that

$$t\,f(x) - x\,f(t) = [f(c) - cf'(c)](t - x), \tag{3.1}$$

$$t\,g(x) - x\,g(t) = [g(d) - dg'(d)](t - x). \tag{3.2}$$

Multiplying (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we have

$$\begin{aligned} & 2t\,f(x)\,g(x) - x\,g(x)\,f(t) - x\,f(x)\,g(t) \\ & \qquad = [f(c) - cf'(c)](t - x)\,g(x) + [g(d) - dg'(d)](t - x)\,f(x). \end{aligned} \tag{3.3}$$

Integrating both sides of (3.3) with respect to $t$ over $[a, b]$ we have

$$\begin{aligned} & \left(b^2 - a^2\right) f(x)\,g(x) - x\,g(x) \int_a^b f(t)\,dt - x\,f(x) \int_a^b g(t)\,dt \\ & \qquad = [f(c) - cf'(c)] \left\{ \frac{b^2 - a^2}{2} g(x) - x\,g(x)(b - a) \right\} \\ & \qquad\qquad + [g(d) - dg'(d)] \left\{ \frac{b^2 - a^2}{2} f(x) - x\,f(x)(b - a) \right\}. \end{aligned} \tag{3.4}$$

Now, integrating both sides of (3.4) with respect to $x$ over $[a,b]$ we have

$$(3.5)\quad \left(b^2-a^2\right)\int_a^b f(x)\,g(x)\,dx - \left(\int_a^b f(t)\,dt\right)\left(\int_a^b xg(x)\,dx\right) - \left(\int_a^b g(t)\,dt\right)\left(\int_a^b xf(x)\,dx\right)$$
$$= \left[f(c)-cf'(c)\right]\left\{\frac{\left(b^2-a^2\right)}{2}\int_a^b g(x)\,dx-(b-a)\int_a^b xg(x)\,dx\right\}$$
$$+\left[g(d)-dg'(d)\right]\left\{\frac{\left(b^2-a^2\right)}{2}\int_a^b f(x)\,dx-(b-a)\int_a^b xf(x)\,dx\right\}.$$

Rewriting (3.5) we have

$$(3.6)\quad G[f,g]=\left[f(c)-cf'(c)\right]\int_a^b g(x)\left\{\frac{1}{2}-\frac{x}{a+b}\right\}dx$$
$$+\left[g(d)-dg'(d)\right]\int_a^b f(x)\left\{\frac{1}{2}-\frac{x}{a+b}\right\}dx.$$

Using the properties of modulus, from (3.6) we have

$$|G[f,g]|\le \left\|f-\mathit{l}f'\right\|_\infty\int_a^b |g(x)|\left|\frac{1}{2}-\frac{x}{a+b}\right|dx$$
$$+\left\|g-\mathit{l}g'\right\|_\infty\int_a^b |f(x)|\left|\frac{1}{2}-\frac{x}{a+b}\right|dx.$$

This completes the proof of Theorem 2.2.

Multiplying the left sides and right sides of (3.1) and (3.2) we get

$$(3.7)\quad t^2 f(x)g(x) - (xf(x))(tg(t)) - (xg(x))(tf(t)) + x^2 f(t)g(t) = [f(c) - cf'(c)][g(d) - dg'(d)](t-x)^2.$$

Integrating both sides of (3.7) with respect to $t$ over $[a,b]$ we have

$$(3.8)\quad \frac{(b^3-a^3)}{3} f(x)g(x) - xf(x)\int_a^b tg(t)\,dt - xg(x)\int_a^b tf(t)\,dt + x^2\int_a^b f(t)g(t)\,dt = [f(c) - cf'(c)][g(d) - dg'(d)]\left\{\frac{(b^3-a^3)}{3} - x(b^2-a^2) + x^2(b-a)\right\}.$$

Now, integrating both sides of (3.8) with respect to $x$ over $[a,b]$ we have

$$(3.9)\quad \frac{(b^3-a^3)}{3}\int_a^b f(x)g(x)\,dx - \left(\int_a^b xf(x)\,dx\right)\left(\int_a^b tg(t)\,dt\right) - \left(\int_a^b xg(x)\,dx\right)\left(\int_a^b tf(t)\,dt\right) + \frac{(b^3-a^3)}{3}\int_a^b f(t)g(t)\,dt = [f(c) - cf'(c)][g(d) - dg'(d)] \times \left\{\frac{(b^3-a^3)}{3}(b-a) - (b^2-a^2)\frac{(b^2-a^2)}{2} + (b-a)\frac{(b^3-a^3)}{3}\right\}.$$

Rewriting (3.9) we have

$$(3.10)\quad H[f,g] = [f(c) - cf'(c)][g(d) - dg'(d)]M.$$

Using the properties of modulus, from (3.10) we have

$$|H[f,g]| \leq \|f - lf'\|_\infty \|g - lg'\|_\infty |M| .$$

The proof of Theorem 2.3 is complete.

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