

A CERTAIN CLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS DEFINED BY MEANS OF A LINEAR OPERATOR

DING-GONG YANG

Department of Mathematics
Suzhou University
Suzhou, Jiangsu 215006,
P.R. China.

N-ENG XU

Department of Mathematics
Changshu Institute of Technology
Changshu, Jiangsu 215500,
P.R. China.
EMail: xuneng11@pub.sz.jsinfo.net

SHIGEYOSHI OWA

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
EMail: owa@math.kindai.ac.jp

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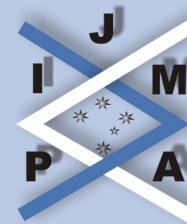
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Abstract:

Making use of a linear operator, which is defined here by means of the Hadamard product (or convolution), we introduce a class $Q_p(a, c; h)$ of analytic and multivalent functions in the open unit disk. An inclusion relation and a convolution property for the class $Q_p(a, c; h)$ are presented. Some integral-preserving properties are also given.



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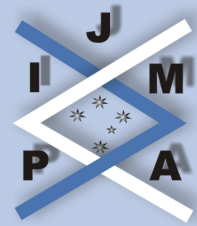
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1. Introduction and Preliminaries

Let the functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^{p+k} \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

be analytic in the open unit disk $U = \{z : |z| < 1\}$. Then the Hadamard product (or convolution) $(f * g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(1.1) \quad (f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^{p+k} = (g * f)(z).$$

Let A_p denote the class of functions $f(z)$ normalized by

$$(1.2) \quad f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{p+k} \quad (p \in \mathbb{N}),$$

which are analytic in U . A function $f(z) \in A_p$ is said to be in the class $S_p^*(\alpha)$ if it satisfies

$$(1.3) \quad \operatorname{Re} \frac{z f'(z)}{f(z)} > p\alpha \quad (z \in U)$$

for some $\alpha (\alpha < 1)$. When $0 \leq \alpha < 1$, $S_p^*(\alpha)$ is the class of p -valently starlike functions of order α in U . Also we write $A_1 = A$ and $S_1^*(\alpha) = S^*(\alpha)$. A function $f(z) \in A$ is said to be prestarlike of order $\alpha (\alpha < 1)$ in U if

$$(1.4) \quad \frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha).$$

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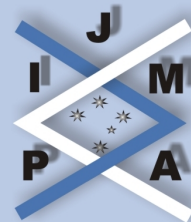
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We denote this class by $R(\alpha)$ (see [9]). It is clear that a function $f(z) \in A$ is in the class $R(0)$ if and only if $f(z)$ is convex univalent in U and

$$R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right).$$

We now define the function $\varphi_p(a, c; z)$ by

$$(1.5) \quad \varphi_p(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \quad (z \in U),$$

where

$$c \notin \{0, -1, -2, \dots\} \quad \text{and} \quad (x)_k = x(x+1) \cdots (x+k-1) \quad (k \in \mathbb{N}).$$

Corresponding to the function $\varphi_p(a, c; z)$, Saitoh [10] introduced and studied a linear operator $L_p(a, c)$ on A_p by the following Hadamard product (or convolution):

$$(1.6) \quad L_p(a, c)f(z) = \varphi_p(a, c; z) * f(z) \quad (f(z) \in A_p).$$

For $p = 1$, $L_1(a, c)$ on A was first defined by Carlson and Shaffer [1]. We remark in passing that a much more general convolution operator than the operator $L_p(a, c)$ was introduced by Dziok and Srivastava [2].

It is known [10] that

$$(1.7) \quad z(L_p(a, c)f(z))' = aL_p(a+1, c)f(z) - (a-p)L_p(a, c)f(z) \quad (f(z) \in A_p).$$

Setting $a = n + p > 0$ and $c = 1$ in (1.6), we have

$$(1.8) \quad L_p(n+p, 1)f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) = D^{n+p-1}f(z) \quad (f(z) \in A_p).$$

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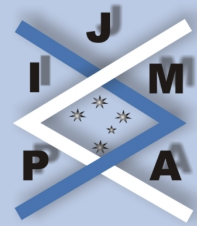
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The operator D^{n+p-1} when $p = 1$ was first introduced by Ruscheweyh [8], and D^{n+p-1} was introduced by Goel and Sohi [3]. Thus we name D^{n+p-1} as the Ruscheweyh derivative of $(n + p - 1)$ th order.

For functions $f(z)$ and $g(z)$ analytic in U , we say that $f(z)$ is subordinate to $g(z)$ in U , and write $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in U such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in U).$$

Furthermore, if the function $g(z)$ is univalent in U , then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let P be the class of analytic functions $h(z)$ with $h(0) = p$, which are convex univalent in U and for which

$$\operatorname{Re} h(z) > 0 \quad (z \in U).$$

In this paper we introduce and investigate the following subclass of A_p .

Definition 1.1. A function $f(z) \in A_p$ is said to be in the class $Q_p(a, c; h)$ if it satisfies

$$(1.9) \quad \frac{L_p(a+1, c)f(z)}{L_p(a, c)f(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a},$$

where

$$(1.10) \quad a \neq 0, \quad c \notin \{0, -1, -2, \dots\} \quad \text{and} \quad h(z) \in P.$$

It is easy to see that, if $f(z) \in Q_p(a, c; h)$, then $L_p(a, c)f(z) \in S_p^*(0)$.

For $a = n + p$ ($n > -p$), $c = 1$ and

$$(1.11) \quad h(z) = p + \frac{(A - B)z}{1 + Bz} \quad (-1 \leq B < A \leq 1),$$



Yang [12] introduced and studied the class

$$Q_p(n + p, 1; h) = S_{n,p}(A, B).$$

For $h(z)$ given by (1.11), the class

$$(1.12) \quad Q_p(a, c; h) = H_{a,c,p}(A, B)$$

has been considered by Liu and Owa [5].

For $p = 1$, $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$, Kim and Srivastava [4] have shown some properties of the class $H_{a,c,1}(1 - 2\alpha, -1)$.

In the present paper, we shall establish an inclusion relation and a convolution property for the class $Q_p(a, c; h)$. Integral transforms of functions in this class are also discussed. We observe that the proof of each of the results in [5] is much akin to that of the corresponding assertion made by Yang [12] in the case of $a = n + p$ and $c = 1$. However, the methods used in [5, 12] do not work for the general function class $Q_p(a, c; h)$.

We need the following lemmas in order to derive our main results for the class $Q_p(a, c; h)$.

Lemma 1.2 (Ruscheweyh [9]). *Let $\alpha < 1$, $f(z) \in S^*(\alpha)$ and $g(z) \in R(\alpha)$. Then, for any analytic function $F(z)$ in U ,*

$$\frac{g * (fF)}{g * f}(U) \subset \overline{co}(F(U)),$$

where $\overline{co}(F(U))$ denotes the closed convex hull of $F(U)$.

Lemma 1.3 (Miller and Mocanu [6]). *Let β ($\beta \neq 0$) and γ be complex numbers and let $h(z)$ be analytic and convex univalent in U with*

$$\operatorname{Re}(\beta h(z) + \gamma) > 0 \quad (z \in U).$$

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If $q(z)$ is analytic in U with $q(0) = h(0)$, then the subordination

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z)$$

implies that $q(z) \prec h(z)$.



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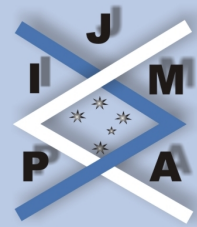
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2. Main Results

Theorem 2.1. Let $h(z) \in P$ and

$$(2.1) \quad \operatorname{Re} h(z) > \beta \quad (z \in U; 0 \leq \beta < p).$$

If

$$(2.2) \quad 0 < a_1 < a_2 \quad \text{and} \quad a_2 \geq 2(p - \beta),$$

then

$$Q_p(a_2, c; h) \subset Q_p(a_1, c; h).$$

Proof. Define

$$g(z) = z + \sum_{k=1}^{\infty} \frac{(a_1)_k}{(a_2)_k} z^{k+1} \quad (z \in U; 0 < a_1 < a_2).$$

Then

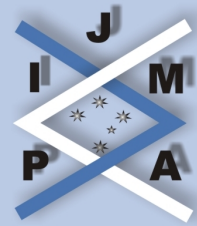
$$(2.3) \quad \frac{\varphi_p(a_1, a_2; z)}{z^{p-1}} = g(z) \in A,$$

where $\varphi_p(a_1, a_2; z)$ is defined as in (1.5), and

$$(2.4) \quad \frac{z}{(1-z)^{a_2}} * g(z) = \frac{z}{(1-z)^{a_1}}.$$

From (2.4) we have

$$\frac{z}{(1-z)^{a_2}} * g(z) \in S^* \left(1 - \frac{a_1}{2}\right) \subset S^* \left(1 - \frac{a_2}{2}\right)$$



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for $0 < a_1 < a_2$, which implies that

$$(2.5) \quad g(z) \in R\left(1 - \frac{a_2}{2}\right).$$

Since

$$(2.6) \quad L_p(a_1, c)f(z) = \varphi_p(a_1, a_2; z) * L_p(a_2, c)f(z) \quad (f(z) \in A_p),$$

we deduce from (1.7) and (2.6) that

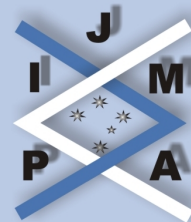
$$(2.7) \quad \begin{aligned} & a_1 L_p(a_1 + 1, c)f(z) \\ &= z(L_p(a_1, c)f(z))' + (a_1 - p)L_p(a_1, c)f(z) \\ &= \varphi_p(a_1, a_2; z) * (z(L_p(a_2, c)f(z))' + (a_1 - p)L_p(a_2, c)f(z)) \\ &= \varphi_p(a_1, a_2; z) * (a_2 L_p(a_2 + 1, c)f(z) + (a_1 - a_2)L_p(a_2, c)f(z)). \end{aligned}$$

By using (2.3), (2.6) and (2.7), we find that

$$(2.8) \quad \begin{aligned} & \frac{L_p(a_1 + 1, c)f(z)}{L_p(a_1, c)f(z)} \\ &= \frac{(z^{p-1}g(z)) * \left(\frac{a_2}{a_1}L_p(a_2 + 1, c)f(z) + \left(1 - \frac{a_2}{a_1}\right)L_p(a_2, c)f(z)\right)}{(z^{p-1}g(z)) * L_p(a_2, c)f(z)} \\ &= \frac{g(z) * \left(\frac{a_2}{a_1} \frac{L_p(a_2+1, c)f(z)}{z^{p-1}} + \left(1 - \frac{a_2}{a_1}\right) \frac{L_p(a_2, c)f(z)}{z^{p-1}}\right)}{g(z) * \frac{L_p(a_2, c)f(z)}{z^{p-1}}} \\ &= \frac{g(z) * (q(z)F(z))}{g(z) * q(z)} \quad (f(z) \in A_p), \end{aligned}$$

where

$$q(z) = \frac{L_p(a_2, c)f(z)}{z^{p-1}} \in A$$



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and

$$F(z) = \frac{a_2 L_p(a_2 + 1, c) f(z)}{a_1 L_p(a_2, c) f(z)} + 1 - \frac{a_2}{a_1}.$$

Let $f(z) \in Q_p(a_2, c; h)$. Then

$$\begin{aligned} F(z) &< \frac{a_2}{a_1} \left(1 - \frac{p}{a_2} + \frac{h(z)}{a_2} \right) + 1 - \frac{a_2}{a_1} \\ (2.9) \quad &= 1 - \frac{p}{a_1} + \frac{h(z)}{a_1} = h_1(z) \quad (\text{say}), \end{aligned}$$

where $h_1(z)$ is convex univalent in U , and, by (1.7),

$$\begin{aligned} \frac{zq'(z)}{q(z)} &= \frac{z(L_p(a_2, c) f(z))'}{L_p(a_2, c) f(z)} + 1 - p \\ &= a_2 \frac{L_p(a_2 + 1, c) f(z)}{L_p(a_2, c) f(z)} + 1 - a_2 \\ (2.10) \quad &< 1 - p + h(z). \end{aligned}$$

By using (2.1), (2.2) and (2.10), we get

$$\operatorname{Re} \frac{zq'(z)}{q(z)} > 1 - p + \beta \geq 1 - \frac{a_2}{2} \quad (z \in U),$$

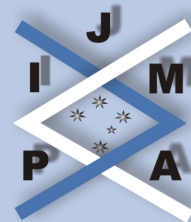
that is,

$$(2.11) \quad q(z) \in S^* \left(1 - \frac{a_2}{2} \right).$$

Consequently, in view of (2.5), (2.8), (2.9) and (2.11), an application of Lemma 1.2 yields

$$\frac{L_p(a_1 + 1, c) f(z)}{L_p(a_1, c) f(z)} < h_1(z).$$

Thus $f(z) \in Q_p(a_1, c; h)$ and the proof of Theorem 2.1 is completed. \square



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By carefully selecting the function $h(z)$ involved in Theorem 2.1, we can obtain a number of useful consequences.

Corollary 2.2. *Let*

$$(2.12) \quad h(z) = p - 1 + \left(\frac{1 + Az}{1 + Bz} \right)^\gamma \quad (z \in U; 0 < \gamma \leq 1; -1 \leq B < A \leq 1).$$

If

$$0 < a_1 < a_2 \quad \text{and} \quad a_2 \geq 2 \left(1 - \left(\frac{1 - A}{1 - B} \right)^\gamma \right),$$

then

$$Q_p(a_2, c; h) \subset Q_p(a_1, c; h).$$

Proof. The analytic function $h(z)$ defined by (2.12) is convex univalent in U (cf. [11]), $h(0) = p$, and $h(U)$ is symmetric with respect to the real axis. Thus $h(z) \in P$ and

$$\operatorname{Re} h(z) > \beta = h(-1) = p - 1 + \left(\frac{1 - A}{1 - B} \right)^\gamma \geq 0 \quad (z \in U).$$

Hence the desired result follows from Theorem 2.1 at once. \square

If we let $\gamma = 1$, then Corollary 2.2 yields the following.

Corollary 2.3. *Let $h(z)$ be given by (1.11). If a , A and B ($-1 \leq B < A \leq 1$) satisfy either*

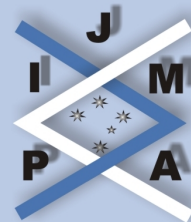
$$(i) \quad a \geq 1 - 2 \left(\frac{1 - A}{1 - B} \right) > 0$$

or

$$(ii) \quad a > 0 \geq 1 - 2 \left(\frac{1 - A}{1 - B} \right),$$

then

$$Q_p(a + 1, c; h) \subset Q_p(a, c; h).$$



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Using Jack's Lemma, Liu and Owa [5, Theorem 1] proved that, if $a \geq \frac{A-B}{1-B}$, then

$$H_{a+1,c,p}(A, B) \subset H_{a,c,p}(A, B).$$

Since

$$\frac{A-B}{1-B} \geq 1 - 2 \left(\frac{1-A}{1-B} \right) \quad (-1 \leq B < A \leq 1)$$

and the equality occurs only when $A = 1$, we see that Corollary 2.3 is better than the result of [5].

Corollary 2.4. *Let*

$$(2.13) \quad h(z) = p + \sum_{k=1}^{\infty} \left(\frac{\gamma+1}{\gamma+k} \right) \delta^k z^k \quad (z \in U; 0 < \delta \leq 1; \gamma \geq 0).$$

If

$$0 < a_1 < a_2 \quad \text{and} \quad a_2 \geq 2 \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{\gamma+1}{\gamma+k} \right) \delta^k,$$

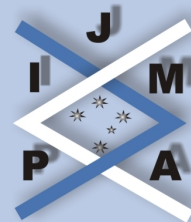
then

$$Q_p(a_2, c; h) \subset Q_p(a_1, c; h).$$

Proof. The function $h(z)$ defined by (2.13) is in the class P (cf. [8]) and satisfies $h(\bar{z}) = \overline{h(z)}$. Thus

$$\operatorname{Re} h(z) > \beta = h(-1) = p + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\gamma+1}{\gamma+k} \right) \delta^k > p - \delta \geq 0 \quad (z \in U).$$

Therefore we have the corollary by using Theorem 2.1. □



Corollary 2.5. *Let*

$$(2.14) \quad h(z) = p + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{\gamma z}}{1 - \sqrt{\gamma z}} \right) \right)^2 \quad (z \in U; 0 < \gamma \leq 1).$$

If

$$0 < a_1 < a_2 \quad \text{and} \quad a_2 \geq \frac{16}{\pi^2} (\arctan \sqrt{\gamma})^2,$$

then

$$Q_p(a_2, c; h) \subset Q_p(a_1, c; h).$$

Proof. The function $h(z)$ defined by (2.14) belongs to the class P (cf. [7]) and satisfies $h(\bar{z}) = \overline{h(z)}$. Thus

$$\operatorname{Re} h(z) > \beta = h(-1) = p - \frac{8}{\pi^2} (\arctan \sqrt{\gamma})^2 \geq p - \frac{1}{2} > 0 \quad (z \in U).$$

Hence an application of Theorem 2.1 yields the desired result. \square

For $\gamma = 1$, Corollary 2.5 leads to

Corollary 2.6. *Let*

$$h(z) = p + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \quad (z \in U).$$

Then, for $a > 0$,

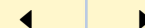
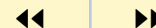
$$Q_p(a + 1, c; h) \subset Q_p(a, c; h).$$

Theorem 2.7. *Let* $h(z) \in P$ *and*

$$(2.15) \quad \operatorname{Re} h(z) > p - 1 + \alpha \quad (z \in U; \alpha < 1).$$

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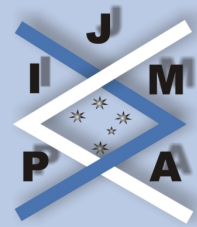
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If $f(z) \in Q_p(a, c; h)$,

$$(2.16) \quad g(z) \in A_p \quad \text{and} \quad \frac{g(z)}{z^{p-1}} \in R(\alpha) \quad (\alpha < 1),$$

then

$$(f * g)(z) \in Q_p(a, c; h).$$

Proof. Let $f(z) \in Q_p(a, c; h)$ and suppose that

$$(2.17) \quad q(z) = \frac{L_p(a, c)f(z)}{z^{p-1}}.$$

Then

$$(2.18) \quad F(z) = \frac{L_p(a+1, c)f(z)}{L_p(a, c)f(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a},$$

$q(z) \in A$ and

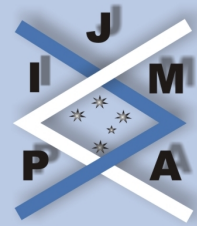
$$(2.19) \quad \frac{zq'(z)}{q(z)} \prec 1 - p + h(z)$$

(see (2.10) used in the proof of Theorem 2.1). By (2.15) and (2.19), we see that

$$(2.20) \quad q(z) \in S^*(\alpha).$$

For $g(z) \in A_p$, it follows from (2.17) and (2.18) that

$$(2.21) \quad \begin{aligned} \frac{L_p(a+1, c)(f * g)(z)}{L_p(a, c)(f * g)(z)} &= \frac{g(z) * L_p(a+1, c)f(z)}{g(z) * L_p(a, c)f(z)} \\ &= \frac{\frac{g(z)}{z^{p-1}} * (q(z)F(z))}{\frac{g(z)}{z^{p-1}} * q(z)} \quad (z \in U). \end{aligned}$$



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Now, by using (2.16), (2.18), (2.20) and (2.21), an application of Lemma 1.2 leads to

$$\frac{L_p(a+1, c)(f * g)(z)}{L_p(a, c)(f * g)(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a}.$$

This shows that $(f * g)(z) \in Q_p(a, c; h)$. □

For $\alpha = 0$ and $\alpha = \frac{1}{2}$, Theorem 2.7 reduces to

Corollary 2.8. *Let $h(z) \in P$ and $g(z) \in A_p$ satisfy either*

(i) $\frac{g(z)}{z^{p-1}}$ is convex univalent in U and

$$\operatorname{Re} h(z) > p - 1 \quad (z \in U)$$

or

(ii) $\frac{g(z)}{z^{p-1}} \in S^*\left(\frac{1}{2}\right)$ and

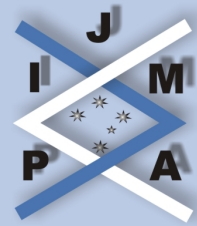
$$\operatorname{Re} h(z) > p - \frac{1}{2} \quad (z \in U).$$

If $f(z) \in Q_p(a, c; h)$, then

$$(f * g)(z) \in Q_p(a, c; h).$$

Theorem 2.9. *Let $h(z) \in P$ and*

$$(2.22) \quad \operatorname{Re} h(z) > -\operatorname{Re} \lambda \quad (z \in U),$$



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where λ is a complex number such that $\operatorname{Re} \lambda > -p$. If $f(z) \in Q_p(a, c; h)$, then the function

$$(2.23) \quad g(z) = \frac{\lambda + p}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt$$

is also in the class $Q_p(a, c; h)$.

Proof. For $f(z) \in A_p$ and $\operatorname{Re} \lambda > -p$, it follows from (1.7) and (2.23) that $g(z) \in A_p$ and

$$(2.24) \quad \begin{aligned} (\lambda + p)L_p(a, c)f(z) &= \lambda L_p(a, c)g(z) + z(L_p(a, c)g(z))' \\ &= aL_p(a + 1, c)g(z) + (\lambda + p - a)L_p(a, c)g(z). \end{aligned}$$

If we let

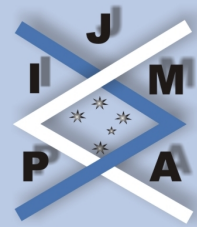
$$(2.25) \quad q(z) = \frac{L_p(a + 1, c)g(z)}{L_p(a, c)g(z)},$$

then (2.24) and (2.25) lead to

$$(2.26) \quad aq(z) + \lambda + p - a = (\lambda + p) \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)}.$$

Differentiating both sides of (2.26) logarithmically and using (1.7) and (2.25), we obtain

$$(2.27) \quad \begin{aligned} \frac{zq'(z)}{aq(z) + \lambda + p - a} &= \frac{1}{a} \left(\frac{z(L_p(a, c)f(z))'}{L_p(a, c)f(z)} - \frac{z(L_p(a, c)g(z))'}{L_p(a, c)g(z)} \right) \\ &= \frac{L_p(a + 1, c)f(z)}{L_p(a, c)f(z)} - q(z). \end{aligned}$$



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Let $f(z) \in Q_p(a, c; h)$. Then it follows from (2.27) that

$$(2.28) \quad q(z) + \frac{zq'(z)}{aq(z) + \lambda + p - a} \prec 1 - \frac{p}{a} + \frac{h(z)}{a}.$$

Also, in view of (2.22), we have

$$(2.29) \quad \operatorname{Re} \left\{ a \left(1 - \frac{p}{a} + \frac{h(z)}{a} \right) + \lambda + p - a \right\} = \operatorname{Re} h(z) + \operatorname{Re} \lambda > 0 \quad (z \in U).$$

Therefore, it follows from (2.28), (2.29) and Lemma 1.3 that

$$q(z) \prec 1 - \frac{p}{a} + \frac{h(z)}{a}.$$

This proves that $g(z) \in Q_p(a, c; h)$. □

From Theorem 2.9 we have the following corollaries.

Corollary 2.10. *Let $h(z)$ be defined as in Corollary 2.2. If $f(z) \in Q_p(a, c; h)$ and*

$$\operatorname{Re} \lambda \geq 1 - p - \left(\frac{1 - A}{1 - B} \right)^\gamma \quad (0 < \gamma \leq 1; -1 \leq B < A \leq 1),$$

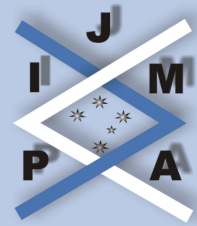
then the function $g(z)$ given by (2.23) is also in the class $Q_p(a, c; h)$.

In the special case when $\gamma = 1$, Corollary 2.10 was obtained by Liu and Owa [5, Theorem 2] using Jack's Lemma.

Corollary 2.11. *Let $h(z)$ be defined as in Corollary 2.4. If $f(z) \in Q_p(a, c; h)$ and*

$$\operatorname{Re} \lambda \geq \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{\gamma + 1}{\gamma + k} \right) \delta^k - p \quad (0 < \delta \leq 1; \gamma \geq 0),$$

then the function $g(z)$ given by (2.23) is also in the class $Q_p(a, c; h)$.



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Corollary 2.12. Let $h(z)$ be defined as in Corollary 2.5. If $f(z) \in Q_p(a, c; h)$ and

$$\operatorname{Re} \lambda \geq \frac{8}{\pi^2} (\arctan \sqrt{\gamma})^2 - p \quad (0 < \gamma \leq 1),$$

then the function $g(z)$ given by (2.23) is also in the class $Q_p(a, c; h)$.

Theorem 2.13. Let $h(z) \in P$ and

$$(2.30) \quad \operatorname{Re} h(z) > -\frac{\operatorname{Re} \lambda}{\beta} \quad (z \in U),$$

where $\beta > 0$ and λ is a complex number such that $\operatorname{Re} \lambda > -p\beta$. If $f(z) \in Q_p(a, c; h)$, then the function $g(z) \in A_p$ defined by

$$(2.31) \quad L_p(a, c)g(z) = \left(\frac{\lambda + p\beta}{z^\lambda} \int_0^z t^{\lambda-1} (L_p(a, c)f(t))^\beta dt \right)^{\frac{1}{\beta}}$$

is also in the class $Q_p(a, c; h)$.

Proof. Let $f(z) \in Q_p(a, c; h)$. From the definition of $g(z)$ we have

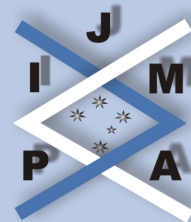
$$(2.32) \quad z^\lambda (L_p(a, c)g(z))^\beta = (\lambda + p\beta) \int_0^z t^{\lambda-1} (L_p(a, c)f(t))^\beta dt.$$

Differentiating both sides of (2.32) logarithmically and using (1.7), we get

$$(2.33) \quad \lambda + \beta(aq(z) + p - a) = (\lambda + p\beta) \left(\frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} \right)^\beta,$$

where

$$(2.34) \quad q(z) = \frac{L_p(a+1, c)g(z)}{L_p(a, c)g(z)}.$$



Also, differentiating both sides of (2.33) logarithmically and using (1.7), we arrive at

$$(2.35) \quad q(z) + \frac{zq'(z)}{a\beta q(z) + \lambda + \beta(p-a)} = \frac{L_p(a+1, c)f(z)}{L_p(a, c)f(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a}.$$

Noting that (2.30) and $\beta > 0$, we see that

$$(2.36) \quad \operatorname{Re} \left\{ a\beta \left(1 - \frac{p}{a} + \frac{h(z)}{a} \right) + \lambda + \beta(p-a) \right\} \\ = \beta \operatorname{Re} h(z) + \operatorname{Re} \lambda > 0 \quad (z \in U).$$

Now, in view of (2.34), (2.35) and (2.36), an application of Lemma 1.3 yields

$$\frac{L_p(a+1, c)g(z)}{L_p(a, c)g(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a},$$

that is, $g(z) \in Q_p(a, c; h)$. □

Corollary 2.14. *Let $h(z)$ be defined as in Corollary 2.2. If $f(z) \in Q_p(a, c; h)$ and*

$$\operatorname{Re} \lambda \geq \beta \left(1 - p - \left(\frac{1-A}{1-B} \right)^\gamma \right) \quad (0 < \gamma \leq 1; -1 \leq B < A \leq 1; \beta > 0),$$

then the function $g(z) \in A_p$ defined by (2.31) is also in the class $Q_p(a, c; h)$.

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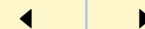
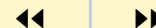
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