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ON THE ABSOLUTE CONVERGENCE OF SMALL GAPS FOURIER SERIES OF FUNCTIONS OF $\bigwedge BV^{(p)}$

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ABSTRACT. Let f be a 2π periodic function in $L^1[0,2\pi]$ and $\sum_{k=-\infty}^{\infty} \widehat{f}(n_k)e^{in_k x}$ be its Fourier series with 'small' gaps $n_{k+1} - n_k \geq q \geq 1$. Here we have obtained sufficiency conditions for the absolute convergence of such series if f is of $\bigwedge BV^{(p)}$ locally. We have also obtained a beautiful interconnection between lacunary and non-lacunary Fourier series.

Key words and phrases: Fourier series with small gaps, Absolute convergence of Fourier series and p-$\bigwedge$-bounded variation.

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1. INTRODUCTION

Let f be a 2π periodic function in $L^1[0,2\pi]$ and $\widehat{f}(n)$, $n \in \mathbb{Z}$, be its Fourier coefficients. The series

$$\sum_{k \in Z} \widehat{f}(n_k)e^{in_k x}, \tag{1.1}$$

wherein $\{n_k\}_1^\infty$ is a strictly increasing sequence of natural numbers and $n_{-k} = -n_k$, for all k, satisfy an inequality

$$(n_{k+1} - n_k) \geq q \geq 1 \quad \text{for all } \; k = 0, 1, 2, \dots, \tag{1.2}$$

is called the Fourier series of f with 'small' gaps.

Obviously, if $n_k = k$, for all k, (i.e. $n_{k+1} - n_k = q = 1$, for all k), then we get non-lacunary Fourier series and if $\{n_k\}$ is such that

$$(n_{k+1} - n_k) \to \infty \; \text{ as } \; k \to \infty, \tag{1.3}$$

then (1.1) is said to be the lacunary Fourier series.

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By applying the Wiener-Ingham result [1, Vol. I, p. 222] for the finite trigonometric sums with small gap (1.2) we have studied the sufficiency condition for the convergence of the series $\sum_{k\in Z}\left|\widehat{f}(n_k)\right|^{\beta}$ $(0<\beta\le 2)$ in terms of $\bigwedge BV$ and the modulus of continuity [2, Theorem 3]. Here we have generalized this result and we have also obtained a sufficiency condition if function f is of $\bigwedge BV^{(p)}$. In 1980 Shiba [4] generalized the class $\bigwedge BV$. He introduced the class $\bigwedge BV^{(p)}$.

Definition 1.1. Given an interval I, a sequence of non-decreasing positive real numbers $\bigwedge = \{\lambda_m\}$ $(m=1,2,\dots)$ such that $\sum_m \frac{1}{\lambda_m}$ diverges and $1\le p<\infty$ we say that $f\in\bigwedge BV^{(p)}$ (that is f is a function of $p-\bigwedge$-bounded variation over (I)) if

$$V_{\Lambda_p}(f,I)=\sup_{\{I_m\}}\{V_{\Lambda_p}(\{I_m\},f,I)\}<\infty,$$

where

$$V_{\Lambda_p}(\{I_m\},f,I)=\left(\sum_m \frac{|f(b_m)-f(a_m)|^p}{\lambda_m}\right)^{\frac{1}{p}},$$

and $\{I_m\}$ is a sequence of non-overlapping subintervals $I_m=[a_m,b_m]\subset I=[a,b]$.

Note that, if $p=1$, one gets the class $\bigwedge BV(I)$; if $\lambda_m\equiv 1$ for all m, one gets the class $BV^{(p)}$; if $p=1$ and $\lambda_m\equiv m$ for all m, one gets the class Harmonic $BV(I)$. if $p=1$ and $\lambda_m\equiv 1$ for all m, one gets the class $BV(I)$.

Definition 1.2. For $p\ge 1$, the $p-$integral modulus of continuity $\omega^{(p)}(\delta,f,I)$ of f over I is defined as

$$\omega^{(p)}(\delta,f,I)=\sup_{0\le h\le\delta}\|(T_hf-f)(x)\|_{p,I},$$

where $T_hf(x)=f(x+h)$ for all x and $\|(\cdot)\|_{p,I}=\|(\cdot)\chi_I\|_p$ in which χ_I is the characteristic function of I and $\|(\cdot)\|_p$ denotes the L^p-norm. $p=\infty$ gives the modulus of continuity $\omega(\delta,f,I)$.

We prove the following theorems.

Theorem 1.1. *Let $f\in L[-\pi,\pi]$ possess a Fourier series with 'small' gaps (1.2) and I be a subinterval of length $\delta_1>\frac{2\pi}{q}$. If $f\in\bigwedge BV(I)$ and*

$$\sum_{k=1}^{\infty}\left(\frac{\omega(\frac{1}{n_k},f,I)}{k\left(\sum_{j=1}^{n_k}\frac{1}{\lambda_j}\right)}\right)^{\frac{\beta}{2}}<\infty,$$

then

$$\sum_{k\in Z}\left|\widehat{f}(n_k)\right|^{\beta}<\infty \qquad (0<\beta\le 2). \tag{1.4}$$

Since $\{\lambda_j\}$ is non-decreasing, one gets $\sum_{j=1}^{n_k}\frac{1}{\lambda_j}\ge\frac{n_k}{\lambda_{n_k}}$ and hence our earlier theorem [2, Theorem 3] follows from Theorem 1.1.

Theorem 1.1 with $\beta=1$ and $\lambda_n\equiv 1$ shows that the Fourier series of f with 'small' gaps condition (1.2) (respectively (1.3)) converges absolutely if the hypothesis of the Stechkin theorem [5, Vol. II, p. 196] is satisfied only in a subinterval of $[0,2\pi]$ of length $>\frac{2\pi}{q}$ (respectively of arbitrary positive length).

Theorem 1.2. *Let f and I be as in Theorem 1.1. If $f \in \bigwedge BV^{(p)}(I)$, $1 \le p < 2r$, $1 < r < \infty$ and*

$$\sum_{k=1}^{\infty}\left(\frac{\left(\omega^{((2-p)s+p)}\left(\frac{1}{n_k}, f, I\right)\right)^{2-p/r}}{k\left(\sum_{j=1}^{n_k}\left(\frac{1}{\lambda_j}\right)\right)^{\frac{1}{r}}}\right)^{\frac{\beta}{2}} < \infty,$$

where $\frac{1}{r}+\frac{1}{s}=1$, then (1.4) holds.

Theorem 1.2 with $\beta = 1$ is a 'small' gaps analogue of the Schramm and Waterman result [3, Theorem 1].

We need the following lemmas to prove the theorems.

Lemma 1.3 ([2, Lemma 4])**.** *Let f and I be as in Theorem 1.1. If $f \in L^2(I)$ then*

(1.5) $$\sum_{k\in Z}\left|\widehat{f}(n_k)\right|^2 \le A_\delta \left|I\right|^{-1}\left\|f\right\|_{2,I}^2,$$

where A_δ depends only on δ.

Lemma 1.4. *If $|n_k| > p$ then for $t \in \mathbb{N}$ one has*

$$\int_0^{\frac{\pi}{p}} \sin^{2t}|n_k|\, h\, dh \ge \frac{\pi}{2^{t+1}p}.$$

Proof. Obvious. □

Lemma 1.5 (Stechkin, refer to [6])**.** *If $u_n \ge 0$ for $n \in \mathbb{N}$, $u_n \ne 0$ and a function $F(u)$ is concave, increasing, and $F(0)=0$, then*

$$\sum_1^{\infty} F(u_n) \le 2\sum_1^{\infty} F\left(\frac{u_n+u_{n+1}+\cdots}{n}\right).$$

Lemma 1.6. *If $f \in \bigwedge BV^{(p)}(I)$ implies f is bounded over I.*

Proof. Observe that

$$\begin{aligned}|f(x)|^p &\le 2^p\left(|f(a)|^p + \lambda_1\frac{|f(x)-f(a)|^p}{\lambda_1} + \lambda_2\frac{|f(b)-f(x)|^p}{\lambda_2}\right)\\ &\le 2^p\left(|f(a)|^p + \lambda_2 V_{\wedge_p}(f,I)\right)\end{aligned}$$

Hence the lemma follows. □

Proof of Theorem 1.1. Let $I = \left[x_0 - \frac{\delta_1}{2}, x_0 + \frac{\delta_1}{2}\right]$ for some x_0 and δ_2 be such that $0 < \frac{2\pi}{q} < \delta_2 < \delta_1$. Put $\delta_3 = \delta_1 - \delta_2$ and $J = \left[x_0 - \frac{\delta_2}{2}, x_0 + \frac{\delta_2}{2}\right]$. Suppose integers T and j satisfy

(1.6) $$|n_T| > \frac{4\pi}{\delta_3} \quad \text{and} \quad 0 \le j \le \frac{\delta_3\,|n_T|}{4\pi}.$$

Since $f \in \bigwedge BV(I)$ implies f is bounded over I by Lemma 1.6 (for $p=1$), we have $f \in L^2(I)$, so that (1.5) holds and $f \in L^2[-\pi,\pi]$. If we put $f_j = (T_{2jh}f - T_{(2j-1)h}f)$ then $f_j \in L^2(I)$ and the Fourier series of f_j also possesses gaps (1.2). Hence by Lemma 1.3 we get

(1.7) $$\sum_{k\in Z}\left|\hat{f}(n_k)\right|^2 \sin^2\left(\frac{n_k h}{2}\right) = O\left(\|f_j\|_{2,J}^2\right)$$

because

$$\hat{f}_j(n_k) = 2i\hat{f}(n_k)e^{in_k(2j-\frac{1}{2}h)}\sin\left(\frac{n_k h}{2}\right).$$

Integrating both the sides of (1.7) over $(0, \frac{\pi}{n_T})$ with respect to h and using Lemma 1.4, we get

$$\sum_{|n_k|\geq n_T}^{\infty} \left|\hat{f}(n_k)\right|^2 = O(n_T) \int_0^{\frac{\pi}{n_T}} \left(\| f_j \|_{2,J}^2\right) dh. \tag{1.8}$$

Multiplying both the sides of the equation by $\frac{1}{\lambda_j}$ and then taking summation over j, we get

$$\left(\sum_j \frac{1}{\lambda_j}\right)\left(\sum_{|n_k|\geq n_T}^{\infty} \left|\hat{f}(n_k)\right|^2\right) = O(n_T) \int_0^{\frac{\pi}{n_T}} \left(\left\|\sum_j \frac{|f_j|^2}{\lambda_j}\right\|_{1,J}\right) dh. \tag{1.9}$$

Now, since $x \in J$ and $h \in (0, \frac{\pi}{n_T})$ we have $|f_j(x)| = O(\omega(\frac{1}{n_T}, f, I))$, for each j of the summation; since $x \in J$ and $f \in \bigwedge BV(I)$ we have $\sum_j \frac{|f_j(x)|}{\lambda_j} = O(1)$ because for each j the points $x + 2jh$ and $x + (2j-1)h$ lie in I for $h \in (0, \frac{\pi}{n_T})$ and $x \in J \subset I$. Therefore

$$\begin{aligned}\left(\sum_j \frac{|f_j(x)|^2}{\lambda_j}\right) &= O\left(\omega\left(\frac{1}{n_T}, f, I\right)\right)\left(\sum_j \frac{|f_j(x)|}{\lambda_j}\right)\\ &= O\left(\omega\left(\frac{1}{n_T}, f, I\right)\right).\end{aligned}$$

It follows now from (1.9) that

$$R_{n_T} = \sum_{|n_k \geq n_T} \left|\hat{f}(n_k)\right|^2 = O\left(\frac{\omega\left(\frac{1}{n_T}, f, I\right)}{\sum_{j=1}^{n_T} \frac{1}{\lambda_j}}\right).$$

Finally, Lemma 1.5 with $u_k = \left|\hat{f}(n_k)\right|^2$ $(k \in Z)$ and $F(u) = u^{\beta/2}$ gives

$$\begin{aligned}\sum_{|k|=1}^{\infty} \left|\hat{f}(n_k)\right|^{\beta} &= 2\sum_{k=1}^{\infty} F\left(\left|\hat{f}(n_k)\right|^2\right)\\ &\leq 4\sum_{k=1}^{\infty} F\left(\frac{R_{n_k}}{k}\right)\\ &\leq 4\sum_{k=1}^{\infty} \left(\frac{R_{n_k}}{k}\right)^{\beta/2}\\ &= O(1)\left(\sum_{k=1}^{\infty}\left(\frac{\omega(\frac{1}{n_k}, f, I)}{k(\sum_{j=1}^{n_k}\frac{1}{\lambda_j})}\right)^{(\beta/2)}\right).\end{aligned}$$

This proves the theorem. □

Proof of Theorem 1.2. Since $f \in \bigwedge BV^{(p)}(I)$, Lemma 1.6 implies f is bounded over I. Therefore $f \in L^2(I)$, and hence (1.5) holds so that $f \in L^2[-\pi, \pi]$. Using the notations and procedure of Theorem 1.1 we get (1.9). Since $2 = \frac{(2-p)s+p}{s} + \frac{p}{r}$, by using Hölder's inequality, we get from

(1.9)
$$\int_J |f_j(x)|^2\,dx \le \left(\int_J |f_j(x)|^{(2-p)s+p}\,dx\right)^{\frac{1}{s}} \left(\int_J |f_j(x)|^p\,dx\right)^{\frac{1}{r}}$$
$$\le \Omega_{h,J}^{1/r}\left(\int_J |f_j(x)|^p\,dx\right)^{\frac{1}{r}},$$

where $\Omega_{h,J} = (\omega^{(2-p)s+p}(h,f,J))^{2r-p}$.

This together with (1.9) implies, putting

$$B = \sum_{k\in Z}\left|\widehat{f}(n_k)\right|^2 \sin^2\left(\frac{n_k h}{2}\right),$$

that

$$B \le \Omega_{h,J}^{1/r}\left(\int_J |f_j(x)|^p\,dx\right)^{\frac{1}{r}}.$$

Thus

$$B^r \le \Omega_{h,J}\left(\int_J |f_j(x)|^p\,dx\right).$$

Now multiplying both the sides of the equation by $\frac{1}{\lambda_j}$ and then taking the summation over $j = 1$ to n_T $(T \in \mathbb{N})$ we get

$$B^r \le \frac{\Omega_{h,J}\left(\int_J \left(\sum_j \frac{|f_j(x)|^p}{\lambda_j}\right)dx\right)}{\sum_j \frac{1}{\lambda_j}}.$$

Therefore

$$B \le \left(\frac{\Omega_{h,J}}{\sum_j \frac{1}{\lambda_j}}\right)^{\frac{1}{r}}\left(\int_J\left(\sum_j \frac{|f_j(x)|^p}{\lambda_j}\right)dx\right)^{\frac{1}{r}}.$$

Substituting back the value of B and then integrating both the sides of the equation with respect to h over $(0, \frac{\pi}{n_T})$, we get

(1.10)
$$\sum_{k\in Z}\left|\widehat{f}(n_k)\right|^2 \int_0^{\pi/n_T}\left(\sin^2\left(\frac{|n_k|\,h}{2}\right)\right)dh$$
$$= O\left(\frac{\Omega_{1/n_T,J}}{\left(\sum_j \frac{1}{\lambda_j}\right)}\right)^{\frac{1}{r}}\int_0^{\pi/n_T}\left(\int_J\left(\sum_j \frac{|f_j(x)|^p}{\lambda_j}\right)dx\right)^{\frac{1}{r}}dh.$$

Observe that for x in J, h in $(0, \frac{\pi}{n_T})$ and for each j of the summation the points $x+2jh$ and $x+(2j-1)h$ lie in I; moreover,$f \in \bigwedge BV^{(p)}(I)$ implies

$$\sum_j \frac{|f_j(x)|^p}{\lambda_j} = O(1).$$

Therefore, it follows from (1.10) and Lemma 1.4 that

$$R_{n_T} \equiv \sum_{|n_k|\ge n_T}\left|\widehat{f}(n_k)\right|^2 = O\left(\frac{\Omega_{1/n_T,I}}{\sum_{j=1}^{n_T}\frac{1}{\lambda_j}}\right)^{\frac{1}{r}}.$$

Thus

$$R_{n_T} = O\left(\frac{\omega^{(2-p)s+p}\left(\frac{1}{n_T}, f, I\right)^{2-p/r}}{\left(\sum_{j=1}^{n_T} \frac{1}{\lambda_j}\right)^{\frac{1}{r}}}\right).$$

Now proceeding as in the proof of Theorem 1.1, the theorem is proved using Lemma 1.5. □

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