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SOME INEQUALITIES CONNECTED WITH AN APPROXIMATE INTEGRATION

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ABSTRACT. Some classical and new inequalities of an approximate integration are obtained with use of Hadamard type inequalities and delta–convex functions of higher orders.

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1. INTRODUCTION

One of the most famous inequalities in analysis is the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for a convex function $f : [a,b] \to \mathbb{R}$. Using this inequality and some properties of delta–convex functions (cf. an exhaustive study of this class of functions given by Veselý and Zajíček [8]; cf. also [2], where, independently of [8], the authors introduced the concept of convex–dominated functions which coincides with the notion of delta–convex functions) Dragomir, Pearce and Pečarić proved recently the following result.

Theorem 1.1. [3, Remark 1] *Let f be twice differentiable on $[a,b]$ and suppose that $M := \sup_{x\in[a,b]} \left|f''(x)\right| < \infty$. Then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{M}{24}(b-a)^2 \quad \textit{and}$$
$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{M}{12}(b-a)^2.$$

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By multiplying both sides of these inequalities by $b-a$ the simplest cases of the inequalities estimating the accuracy of the Midpoint and Trapezoidal Rules of an approximate integration can be recognized.

In this paper we give some results related to Theorem 1.1. Some new inequalities are obtained and some known inequalities are reproved. To obtain these results we make use of an important extension of convex functions, i.e. convex functions of higher orders (studied among others by Popoviciu [6]). Let us recall this notion. It is not difficult to notice that a function $f : I \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval) is convex if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ f(x) & f(y) & f(z) \end{vmatrix} \geq 0 \tag{1.2}$$

for any $x, y, z \in I$ such that $x < y < z$. Following this observation we define the function $f : I \to \mathbb{R}$ to be n*–convex* ($n \in \mathbb{N}$) if and only if

$$D_{n+1}(x_0, x_1, \dots, x_{n+1}; f) := \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & \dots & x_{n+1}^n \\ f(x_0) & f(x_1) & \dots & f(x_{n+1}) \end{vmatrix} \geq 0$$

for any $x_0, x_1, \dots, x_{n+1} \in I$ such that $x_0 < x_1 < \dots < x_{n+1}$. Obviously 1–convex functions are convex in the classical sense. For more information about the definition and the properties of convex functions of higher orders the reader is referred to [5], [6], [7].

The following theorem (due to Popoviciu [6]) characterizes n–convexity of $n+1$ times differentiable functions (cf. also [5], [1, Theorem A]).

Theorem 1.2. *Assume that* $f : (a, b) \to \mathbb{R}$ *is an* $n+1$ *times differentiable function. Then* f *is* n*–convex if and only if* $f^{(n+1)}(x) \geq 0$*,* $x \in (a, b)$*.*

The result similar to the if part of Theorem 1.2 is true for $f : [a, b] \to \mathbb{R}$. The expression "$f : [a, b] \to \mathbb{R}$ is continuous" means, as usual, that f is continuous on (a, b), continuous on the right at a and continuous on the left at b.

Theorem 1.3. *Assume that* $f : [a, b] \to \mathbb{R}$ *is* $n+1$ *times differentiable on* (a, b) *and continuous on* $[a, b]$*. If* $f^{(n+1)}(x) \geq 0$*,* $x \in (a, b)$*, then* f *is* n*–convex.*

Proof. The result follows by Theorem 1.2 and by the fact that the functions $D_{n+1}(\cdot, x_1, \dots, x_{n+1}; f)$ and $D_{n+1}(x_0, \dots, x_n, \cdot; f)$ are continuous on the right at a and on the left at b, respectively. □

In [1] Bessenyei and Páles recently obtained some extensions of Hadamard's inequality (1.1) for convex functions of higher orders ([1, Theorems 6 and 7]). Since the notations of these results will be used very often in the present paper, we quote these theorems in extenso. Let us remark that in [1] the name n*–monotone functions* is used for $(n-1)$–convex functions. For reader's convenience we consequently use this last name.

Theorem 1.4. [1, Theorem 6] *Let, for* $n \geq 0$*,*

$$p_n(x) := \begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ x & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ x^n & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{vmatrix},$$

then p_n *has* n *pairwise distinct roots in* $(0,1)$. *Denote these roots by* $\lambda_1,\dots,\lambda_n$ *and*

$$\alpha_0 := \frac{1}{p_n^2(0)}\int_0^1 p_n^2(x)dx,$$
$$\alpha_k := \frac{1}{\lambda_k}\int_0^1 \frac{xp_n(x)}{(x-\lambda_k)p_n'(\lambda_k)}dx \quad (k=1,\dots,n).$$

Then the following inequalities hold for any $2n$*–convex function* $f:[a,b]\to\mathbb{R}$*:*

$$\alpha_0 f(a) + \sum_{k=1}^{n}\alpha_k f\big((1-\lambda_k)a+\lambda_k b\big) \le \frac{1}{b-a}\int_a^b f(x)dx \quad \text{and} \tag{1.3}$$

$$\frac{1}{b-a}\int_a^b f(x)dx \le \sum_{k=1}^{n}\alpha_k f\big(\lambda_k a+(1-\lambda_k)b\big) + \alpha_0 f(b). \tag{1.4}$$

Theorem 1.5. [1, Theorem 7] *Let, for* $n\ge 1$*,*

$$p_n(x) := \begin{vmatrix} 1 & 1 & \cdots & \frac{1}{n} \\ x & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x^n & \frac{1}{n+1} & \cdots & \frac{1}{2n} \end{vmatrix}, \quad q_n(x) := \begin{vmatrix} 1 & \frac{1}{2\cdot 3} & \cdots & \frac{1}{n(n+1)} \\ x & \frac{1}{3\cdot 4} & \cdots & \frac{1}{(n+1)(n+2)} \\ \vdots & \vdots & \ddots & \vdots \\ x^{n-1} & \frac{1}{(n+1)(n+2)} & \cdots & \frac{1}{(2n-1)2n} \end{vmatrix},$$

then p_n *has* n*, and* q_n *has* $n-1$ *pairwise distinct roots in* $(0,1)$. *Denote these roots by* $\lambda_1,\dots,\lambda_n$ *and* $\mu_1,\dots,\mu_{n-1}$*, respectively. Let*

$$\alpha_k := \int_0^1 \frac{p_n(x)}{(x-\lambda_k)p_n'(\lambda_k)}dx \quad (k=1,\dots,n) \quad \text{and}$$
$$\beta_0 := \frac{1}{q_n^2(0)}\int_0^1 (1-x)q_n^2(x)dx,$$
$$\beta_k := \frac{1}{(1-\mu_k)\mu_k}\int_0^1 \frac{x(1-x)q_n(x)}{(x-\mu_k)q_n'(\mu_k)}dx \quad (k=1,\dots,n-1),$$
$$\beta_n := \frac{1}{q_n^2(1)}\int_0^1 xq_n^2(x)dx.$$

Then the following inequalities hold for any $(2n-1)$*–convex function* $f:[a,b]\to\mathbb{R}$*:*

$$\sum_{k=1}^{n}\alpha_k f\big((1-\lambda_k)a+\lambda_k b\big) \le \frac{1}{b-a}\int_a^b f(x)dx \quad \text{and} \tag{1.5}$$

$$\frac{1}{b-a}\int_a^b f(x)dx \le \beta_0 f(a) + \sum_{k=1}^{n-1}\beta_k f\big((1-\mu_k)a+\mu_k b\big) + \beta_n f(b). \tag{1.6}$$

Remark 1.6. (cf. [1, Corollary 1]) For $n = 1$ we obtain by Theorem 1.5 the classical Hadamard inequality. Indeed, it is easy to compute

$$p_1(x) = \begin{vmatrix} 1 & \frac{1}{1} \\ x & \frac{1}{2} \end{vmatrix} = \frac{1}{2} - x,$$

$$q_1(x) = 1, \quad \lambda_1 = \frac{1}{2},$$

$$\alpha_1 = \int_0^1 \frac{\frac{1}{2} - x}{\left(x - \frac{1}{2}\right) \cdot (-1)} dx = 1,$$

$$\beta_0 = \int_0^1 (1 - x) dx = \frac{1}{2},$$

$$\beta_1 = \int_0^1 x dx = \frac{1}{2}.$$

Then using (1.5) and (1.6) for a 1–convex (i.e. convex) function $f : [a, b] \to \mathbb{R}$ we get (1.1).

Now let us recall the notion of delta–convexity. Let $g : I \to \mathbb{R}$ be a convex function. It is well known (cf. e.g. [4], [8]) that $f : I \to \mathbb{R}$ is *delta–convex with a control function* g (briefly g–delta–convex) if and only if the functions $g + f$ and $g - f$ are convex. Combining this fact with (1.2) we obtain that the function f is g–delta–convex if and only if

$$\left| \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ f(x) & f(y) & f(z) \end{vmatrix} \right| \le \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ g(x) & g(y) & g(z) \end{vmatrix}$$

for any $x, y, z \in I$ such that $x < y < z$.

In the paper [4], Ger proposed to consider delta–convex functions of higher orders. For a definition and a discussion of this notion the reader is referred to [4]. In this paper we use the following definition. Let $g : I \to \mathbb{R}$ be an n–convex function. The function $f : I \to \mathbb{R}$ is said to be *n–delta–convex with a control function* g (n–g–delta–convex for short) if and only if the inequality

$$\left| D_{n+1}(x_0, x_1, \dots, x_{n+1}; f) \right| \le D_{n+1}(x_0, x_1, \dots, x_{n+1}; g)$$

holds for any $x_0, x_1, \dots, x_{n+1} \in I$ such that $x_0 < x_1 < \dots < x_{n+1}$. Obviously 1–$g$–delta–convex functions are g–delta–convex.

Using the properties of determinants we obtain the following theorem (cf. [4, Proposition 1]).

Theorem 1.7. *Let $g : I \to \mathbb{R}$ be an n–convex function. The function $f : I \to \mathbb{R}$ is n–g–delta–convex if and only if the functions $g + f$ and $g - f$ are n–convex.*

The next result follows immediately from Theorems 1.7 and 1.3.

Theorem 1.8. *Assume that the functions $f, g : [a, b] \to \mathbb{R}$ are $n+1$ times differentiable on (a, b) and continuous on $[a, b]$. If the inequality $\left| f^{(n+1)}(x) \right| \le g^{(n+1)}(x)$ holds for any $x \in (a, b)$, then f is n–g–delta–convex.*

2. MAIN RESULTS

Theorem 2.1. *Let, for $n \ge 0$, $g : [a, b] \to \mathbb{R}$ be a $2n$–convex function and let $f : [a, b] \to \mathbb{R}$ be a $2n$–g–delta–convex. Then, under the notations of Theorem 1.4, the following inequalities* *hold:*

$$\left|\alpha_0 f(a) + \sum_{k=1}^{n} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq \frac{1}{b-a}\int_a^b g(x)dx - \alpha_0 g(a) - \sum_{k=1}^{n} \alpha_k g\big((1-\lambda_k)a + \lambda_k b\big) \tag{2.1}$$

and

$$\left|\sum_{k=1}^{n} \alpha_k f\big(\lambda_k a + (1-\lambda_k)b\big) + \alpha_0 f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq \sum_{k=1}^{n} \alpha_k g\big(\lambda_k a + (1-\lambda_k)b\big) + \alpha_0 g(b) - \frac{1}{b-a}\int_a^b g(x)dx. \tag{2.2}$$

Proof. Since f is $2n$–g–delta–convex, the functions $g+f$ and $g-f$ are $2n$–convex. Using (1.3) for $g+f$ we obtain

$$\alpha_0 g(a) + \alpha_0 f(a) + \sum_{k=1}^{n} \alpha_k g\big((1-\lambda_k)a + \lambda_k b\big) + \sum_{k=1}^{n} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) \\ \leq \frac{1}{b-a}\int_a^b g(x)dx + \frac{1}{b-a}\int_a^b f(x)dx.$$

Then

$$\alpha_0 f(a) + \sum_{k=1}^{n} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) - \frac{1}{b-a}\int_a^b f(x)dx \\ \leq \frac{1}{b-a}\int_a^b g(x)dx - \alpha_0 g(a) - \sum_{k=1}^{n} \alpha_k g\big((1-\lambda_k)a + \lambda_k b\big). \tag{2.3}$$

Using (1.3) for $g-f$ we get

$$\alpha_0 g(a) - \alpha_0 f(a) + \sum_{k=1}^{n} \alpha_k g\big((1-\lambda_k)a + \lambda_k b\big) - \sum_{k=1}^{n} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) \\ \leq \frac{1}{b-a}\int_a^b g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx.$$

Then

$$\alpha_0 f(a) + \sum_{k=1}^{n} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) - \frac{1}{b-a}\int_a^b f(x)dx \\ \geq -\left(\frac{1}{b-a}\int_a^b g(x)dx - \alpha_0 g(a) - \sum_{k=1}^{n} \alpha_k g\big((1-\lambda_k)a + \lambda_k b\big)\right) \tag{2.4}$$

and the inequality (2.1) follows by (2.3) and (2.4).

The proof of (2.2) is analogous: it is enough to use (1.4) for $2n$–convex functions $g+f$ and $g-f$. □

Theorem 2.2. *Let, for $n \geq 1$, $g : [a,b] \to \mathbb{R}$ be a $(2n-1)$–convex function and let $f : [a,b] \to \mathbb{R}$ be $(2n-1)$–g–delta–convex. Then, under the notations of Theorem 1.5, the following inequalities hold:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=1}^n \alpha_k f\big((1-\lambda_k)a+\lambda_k b\big)\right| \leq \frac{1}{b-a}\int_a^b g(x)dx - \sum_{k=1}^n \alpha_k g\big((1-\lambda_k)a+\lambda_k b\big) \tag{2.5}$$

and

$$\left|\beta_0 f(a) + \sum_{k=1}^{n-1}\beta_k f\big((1-\mu_k)a+\mu_k b\big) + \beta_n f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \beta_0 g(a) + \sum_{k=1}^{n-1}\beta_k g\big((1-\mu_k)a+\mu_k b\big) + \beta_n g(b) - \frac{1}{b-a}\int_a^b g(x)dx. \tag{2.6}$$

Proof. Our argument is similar to the one in the proof of Theorem 2.1. Since f is $(2n-1)$–g–delta–convex, the functions $g+f$ and $g-f$ are $(2n-1)$–convex. Using (1.5) for $g+f$ and $g-f$ we obtain (2.5). Using (1.6) for $g+f$ and $g-f$ we get (2.6). □

3. APPLICATIONS

By an appropriate specification of the control function g in Theorems 2.1 and 2.2 we can obtain some inequalities which estimate the accuracy of some formulae of an approximate integration. Both classical and new inequalities can be derived. Let us start with the following remark.

Remark 3.1. Let f be k times differentiable on $[a,b]$ and assume that

$$M_k(f) := \sup_{x\in[a,b]}\left|f^{(k)}(x)\right| < \infty.$$

Then for $g(x) = \frac{M_k(f)x^k}{k!}$ we have $g^{(k)}(x) = M_k(f)$ and $\left|f^{(k)}(x)\right| \leq g^{(k)}(x)$, $x \in [a,b]$. By Theorem 1.8 f is $(k-1)$–g–delta–convex.

Now we are going to discuss the accuracy of the Midpoint and Trapezoidal rules in approximate integration. We recall these rules.

Midpoint Rule. Let f be twice differentiable on $[a,b]$ and assume that $M_2(f) < \infty$. Let $m \in \mathbb{N}$, $x_i = a + i\dfrac{b-a}{m}$, $i = 0,\dots,m$ and let $y_i = f\left(\dfrac{x_{i-1}+x_i}{2}\right)$, $i = 1,\dots,m$. Then

$$\left|\int_a^b f(x)dx - \frac{b-a}{m}(y_1+\cdots+y_m)\right| \leq \frac{M_2(f)(b-a)^3}{24m^2}.$$

Observe that for $m=1$ we get

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right)\right| \leq \frac{M_2(f)(b-a)^3}{24}. \tag{3.1}$$

Trapezoidal Rule. Let f be twice differentiable on $[a,b]$ and assume that $M_2(f) < \infty$. Let $m \in \mathbb{N}$, $x_i = a + i\dfrac{b-a}{m}$, $y_i = f(x_i)$, $i = 0,\dots,m$. Then

$$\left|\int_a^b f(x)dx - \frac{b-a}{2m}\big(y_0 + y_m + 2(y_1 + y_2 + \cdots + y_{m-1})\big)\right| \le \frac{M_2(f)(b-a)^3}{12m^2}.$$

For $m = 1$ we get

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}\big(f(a)+f(b)\big)\right| \le \frac{M_2(f)(b-a)^3}{12}. \tag{3.2}$$

Now we derive (3.1) and (3.2) from Theorem 2.2 (cf. [3, Remark 1] and Theorem 1.1).

Corollary 3.2. *Let f be twice differentiable on $[a,b]$ and assume that $M_2(f) < \infty$. Then the inequalities* (3.1) *and* (3.2) *hold.*

Proof. Let $n = 1$. We use the notations of Theorem 1.5. By Remark 1.6 we have $p_1(x) = \frac{1}{2} - x$, $q_1(x) = 1$, $\lambda_1 = \frac{1}{2}$, $\alpha_1 = 1$, $\beta_0 = \beta_1 = \frac{1}{2}$. Let $g(x) = \frac{M_2(f)x^2}{2}$. Then by Remark 3.1 f is g–delta–convex and by (2.5) we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{b-a}\int_a^b \frac{M_2(f)x^2}{2}dx - \frac{M_2(f)\left(\frac{a+b}{2}\right)^2}{2}.$$

Multiplying both sides of this inequality by $b-a$ we compute

$$\begin{aligned}\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right)\right| &\le \frac{M_2(f)}{2}\left(\frac{b^3-a^3}{3} - (b-a)\frac{(a+b)^2}{4}\right)\\ &= \frac{M_2(f)(b-a)^3}{24},\end{aligned}$$

which gives (3.1). By (2.6) we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{M_2(f)}{2}\left(\frac{a^2+b^2}{2}\right) - \frac{1}{b-a}\int_a^b \frac{M_2(f)x^2}{2}dx$$

Multiplying both sides of this inequality by $b-a$ we obtain (3.2). □

As an example of some new inequalities we give the following

Corollary 3.3. *Let f be three times differentiable on $[a,b]$ and assume that $M_3(f) < \infty$. Then*

$$\left|\int_a^b f(x)dx - \frac{b-a}{4}\left(f(a) + 3f\left(\frac{a+2b}{3}\right)\right)\right| \le \frac{M_3(f)(b-a)^4}{216} \quad \textit{and} \tag{3.3}$$

$$\left|\int_a^b f(x)dx - \frac{b-a}{4}\left(f(b) + 3f\left(\frac{2a+b}{3}\right)\right)\right| \le \frac{M_3(f)(b-a)^4}{216}. \tag{3.4}$$

Proof. Let $n = 1$. Under the notations of Theorem 1.4 we compute

$$p_1(x) = \begin{vmatrix} 1 & \frac{1}{2} \\ x & \frac{1}{3}\end{vmatrix} = \frac{1}{3} - \frac{1}{2}x, \qquad \lambda_1 = \frac{2}{3},$$

$$\alpha_0 = 9\int_0^1 \left(\frac{1}{3} - \frac{1}{2}x\right)^2 dx = \frac{1}{4},$$

$$\alpha_1 = \frac{3}{2}\int_0^1 \frac{x(\frac{1}{3} - \frac{1}{2}x)}{(x-\frac{2}{3})\cdot(-\frac{1}{2})}dx = \frac{3}{2}\int_0^1 xdx = \frac{3}{4}.$$

Let $g(x) = \frac{M_3(f)x^3}{6}$. By Remark 3.1 f is 2–g–delta–convex. By the inequality (2.1) of Theorem 2.1 we infer

$$\left|\frac{1}{4}f(a) + \frac{3}{4}f\left(\frac{1}{3}a + \frac{2}{3}b\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{b-a}\int_a^b \frac{M_3(f)x^3}{6} - \frac{1}{4}\cdot\frac{M_3(f)a^3}{6} - \frac{3}{4}\cdot\frac{M_3(f)\left(\frac{1}{3}a+\frac{2}{3}b\right)^3}{6}.$$

Multiplying both sides of this inequality by $b-a$ and computing the right hand side we get (3.3). The inequality (3.4) we obtain similarly using (2.2). □

Let us now discuss the accuracy of Simpson's Rule in approximate integration. Recall that this rule reads as follows.

Simpson's Rule. Let f be four times differentiable on $[a,b]$ and assume that $M_4(f) < \infty$. Let $m \in \mathbb{N}$, $x_i = a + i\frac{b-a}{2m}$, $y_i = f(x_i)$, $i = 0,\ldots,2m$. Then

$$\left|\int_a^b f(x)dx - \frac{b-a}{6m}\big(y_0 + y_{2m} + 2(y_2 + y_4 + \cdots + y_{2m-2}) + 4(y_1 + y_3 + \cdots + y_{2m-1})\big)\right| \leq \frac{M_4(f)(b-a)^5}{2880m^4}.$$

For $m = 1$ we obtain

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leq \frac{M_4(f)(b-a)^5}{2880}. \tag{3.5}$$

We can derive (3.5) from Theorem 2.2.

Corollary 3.4. *Let f be four times differentiable on $[a,b]$ and assume that $M_4(f) < \infty$. Then the inequality* (3.5) *holds.*

Proof. Let $n = 2$. Using the notations of Theorem 1.5 we compute

$$q_2(x) = \begin{vmatrix} 1 & \frac{1}{6} \\ x & \frac{1}{12} \end{vmatrix} = \frac{1}{12}(1-2x), \qquad \mu_1 = \frac{1}{2},$$

$$\beta_0 = 144\int_0^1 (1-x)\cdot\frac{1}{144}(1-2x)^2 dx = \frac{1}{6},$$

$$\beta_1 = 4\int_0^1 \frac{x(1-x)\frac{1}{12}(1-2x)}{\left(x-\frac{1}{2}\right)\cdot\left(-\frac{1}{6}\right)}dx = 4\int_0^1 x(1-x)dx = \frac{2}{3},$$

$$\beta_2 = 144\int_0^1 x\cdot\frac{1}{144}(1-2x)^2 dx = \frac{1}{6}.$$

Let $g(x) = \frac{M_4(f)x^4}{24}$. By Remark 3.1 f is 3–g–delta–convex. By the inequality (2.6) of Theorem 2.2 we obtain

$$\left|\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{6}\cdot\frac{M_4(f)a^4}{24} + \frac{2}{3}\cdot\frac{M_4(f)}{24}\left(\frac{a+b}{2}\right)^4 + \frac{1}{6}\cdot\frac{M_4(f)b^4}{24} - \frac{1}{b-a}\int_a^b \frac{M_4(f)x^4}{24}dx,$$

from which the inequality (3.5) follows. □

Other examples of the roots of polynomials of Theorems 1.4 and 1.5 are given in [1]. Then the integral inequalities similar to (3.1), (3.2), (3.3), (3.4) and (3.5) can be obtained by Theorems 2.1 and 2.2.

REFERENCES

[1] M. BESSENYEI AND Zs. PÁLES, Higher-order generalizations of Hadamard's inequality, *Publ. Math. Debrecen,* **61** (2002), 623–643.

[2] S.S. DRAGOMIR AND N.M. IONESCU, On some inequalities for convex–dominated functions, *Anal. Numér. Théor. Approx.,* **19**(1) (1990), 21–27.

[3] S.S. DRAGOMIR, C.E.M. PEARCE AND J. PEČARIĆ, Means, g–convex dominated functions and Hadamard–type inequalities, *Tamsui Oxford Journal of Mathematical Sciences,* **18** (2002), 161–173.

[4] R. GER, Stability of polynomial mappings controlled by n–convex functionals, *World Scientific Publishing Company (WSSIAA),* **3** (1994), 255–268.

[5] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality,* Państwowe Wydawnictwo Naukowe (Polish Scientific Publishers) and Uniwersytet Śląski, Warszawa–Kraków–Katowice 1985.

[6] T. POPOVICIU, Sur quelques propriétés des fonctions d'une ou de deux variables réelles, *Mathematica (Cluj),* **8** (1934), 1–85.

[7] A.W. ROBERTS AND D.E. VARBERG, *Convex Functions,* Academic Press, New York 1973.

[8] L. VESELÝ AND L. ZAJÍČEK, Delta–convex mappings between Banach spaces and applications, *Dissertationes Math.,* **289**, Polish Scientific Publishers, Warszawa 1989.