INEQUALITIES IN THE COMPLEX PLANE

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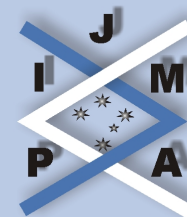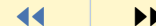|	
Abstract:	A differential inequality is generalised and improved. Several other differential inequalities are considered.

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Róbert Szász

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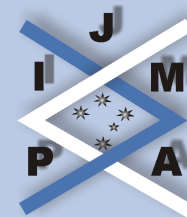Contents

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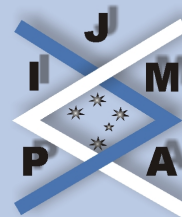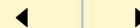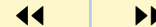
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1. Introduction

Let $\mathcal{H}(U)$ be the set of holomorfic functions defined on the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

In [2, pp. 38 Example 2.4. d] and [3, pp. 192 Example 9.3.4] the authors have proved, as an application of the developed theory, the implication:

If $f \in \mathcal{H}(U)$, $f(0) = 1$ and

$$\operatorname{Re}(f(z) + zf'(z) + z^2 f''(z)) > 0, \ z \in U \ \text{then} \ \operatorname{Re} f(z) > 0, \ z \in U.$$

The aim of this paper is to generalise this inequality and to determine the biggest $\alpha \in R$ for which the implication

$$f(0) = 1, \ \operatorname{Re}(f(z) + zf'(z) + z^2 f''(z)) > 0, \ (\forall)\, z \in U \Rightarrow \operatorname{Re} f(z) > \alpha, \ (\forall)\, z \in U$$

holds true.

In this paper each many-valued function is taken with the principal value.

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2. Preliminaries

In our study we need the following definitions and lemmas:

Let X be a locally convex linear topological space. For a subset $U \subset X$ the closed convex hull of U is defined as the intersection of all closed convex sets containing U and will be denoted by $co(U)$. If $U \subset V \subset X$ then U is called an extremal subset of V provided that whenever $u = tx + (1-t)y$ where $u \in U$, $x, y \in V$ and $t \in (0,1)$ then $x, y \in U$.

An extremal subset of U consisting of just one point is called an extreme point of U.

The set of the extreme points of U will be denoted by EU.

Lemma 2.1 ([1, pp. 45]). *If $J : \mathcal{H}(U) \to \mathbb{R}$ is a real-valued, continuous convex functional and $\mathcal{F}$ is a compact subset of $\mathcal{H}(U)$, then*

$$\max\{J(f) : f \in co(\mathcal{F})\} = \max\{J(f) : f \in \mathcal{F}\} = \max\{J(f) : f \in E(co(\mathcal{F}))\}.$$

In the particular case if J is a linear map then we also have:

$$\min\{J(f) : f \in co(\mathcal{F})\} = \min\{J(f) : f \in \mathcal{F}\} = \min\{J(f) : f \in E(co(\mathcal{F}))\}.$$
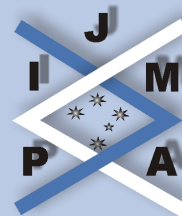
Suppose that $f, g \in \mathcal{H}(U)$. The function f is subordinate to g if there exists a function $\theta \in \mathcal{H}(U)$ such that $\theta(0) = 0$, $|\theta(z)| < 1$, $z \in U$ and $f(z) = g(\theta(z))$, $z \in U$.

The subordination will be denoted by $f \prec g$.

Observation 1. Suppose that $f, g \in \mathcal{H}(U)$ and g is univalent. If $f(0) = g(0)$ and $f(U) \subset g(U)$ then $f \prec g$.
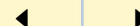
When $F \in \mathcal{H}(U)$ we use the notation

$$s(F) = \{f \in \mathcal{H}(U) : f \prec F\}.$$

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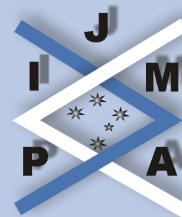
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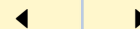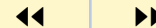
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Lemma 2.2 ([1, pp. 51]). *Suppose that F_α is defined by the equality*

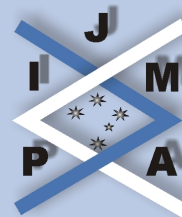$$F_\alpha(z) = \left(\frac{1+cz}{1-z} \right)^\alpha, \quad |c| \le 1, \ c \neq -1.$$

If $\alpha \ge 1$ then $co(s(F_\alpha))$ consists of all functions in $\mathcal{H}(U)$ represented by

$$f(z) = \int_0^{2\pi} \left(\frac{1+cze^{-it}}{1-ze^{-it}} \right)^\alpha d\mu(t)$$

where μ is a positive measure on $[0, 2\pi]$ having the property $\mu([0, 2\pi]) = 1$ and
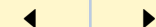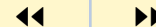
$$E(co(s(F_\alpha))) = \left\{ \frac{1+cze^{-it}}{1-ze^{-it}} \,\middle|\, t \in [0, 2\pi] \right\}.$$

Observation 2. If $L : \mathcal{H}(U) \to \mathcal{H}(U)$ is an invertible linear map and $\mathcal{F} \subset \mathcal{H}(U)$ is a compact subset, then $L(co(\mathcal{F})) = co(L(\mathcal{F}))$ and the set $E(co(\mathcal{F}))$ is in one-to-one correspondence with $EL(co(\mathcal{F}))$.

3. The Main Result

Theorem 3.1. *If $f \in \mathcal{H}(U)$, $f(0) = 1$; $m, p \in \mathbb{N}^*$; $a_k \in \mathbb{R}$, $k = \overline{1, p}$ and*

$$(3.1) \qquad \operatorname{Re} \sqrt[m]{f(z) + a_1 z f'(z) + \cdots + a_p z^p f^{(p)}(z)} > 0, \quad z \in U$$

then

$$(3.2) \quad 1 + \inf_{z \in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1}}{P(n)} z^n \right)$$

$$< \operatorname{Re} f(z) < 1 + \sup_{z \in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1}}{P(n)} z^n \right), \quad z \in U$$

where $P(x) = 1 + a_1 x + a_2 x(x-1) + \cdots + a_p x(x-1)\cdots(x-p+1)$.

Proof. The condition of the theorem can be rewritten in the form

$$\sqrt[m]{f(z) + a_1 z f'(z) + \cdots + a_p z^p f^{(p)}(z)} \prec \frac{1+z}{1-z},$$

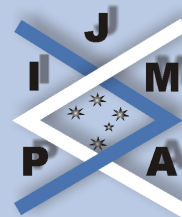which is equivalent to

$$f(z) + a_1 z f'(z) + \cdots + a_p z^p f^{(p)}(z) \prec \left(\frac{1+z}{1-z} \right)^m.$$

According to the results of Lemma 2.2,

$$f(z) + a_1 z f'(z) + \cdots + a_p z^p f^{(p)}(z) = \int_0^{2\pi} \left(\frac{1 + z e^{-it}}{1 - z e^{-it}} \right)^m d\mu(t) = h(z),$$

where $\mu([0, 2\pi]) = 1$.

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If

$$f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in U$$

then

$$f(z) + a_1 z f'(z) + \cdots + a_p z^p f^{(p)}(z) = 1 + \sum_{n=1}^{\infty} b_n P(n) z^n.$$

On the other hand

$$\int_0^{2\pi} \left(\frac{1 + ze^{-it}}{1 - ze^{-it}} \right)^m d\mu(t) = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} C_m^k C_{m+n-k-1}^{m-1} \right) z^n \int_0^{2\pi} e^{-int} d\mu(t),$$

with $C_p^q = 0$ if $q > p$. The equalities $C_p^q = 0$ if $q > p$ imply also that:

$$\sum_{k=0}^{n} C_m^k C_{m+n-k-1}^{m-1} = \sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1}.$$

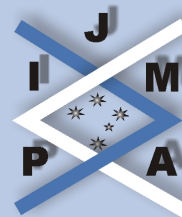The above two developments in power series imply that:

$$1 + \sum_{n=0}^{\infty} b_n P(n) z^n = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1} \right) z^n \int_0^{2\pi} e^{-int} d\mu(t)$$

and

$$b_n = \frac{1}{P(n)} \left(\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1} \right) \int_0^{2\pi} e^{-int} d\mu(t).$$
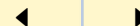
Consequently,

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{P(n)} \left(\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1} \right) z^n \int_0^{2\pi} e^{-int} d\mu(t).$$

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If

$$\mathcal{B} = \left\{ h \in \mathcal{H}(U) \,\Big|\, h(z) = \int_0^{2\pi} \left(\frac{1 + ze^{-it}}{1 - ze^{-it}} \right)^m d\mu(t), \ z \in U, \ \mu([0, 2\pi]) = 1 \right\},$$

$$\mathcal{C} = \left\{ f \in \mathcal{H}(U) \,\Big|\, \operatorname{Re}\left(\sqrt[m]{f(z) + a_1 z f'(z) + \cdots + a_p z^p f^{(p)}(z)} \right) > 0, \ z \in U \right\}$$

then the correspondence $L : \mathcal{B} \to \mathcal{C}$, $L(h) = f$ defines an invertible linear map and according to Observation 2 the extreme points of the class $\mathcal{C}$ are

$$f_t(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{P(n)} \left(\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1} \right) z^n e^{-int}, \quad z \in U, \ t \in [0, 2\pi).$$

This result and Lemma 2.1 implies the assertion of Theorem 3.1. ∎

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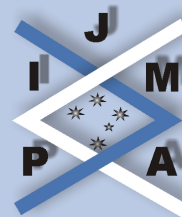
4. Particular Cases

If we put $p = 2, a_1 = a_2 = m = 1$ in Theorem 3.1 then we get:

Corollary 4.1. *If* $f \in \mathcal{H}(U), f(0) = 1$ *and*

(4.1) $$\operatorname{Re}(f(z) + zf'(z) + z^2 f''(z)) > 0, \quad z \in U,$$

then

(4.2) $$\frac{\pi(e^{2\pi} + 1)}{e^{2\pi} - 1} > \operatorname{Re} f(z) > \frac{2\pi e^{\pi}}{e^{2\pi} - 1}, \quad z \in U$$

and these results are sharp in the sense that

$$\sup_{\substack{z \in U \\ f \in \mathcal{C}}} \operatorname{Re} f(z) = \frac{\pi(e^{2\pi} + 1)}{e^{2\pi} - 1} \quad and$$

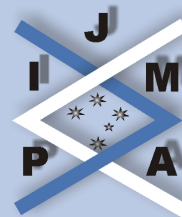$$\inf_{\substack{z \in U \\ f \in \mathcal{C}}} \operatorname{Re} f(z) = \frac{2\pi e^{\pi}}{e^{2\pi} - 1}.$$

Proof. Theorem 3.1 implies the following inequalities:

$$1 + \inf_{z \in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2 + 1} z^n\right) < \operatorname{Re} f(z) < 1 + \sup_{z \in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2 + 1} z^n\right).$$

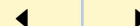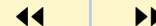The minimum and maximum principle for harmonic functions imply that

$$\sup_{z \in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2 + 1} z^n\right) = \sup_{t \in [0, 2\pi]} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2 + 1} e^{int}\right)$$

$$\inf_{z \in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2 + 1} z^n\right) = \inf_{t \in [0, 2\pi]} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2 + 1} e^{int}\right).$$

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By considering the integral

$$I_n = \int_{|z|=n+\frac{1}{2}} \frac{e^{izt}}{(z^2+1)(e^{2\pi iz}-1)}dz, \quad t \in [0, 2\pi),$$

using the equality $\lim_{n\to\infty} I_n = 0$ and residue theory we deduce that

$$1 + \mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{2}{k^2+1}e^{ikt}\right) = \frac{\pi(e^t + e^{2\pi-t})}{e^{2\pi}-1}, \quad t \in [0, 2\pi)$$

and so we get

$$\frac{\pi(e^{2\pi}+1)}{e^{2\pi}-1} > \mathrm{Re}(f(z)) > \frac{2\pi e^{\pi}}{e^{2\pi}-1}, \quad z \in U.$$

∎

If we put $m = 2$, $a_1 = 0$, $a_2 = 4$, Theorem 3.1 implies

Corollary 4.2. *If $f \in \mathcal{H}(U)$, $f(0) = 1$ and*

(4.3) $$\mathrm{Re}\sqrt{f(z) + 4z^2 f''(z)} > 0, \quad z \in U,$$

then
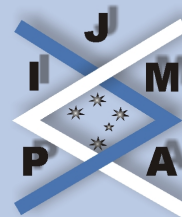
(4.4) $$\mathrm{Re}\, f(z) > 1 + 4\sum_{k=1}^{\infty} \frac{(-1)^k \cdot k}{(2k-1)^2}, \quad z \in U$$
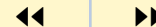
and this result is sharp.

Proof. Theorem 3.1 and the minimum principle imply that

$$\mathrm{Re}\, f(z) > 1 + 4 \inf_{t\in[0,2\pi]} \mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{e^{ikt} \cdot k}{(2k-1)^2}\right).$$

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It is easy to observe that

$$2\sum_{k=1}^{\infty}\frac{ke^{ikt}}{(2k-1)^2} = \sum_{n=1}^{\infty}\frac{e^{ikt}}{2k-1} + \sum_{n=1}^{\infty}\frac{e^{ikt}}{(2k-1)^2}$$

$$= \int_0^1 \sum_{k=1}^{\infty}(x^2)^{k-1}e^{ikt}dx + \int_0^1\int_0^1\sum_{k=1}^{\infty}(x^2y^2)^{k-1}e^{ikt}dxdy$$

$$(4.5)\qquad = \int_0^1\frac{e^{it}}{1-x^2e^{it}}dx + \int_0^1\int_0^1\frac{e^{it}}{1-x^2y^2e^{it}}dxdy, \quad t\in[0,2\pi).$$

Since

$$\operatorname{Re}\frac{e^{it}}{1-x^2e^{it}} \geq \frac{-1}{1+x^2}, \quad x\in[0,1],\ t\in[0,2\pi)$$

and

$$\operatorname{Re}\frac{e^{it}}{1-x^2y^2e^{it}} \geq \frac{-1}{1+x^2\cdot y^2}, \quad x,y\in[0,1],\ t\in[0,2\pi),$$

by integrating we get that

$$\operatorname{Re}\int_0^1\frac{e^{it}}{1-x^2e^{it}}dx \geq -\int_0^1\frac{1}{1+x^2}dx$$

and

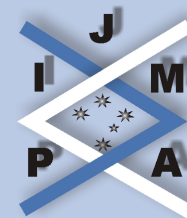$$\operatorname{Re}\int_0^1\frac{e^{it}}{1-x^2y^2e^{it}}dxdy \geq -\int_0^1\frac{1}{1+x^2y^2}dxdy.$$

In the derived inequalities, equality occurs if $t=\pi$, this means that

$$\inf_{t\in[0,2\pi)}\operatorname{Re}\sum_{k=1}^{\infty}\frac{ke^{ikt}}{(2k-1)^2} = \sum_{k=1}^{\infty}\frac{(-1)^k\cdot k}{(2k-1)^2}$$

and the inequality (4.4) holds true. ∎

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[3] P.T. MOCANU, TEODOR BULBOACĂ AND G. Şt. SĂLĂGEAN, *Toeria Geoemtrică a Funcţiilor Univalente*, Casa Cărţii de Ştiinţă1, Cluj-Napoca, 1991.