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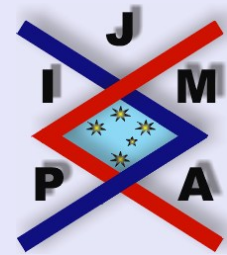
ON THE ABSOLUTE CONVERGENCE OF SMALL GAPS FOURIER SERIES OF FUNCTIONS OF $\bigwedge BV^{(p)}$

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Abstract

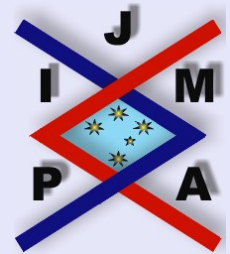
Let f be a 2π periodic function in $L^1[0, 2\pi]$ and $\sum_{k=-\infty}^{\infty} \widehat{f}(n_k) e^{in_k x}$ be its Fourier series with 'small' gaps $n_{k+1} - n_k \geq q \geq 1$. Here we have obtained sufficiency conditions for the absolute convergence of such series if f is of $\wedge BV^{(p)}$ locally. We have also obtained a beautiful interconnection between lacunary and non-lacunary Fourier series.

2000 Mathematics Subject Classification: 42Axx.

Key words: Fourier series with small gaps, Absolute convergence of Fourier series and p - \wedge -bounded variation.

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1. Introduction

Let f be a 2π periodic function in $L^1[0, 2\pi]$ and $\widehat{f}(n)$, $n \in \mathbb{Z}$, be its Fourier coefficients. The series

$$(1.1) \quad \sum_{k \in \mathbb{Z}} \widehat{f}(n_k) e^{in_k x},$$

wherein $\{n_k\}_1^\infty$ is a strictly increasing sequence of natural numbers and $n_{-k} = -n_k$, for all k , satisfy an inequality

$$(1.2) \quad (n_{k+1} - n_k) \geq q \geq 1 \quad \text{for all } k = 0, 1, 2, \dots,$$

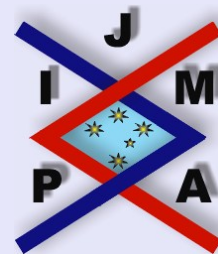
is called the Fourier series of f with ‘small’ gaps.

Obviously, if $n_k = k$, for all k , (i.e. $n_{k+1} - n_k = q = 1$, for all k), then we get non-lacunary Fourier series and if $\{n_k\}$ is such that

$$(1.3) \quad (n_{k+1} - n_k) \rightarrow \infty \text{ as } k \rightarrow \infty,$$

then (1.1) is said to be the lacunary Fourier series.

By applying the Wiener-Ingham result [1, Vol. I, p. 222] for the finite trigonometric sums with small gap (1.2) we have studied the sufficiency condition for the convergence of the series $\sum_{k \in \mathbb{Z}} \left| \widehat{f}(n_k) \right|^\beta$ ($0 < \beta \leq 2$) in terms of $\bigwedge BV$ and the modulus of continuity [2, Theorem 3]. Here we have generalized this result and we have also obtained a sufficiency condition if function f is of $\bigwedge BV^{(p)}$. In 1980 Shiba [4] generalized the class $\bigwedge BV$. He introduced the class $\bigwedge BV^{(p)}$.



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Definition 1.1. Given an interval I , a sequence of non-decreasing positive real numbers $\Lambda = \{\lambda_m\}$ ($m = 1, 2, \dots$) such that $\sum_m \frac{1}{\lambda_m}$ diverges and $1 \leq p < \infty$ we say that $f \in \Lambda BV^{(p)}$ (that is f is a function of $p - \Lambda$ -bounded variation over I) if

$$V_{\Lambda_p}(f, I) = \sup_{\{I_m\}} \{V_{\Lambda_p}(\{I_m\}, f, I)\} < \infty,$$

where

$$V_{\Lambda_p}(\{I_m\}, f, I) = \left(\sum_m \frac{|f(b_m) - f(a_m)|^p}{\lambda_m} \right)^{\frac{1}{p}},$$

and $\{I_m\}$ is a sequence of non-overlapping subintervals $I_m = [a_m, b_m] \subset I = [a, b]$.

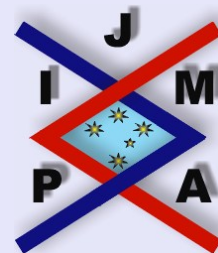
Note that, if $p = 1$, one gets the class $\Lambda BV(I)$; if $\lambda_m \equiv 1$ for all m , one gets the class $BV^{(p)}$; if $p = 1$ and $\lambda_m \equiv m$ for all m , one gets the class Harmonic $BV(I)$. if $p = 1$ and $\lambda_m \equiv 1$ for all m , one gets the class $BV(I)$.

Definition 1.2. For $p \geq 1$, the p -integral modulus of continuity $\omega^{(p)}(\delta, f, I)$ of f over I is defined as

$$\omega^{(p)}(\delta, f, I) = \sup_{0 \leq h \leq \delta} \|(T_h f - f)(x)\|_{p, I},$$

where $T_h f(x) = f(x + h)$ for all x and $\|(\cdot)\|_{p, I} = \|(\cdot)\chi_I\|_p$ in which χ_I is the characteristic function of I and $\|(\cdot)\|_p$ denotes the L^p -norm. $p = \infty$ gives the modulus of continuity $\omega(\delta, f, I)$.

We prove the following theorems.



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Theorem 1.1. Let $f \in L[-\pi, \pi]$ possess a Fourier series with ‘small’ gaps (1.2) and I be a subinterval of length $\delta_1 > \frac{2\pi}{q}$. If $f \in \bigwedge BV(I)$ and

$$\sum_{k=1}^{\infty} \left(\frac{\omega\left(\frac{1}{n_k}, f, I\right)}{k \left(\sum_{j=1}^{n_k} \frac{1}{\lambda_j}\right)} \right)^{\frac{\beta}{2}} < \infty,$$

then

$$(1.4) \quad \sum_{k \in \mathbb{Z}} \left| \widehat{f}(n_k) \right|^{\beta} < \infty \quad (0 < \beta \leq 2).$$

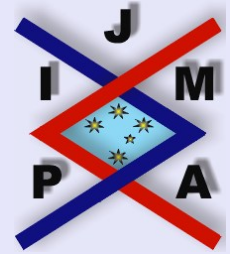
Since $\{\lambda_j\}$ is non-decreasing, one gets $\sum_{j=1}^{n_k} \frac{1}{\lambda_j} \geq \frac{n_k}{\lambda_{n_k}}$ and hence our earlier theorem [2, Theorem 3] follows from Theorem 1.1.

Theorem 1.1 with $\beta = 1$ and $\lambda_n \equiv 1$ shows that the Fourier series of f with ‘small’ gaps condition (1.2) (respectively (1.3)) converges absolutely if the hypothesis of the Stechkin theorem [5, Vol. II, p. 196] is satisfied only in a subinterval of $[0, 2\pi]$ of length $> \frac{2\pi}{q}$ (respectively of arbitrary positive length).

Theorem 1.2. Let f and I be as in Theorem 1.1. If $f \in \bigwedge BV^{(p)}(I)$, $1 \leq p < 2r$, $1 < r < \infty$ and

$$\sum_{k=1}^{\infty} \left(\frac{\left(\omega^{((2-p)s+p)}\left(\frac{1}{n_k}, f, I\right) \right)^{2-p/r}}{k \left(\sum_{j=1}^{n_k} \left(\frac{1}{\lambda_j}\right)\right)^{\frac{1}{r}}} \right)^{\frac{\beta}{2}} < \infty,$$

where $\frac{1}{r} + \frac{1}{s} = 1$, then (1.4) holds.



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Theorem 1.2 with $\beta = 1$ is a ‘small’ gaps analogue of the Schramm and Waterman result [3, Theorem 1].

We need the following lemmas to prove the theorems.

Lemma 1.3 ([2, Lemma 4]). *Let f and I be as in Theorem 1.1. If $f \in L^2(I)$ then*

$$(1.5) \quad \sum_{k \in \mathbb{Z}} \left| \widehat{f}(n_k) \right|^2 \leq A_\delta |I|^{-1} \|f\|_{2,I}^2,$$

where A_δ depends only on δ .

Lemma 1.4. *If $|n_k| > p$ then for $t \in \mathbb{N}$ one has*

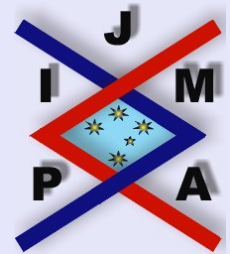
$$\int_0^{\frac{\pi}{p}} \sin^{2t} |n_k| h \, dh \geq \frac{\pi}{2^{t+1} p}.$$

Proof. Obvious. □

Lemma 1.5 (Stechkin, refer to [6]). *If $u_n \geq 0$ for $n \in \mathbb{N}$, $u_n \neq 0$ and a function $F(u)$ is concave, increasing, and $F(0) = 0$, then*

$$\sum_1^\infty F(u_n) \leq 2 \sum_1^\infty F\left(\frac{u_n + u_{n+1} + \dots}{n}\right).$$

Lemma 1.6. *If $f \in \bigwedge BV^{(p)}(I)$ implies f is bounded over I .*



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Proof. Observe that

$$\begin{aligned} |f(x)|^p &\leq 2^p \left(|f(a)|^p + \lambda_1 \frac{|f(x) - f(a)|^p}{\lambda_1} + \lambda_2 \frac{|f(b) - f(x)|^p}{\lambda_2} \right) \\ &\leq 2^p (|f(a)|^p + \lambda_2 V_{\wedge_p}(f, I)) \end{aligned}$$

Hence the lemma follows. \square

Proof of Theorem 1.1. Let $I = [x_0 - \frac{\delta_1}{2}, x_0 + \frac{\delta_1}{2}]$ for some x_0 and δ_2 be such that $0 < \frac{2\pi}{q} < \delta_2 < \delta_1$. Put $\delta_3 = \delta_1 - \delta_2$ and $J = [x_0 - \frac{\delta_2}{2}, x_0 + \frac{\delta_2}{2}]$. Suppose integers T and j satisfy

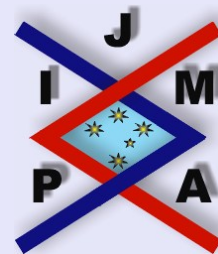
$$(1.6) \quad |n_T| > \frac{4\pi}{\delta_3} \quad \text{and} \quad 0 \leq j \leq \frac{\delta_3 |n_T|}{4\pi}.$$

Since $f \in \wedge BV(I)$ implies f is bounded over I by Lemma 1.6 (for $p = 1$), we have $f \in L^2(I)$, so that (1.5) holds and $f \in L^2[-\pi, \pi]$. If we put $f_j = (T_{2jh}f - T_{(2j-1)h}f)$ then $f_j \in L^2(I)$ and the Fourier series of f_j also possesses gaps (1.2). Hence by Lemma 1.3 we get

$$(1.7) \quad \sum_{k \in \mathbb{Z}} \left| \hat{f}(n_k) \right|^2 \sin^2 \left(\frac{n_k h}{2} \right) = O \left(\|f_j\|_{2,J}^2 \right)$$

because

$$\hat{f}_j(n_k) = 2i \hat{f}(n_k) e^{in_k(2j - \frac{1}{2})h} \sin \left(\frac{n_k h}{2} \right).$$



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Integrating both the sides of (1.7) over $(0, \frac{\pi}{n_T})$ with respect to h and using Lemma 1.4, we get

$$(1.8) \quad \sum_{|n_k| \geq n_T} \left| \hat{f}(n_k) \right|^2 = O(n_T) \int_0^{\frac{\pi}{n_T}} (\|f_j\|_{2,J}^2) dh.$$

Multiplying both the sides of the equation by $\frac{1}{\lambda_j}$ and then taking summation over j , we get

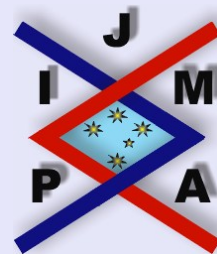
$$(1.9) \quad \left(\sum_j \frac{1}{\lambda_j} \right) \left(\sum_{|n_k| \geq n_T} \left| \hat{f}(n_k) \right|^2 \right) = O(n_T) \int_0^{\frac{\pi}{n_T}} \left(\left\| \sum_j \frac{|f_j|^2}{\lambda_j} \right\|_{1,J} \right) dh.$$

Now, since $x \in J$ and $h \in (0, \frac{\pi}{n_T})$ we have $|f_j(x)| = O(\omega(\frac{1}{n_T}, f, I))$, for each j of the summation; since $x \in J$ and $f \in \wedge BV(I)$ we have $\sum_j \frac{|f_j(x)|}{\lambda_j} = O(1)$ because for each j the points $x + 2jh$ and $x + (2j - 1)h$ lie in I for $h \in (0, \frac{\pi}{n_T})$ and $x \in J \subset I$. Therefore

$$\begin{aligned} \left(\sum_j \frac{|f_j(x)|^2}{\lambda_j} \right) &= O \left(\omega \left(\frac{1}{n_T}, f, I \right) \right) \left(\sum_j \frac{|f_j(x)|}{\lambda_j} \right) \\ &= O \left(\omega \left(\frac{1}{n_T}, f, I \right) \right). \end{aligned}$$

It follows now from (1.9) that

$$R_{n_T} = \sum_{|n_k| \geq n_T} \left| \hat{f}(n_k) \right|^2 = O \left(\frac{\omega \left(\frac{1}{n_T}, f, I \right)}{\sum_{j=1}^{n_T} \frac{1}{\lambda_j}} \right).$$



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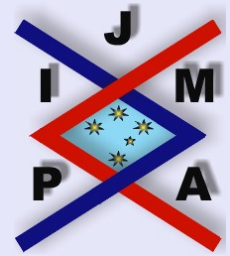
Finally, Lemma 1.5 with $u_k = \left| \hat{f}(n_k) \right|^2$ ($k \in Z$) and $F(u) = u^{\beta/2}$ gives

$$\begin{aligned} \sum_{|k|=1}^{\infty} \left| \hat{f}(n_k) \right|^{\beta} &= 2 \sum_{k=1}^{\infty} F \left(\left| \hat{f}(n_k) \right|^2 \right) \\ &\leq 4 \sum_{k=1}^{\infty} F \left(\frac{R_{n_k}}{k} \right) \\ &\leq 4 \sum_{k=1}^{\infty} \left(\frac{R_{n_k}}{k} \right)^{\beta/2} \\ &= O(1) \left(\sum_{k=1}^{\infty} \left(\frac{\omega \left(\frac{1}{n_k}, f, I \right)}{k \left(\sum_{j=1}^{n_k} \frac{1}{\lambda_j} \right)} \right)^{(\beta/2)} \right). \end{aligned}$$

This proves the theorem. \square

Proof of Theorem 1.2. Since $f \in \bigwedge BV^{(p)}(I)$, Lemma 1.6 implies f is bounded over I . Therefore $f \in L^2(I)$, and hence (1.5) holds so that $f \in L^2[-\pi, \pi]$. Using the notations and procedure of Theorem 1.1 we get (1.9). Since $2 = \frac{(2-p)s+p}{s} + \frac{p}{r}$, by using Hölder's inequality, we get from (1.9)

$$\begin{aligned} \int_J |f_j(x)|^2 dx &\leq \left(\int_J |f_j(x)|^{(2-p)s+p} dx \right)^{\frac{1}{s}} \left(\int_J |f_j(x)|^p dx \right)^{\frac{1}{r}} \\ &\leq \Omega_{h,J}^{1/r} \left(\int_J |f_j(x)|^p dx \right)^{\frac{1}{r}}, \end{aligned}$$



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where $\Omega_{h,J} = (\omega^{(2-p)s+p}(h, f, J))^{2r-p}$.

This together with (1.9) implies, putting

$$B = \sum_{k \in Z} \left| \widehat{f}(n_k) \right|^2 \sin^2 \left(\frac{n_k h}{2} \right),$$

that

$$B \leq \Omega_{h,J}^{1/r} \left(\int_J |f_j(x)|^p dx \right)^{\frac{1}{r}}.$$

Thus

$$B^r \leq \Omega_{h,J} \left(\int_J |f_j(x)|^p dx \right).$$

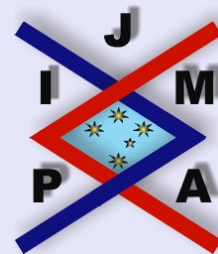
Now multiplying both the sides of the equation by $\frac{1}{\lambda_j}$ and then taking the summation over $j = 1$ to n_T ($T \in \mathbb{N}$) we get

$$B^r \leq \frac{\Omega_{h,J} \left(\int_J \left(\sum_j \frac{|f_j(x)|^p}{\lambda_j} \right) dx \right)}{\sum_j \frac{1}{\lambda_j}}.$$

Therefore

$$B \leq \left(\frac{\Omega_{h,J}}{\sum_j \frac{1}{\lambda_j}} \right)^{\frac{1}{r}} \left(\int_J \left(\sum_j \frac{|f_j(x)|^p}{\lambda_j} \right) dx \right)^{\frac{1}{r}}.$$

Substituting back the value of B and then integrating both the sides of the equa-



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tion with respect to h over $(0, \frac{\pi}{n_T})$, we get

$$(1.10) \quad \sum_{k \in \mathbb{Z}} \left| \widehat{f}(n_k) \right|^2 \int_0^{\pi/n_T} \left(\sin^2 \left(\frac{|n_k| h}{2} \right) \right) dh$$

$$= O \left(\frac{\Omega_{1/n_T, J}}{\left(\sum_j \frac{1}{\lambda_j} \right)} \right)^{\frac{1}{r}} \int_0^{\pi/n_T} \left(\int_J \left(\sum_j \frac{|f_j(x)|^p}{\lambda_j} \right) dx \right)^{\frac{1}{r}} dh.$$

Observe that for x in J , h in $(0, \frac{\pi}{n_T})$ and for each j of the summation the points $x + 2jh$ and $x + (2j - 1)h$ lie in I ; moreover, $f \in \bigwedge BV^{(p)}(I)$ implies

$$\sum_j \frac{|f_j(x)|^p}{\lambda_j} = O(1).$$

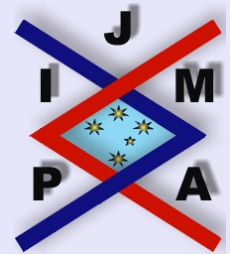
Therefore, it follows from (1.10) and Lemma 1.4 that

$$R_{n_T} \equiv \sum_{|n_k| \geq n_T} \left| \widehat{f}(n_k) \right|^2 = O \left(\frac{\Omega_{1/n_T, I}}{\sum_{j=1}^{n_T} \frac{1}{\lambda_j}} \right)^{\frac{1}{r}}.$$

Thus

$$R_{n_T} = O \left(\frac{\omega^{(2-p)s+p} \left(\frac{1}{n_T}, f, I \right)^{2-p/r}}{\left(\sum_{j=1}^{n_T} \frac{1}{\lambda_j} \right)^{\frac{1}{r}}} \right).$$

Now proceeding as in the proof of Theorem 1.1, the theorem is proved using Lemma 1.5. \square



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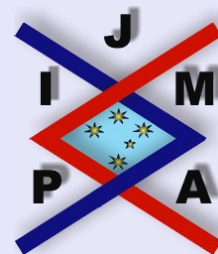
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