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ČEBYŠEV'S INEQUALITY ON TIME SCALES

CHEH-CHIH YEH, FU-HSIANG WONG, AND HORNG-JAAN LI

DEPARTMENT OF INFORMATION MANAGEMENT
LONG-HUA UNIVERSITY OF SCIENCE AND TECHNOLOGY
KUEISHAN TAOYUAN, 33306 TAIWAN
REPUBLIC OF CHINA.
CCYeh@mail.lhu.edu.tw

DEPARTMENT OF MATHEMATICS
NATIONAL TAIPEI TEACHER'S COLLEGE
134, HO-PING E. RD. SEC. 2
TAIPEI 10659, TAIWAN
REPUBLIC OF CHINA.
wong@tea.ntptc.edu.tw

GENERAL EDUCATION CENTER
CHIEN KUO INSTITUTE OF TECHNOLOGY
CHANG-HUA, 50050 TAIWAN
REPUBLIC OF CHINA.
hjli@ckit.edu.tw

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ABSTRACT. In this paper we establish some Čebyšev's inequalities on time scales under suitable conditions.

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1. INTRODUCTION

The purpose of this paper is to establish the well-known Čebyšev's inequality on time scales. To do this, we simply introduce the time scales calculus as follows:

In 1988, Hilger [7] introduced the time scales theory to unify continuous and discrete analysis. A time scale $\mathbb{T}$ is a closed subset of the set $\mathbb{R}$ of the real numbers. We assume that any time scale has the topology that it inherits from the standard topology on $\mathbb{R}$. Since a time scale may or may not be connected, we need the concept of jump operators.

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Definition 1.1. Let $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale. Then the two mappings

$$\sigma,\ \rho : \mathbb{T} \to \mathbb{R}$$

satisfying

$$\sigma(t) = \inf\{\gamma > t | \gamma \in \mathbb{T}\},$$
$$\rho(t) = \sup\{\gamma < t | \gamma \in \mathbb{T}\}$$

are called the jump operators on $\mathbb{T}$.

These jump operators classify the points $\{t\}$ of a time scale $\mathbb{T}$ as right-dense, right-scattered, left-dense and left-scattered according to whether $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ or $\rho(t) < t$, respectively, for $t \in \mathbb{T}$.

Let t be the maximum element of a time scale $\mathbb{T}$. If t is left-scattered, then t is called a generate point of $\mathbb{T}$. Let $\mathbb{T}^{\urcorner}$ denote the set of all non-degenerate points of $\mathbb{T}$. Throughout this paper, we suppose that

(a) $\mathbb{T}$ is a time scale;
(b) an interval means the intersection of a real interval with the given time scale;
(c) $\mathbb{R} = (-\infty, \infty)$.

Definition 1.2. Let $\mathbb{T}$ be a time scale. Then the mapping $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if

(a) f is continuous at each right-dense or maximal point of $\mathbb{T}$;
(b) $\lim_{s \to t^-} f(s) = f(t^-)$ exists for each left-dense point $t \in \mathbb{T}$.

Let $C_{rd}[\mathbb{T}, \mathbb{R}]$ denote the set of all rd-continuous mappings from $\mathbb{T}$ to $\mathbb{R}$.

Definition 1.3. Let $f : \mathbb{T} \to \mathbb{R},\ t \in \mathbb{T}^{\urcorner}$. Then we say that f has the (delta) derivative $f^{\Delta}(t) \in \mathbb{R}$ at t if for each $\epsilon > 0$ there exists a neighborhood U of t such that for all $s \in U$

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \epsilon \left| \sigma(t) - s \right| .$$

In this case, we say that f is (delta) differentiable at t.

Clearly, f^{Δ} is the usual derivative if $\mathbb{T} = \mathbb{R}$, and is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$ (the set of all integers).

Definition 1.4. A function $F : \mathbb{T} \to \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for each $t \in \mathbb{T}^{\urcorner}$. In this case, we define the (Cauchy) integral of f by

$$\int_s^t f(\gamma)\,\Delta\gamma \ = \ F(t) - F(s)$$

for all $s,\ t \in \mathbb{T}$.

It follows from Theorem 1.74 of Bohner and Peterson [3] that every rd-continuous function has an antiderivative. For further results on time scales calculus, we refer to [3, 9].

The purpose of this paper is to establish the well-known Čebyšev inequality [1, 5, 6, 8, 11] on time scales. For other related results, we refer to [4, 10, 12, 13].

2. MAIN RESULTS

We first establish some Čebyšev inequalities which generalize some results of Audréief [1], Beesack and Pečarić [2], Dunkel [4], Fujimara [5, 6], Isayama [8], and Winckler [13]. For other related results, we refer to the book of Mitrinovič [10].

Theorem 2.1. *Suppose that* $p \in C_{rd}([a,b];[0,\infty))$. *Let* $f_1, f_2, k_1, k_2 \in C_{rd}([a,b];\mathbb{R})$ *satisfy the following two conditions:*
(C_1) $f_2(x)k_2(x) > 0$ *on* $[a,b]$*;*
(C_2) $\frac{f_1(x)}{f_2(x)}$ *and* $\frac{k_1(x)}{k_2(x)}$ *are similarly ordered (or oppositely ordered), that is, for all* $x, y \in [a,b]$,

$$\left(\frac{f_1(x)}{f_2(x)} - \frac{f_1(y)}{f_2(y)}\right)\left(\frac{k_1(x)}{k_2(x)} - \frac{k_1(y)}{k_2(y)}\right) \geq 0 \quad (\textit{or} \leq 0),$$

then

$$\frac{1}{2!}\int_a^b\int_a^b p(x)p(y)\begin{vmatrix} f_1(x) & f_1(y) \\ f_2(x) & f_2(y) \end{vmatrix}\begin{vmatrix} k_1(x) & k_1(y) \\ k_2(x) & k_2(y) \end{vmatrix}\Delta x\Delta y = \begin{vmatrix} \int_a^b p(x)f_1(x)k_1(x)\Delta x & \int_a^b p(x)f_1(x)k_2(x)\Delta x \\ \int_a^b p(x)f_2(x)k_1(x)\Delta x & \int_a^b p(x)f_2(x)k_2(x)\Delta x \end{vmatrix} \geq 0 \ (\textit{or} \ \leq 0) \tag{2.1}$$

Proof. Let $x, y \in [a,b]$. Then it follows from (C_1), (C_2) and the identity

$$p(x)p(y)\begin{vmatrix} f_1(x) & f_1(y) \\ f_2(x) & f_2(y) \end{vmatrix}\begin{vmatrix} k_1(x) & k_1(y) \\ k_2(x) & k_2(y) \end{vmatrix} = p(x)p(y)f_2(x)f_2(y)k_2(x)k_2(y)\left(\frac{f_1(x)}{f_2(x)} - \frac{f_1(y)}{f_2(y)}\right)\left(\frac{k_1(x)}{k_2(x)} - \frac{k_1(y)}{k_2(y)}\right)$$

that (2.1) holds. □

Remark 2.2. Suppose that $p, f, g \in C_{rd}([a,b];\mathbb{R})$ with $p(x) \geq 0$ on $[a,b]$. Let f and g be similarly ordered (or oppositely ordered). Taking $f_1(x) = f(x)$, $k_1(x) = g(x)$ and $f_2(x) = k_2(x) = 1$, (2.1) is reduced to the generalized Čebyšev inequality:

$$\int_a^b p(x)f(x)g(x)\Delta x\int_a^b p(x)\Delta x \geq \text{ (or } \leq) \int_a^b p(x)f(x)\Delta x\int_a^b p(x)g(x)\Delta x, \tag{2.2}$$

which generalizes a Winckler's result in [13] if $a = 0$ and $b = x$. Let $\mathbb{T} = \mathbb{Z}$, if $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are similarly ordered (or oppositely ordered), and if $p = (p_1, p_2, \ldots, p_n)$ is a nonnegative sequence, then (2.2) is reduced to

$$\sum_{i=1}^n p_i\sum_{i=1}^n p_ia_ib_i \geq \text{ (or } \leq) \sum_{i=1}^n p_ia_i\sum_{i=1}^n p_ib_i.$$

If $\mathbb{T} = \mathbb{R}$, then (2.2) is reduced to

$$\int_a^b p(x)f(x)g(x)\,dx\int_a^b p(x)\,dx \geq \text{ (or } \leq) \int_a^b p(x)f(x)\,dx\int_a^b p(x)g(x)\,dx.$$

Remark 2.3. Taking $f(x) = \frac{f_1(x)}{f_2(x)}$, $g(x) = \frac{g_1(x)}{g_2(x)}$ and $p(x) = f_2(x)g_2(x)$, inequality (2.2) is reduced to

$$\int_a^b f_1(x)g_1(x)\Delta x\int_a^b f_2(x)g_2(x)\Delta x \geq \text{ (or } \leq) \int_a^b f_1(x)g_2\Delta x\int_a^b f_2(x)g_1\Delta x, \tag{2.3}$$

if $f_2(x)g_2(x) \geq 0$ on $[a,b]$, $\frac{f_1(x)}{f_2(x)}$ and $\frac{g_1(x)}{g_2(x)}$ are both increasing or both decreasing (or one of the functions $\frac{f_1(x)}{f_2(x)}$ or $\frac{g_1(x)}{g_2(x)}$ is nonincreasing and the other nondecreasing). Here f_1, f_2, g_1, $g_2 \in C_{rd}([a,b],\mathbb{R})$ with $f_2(x)g_2(x) \neq 0$ on $[a,b]$. Conversely, if $f_1(x) = f(x)f_2(x)$, $g_1(x) = g(x)g_2(x)$ and $p(x) = f_2(x)g_2(x)$, then inequality (2.3) is reduced to inequality (2.2).

Theorem 2.4. *Let $f \in C_{rd}([a,b],[0,\infty))$ be decreasing (or increasing) with $\int_a^b xp(x)f(x)\Delta x > 0$ and $\int_a^b p(x)f(x)\Delta x > 0$. Then*

$$\frac{\int_a^b xp(x)f^2(x)\Delta x}{\int_a^b xp(x)f(x)\Delta x} \leq (\geq) \frac{\int_a^b p(x)f^2(x)\Delta x}{\int_a^b p(x)f(x)\Delta x}.$$

Proof. Clearly, for any $x, y \in [a,b]$,

$$\int_a^b \int_a^b f(x)f(y)p(x)p(y)(y-x)(f(x)-f(y)\Delta x \Delta y \geq (\leq) 0,$$

which implies that the desired result holds. □

Remark 2.5. Let $f \in C_{rd}([a,b],(0,\infty))$ and n be a positive integer. If p and g are replaced by $\frac{p}{f}$ and f^n respectively, then Čebyšev's inequality (2.2) is reduced to

$$\int_a^b p(x)f^n(x)\Delta x \int_a^b \frac{p(x)}{f(x)}\Delta x \geq \int_a^b p(x)\Delta x \int_a^b p(x)[f(x)]^{n-1}\Delta x,$$

which implies

$$\begin{aligned}\int_a^b p(x)f^n(x)\Delta x \left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^2 &\geq \int_a^b p(x)\Delta x \int_a^b p(x)[f(x)]^{n-1}\Delta x \int_a^b \frac{p(x)}{f(x)}\Delta x \\ &\geq \left(\int_a^b p(x)\Delta x\right)^2 \int_a^b p(x)[f(x)]^{n-2}\Delta x.\end{aligned}$$

provided f and f^n are similarly ordered. Continuing in this way, we get

$$\left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^n \int_a^b p(x)[f(x)]^n \Delta x \geq \left(\int_a^b p(x)\Delta x\right)^{n+1},$$

which extends a result in Dunkel [4].

Remark 2.6. Let $\nu, p \in C_{rd}([a,b],[0,\infty))$. If f and g are similarly ordered (or oppositely ordered), then it follows from Remark 2.2 that

$$\int_a^b p(t)f(\nu(t))g(\nu(t))\Delta t \int_a^b p(t)\Delta t \geq \text{ (or } \leq) \int_a^b p(t)f(\nu(t))\Delta t \int_a^b p(t)g(\nu(t))\Delta t,$$

which is a generalization of a result given in Stein [12].

Remark 2.7. Let $p, f_i \in C_{rd}([a,b],\mathbb{R})$ for each $i = 1, 2, \ldots, n$. Suppose that $f_1, f_2, \ldots, f_n$ are similarly ordered and $p(x) \geq 0$ on $[a,b]$, then it follows from Remark 2.2 that

$$\begin{aligned}&\left(\int_a^b p(x)\Delta x\right)^{n-1} \left(\int_a^b p(x)f_1(x)f_2(x)\cdots f_n(x)\Delta x\right) \\ &\qquad = \left(\int_a^b p(x)\Delta x\right)^{n-2} \left(\int_a^b p(x)\Delta x\right) \left(\int_a^b p(x)f_1(x)f_2(x)\cdots f_n(x)\Delta x\right)\end{aligned}$$

$$\begin{aligned}
&\geq \left(\int_a^b p(x)\Delta x\right)^{n-2}\left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)f_2(x)\cdots f_n(x)\Delta x\right)\\
&\geq \left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)\Delta x\right)^{n-3}\\
&\qquad\times\left(\int_a^b p(x)f_2(x)\Delta x\right)\left(\int_a^b p(x)f_3(x)\cdots f_n(x)\Delta x\right)\\
&\geq \cdots\\
&\geq \left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)f_2(x)\Delta x\right)\cdots\left(\int_a^b p(x)f_n(x)\Delta x\right),
\end{aligned}$$

which is a generalization of a result in Dunkel [4].

In particular, if $f_1(x) = f_2(x) = \cdots = f_n(x) = f(x)$, then

$$\left(\int_a^b p(x)\Delta x\right)^{n-1}\left(\int_a^b p(x)f^n(x)\Delta x\right) \geq \left(\int_a^b p(x)f(x)\Delta x\right)^n.$$

Theorem 2.8. *If $p(x), f(x) \in C_{rd}([a,b],[0,\infty))$ with $f(x) > 0$ on $[a,b]$ and n is a positive integer, then*

$$\left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^n\left(\int_a^b p(x)f^n(x)\Delta x\right) \geq \left(\int_a^b p(x)\Delta x\right)^n.$$

Proof. It follows from $f(x) > 0$ on $[a,b]$ that $f^n(x)$ and $\frac{1}{f(x)}$ are oppositely ordered on $[a,b]$. Hence by (2.2),

$$\begin{aligned}
&\int_a^b p(x)f^n(x)\Delta x\left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^n\\
&\qquad\geq \int_a^b p(x)\Delta x\left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^{n-1}\int_a^b p(x)f^{n-1}(x)\Delta x\\
&\qquad\geq \left(\int_a^b p(x)\Delta x\right)^2\left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^{n-2}\int_a^b p(x)f^{n-2}(x)\Delta x\\
&\qquad\geq \cdots\\
&\qquad\geq \left(\int_a^b p(x)\Delta x\right)^n.
\end{aligned}$$

□

Theorem 2.9. *Let $g_1, g_2, \ldots, g_n \in C_{rd}([a,b],\Re)$ and $p, h_1, h_2, \ldots, h_{n-1} \in C_{rd}([a,b],(0,\infty))$ with $g_n(x) > 0$ on $[a,b]$. If*

$$\frac{g_1(x)g_2(x)\cdots g_{n-1}(x)}{h_1(x)h_2(x)\cdots h_{n-1}(x)} \quad \textit{and} \quad \frac{h_{n-1}(x)}{g_n(x)}$$

are similarly ordered (or oppositely ordered), then

$$\begin{aligned}
(2.4)\quad &\int_a^b p(x)g_n(x)\Delta x\int_a^b \frac{p(x)g_1(x)g_2(x)\cdots g_{n-1}(x)}{h_1(x)h_2(x)\cdots h_{n-2}(x)}\Delta x\\
&\qquad\geq \text{ (or } \leq) \int_a^b p(x)h_{n-1}(x)\Delta x\int_a^b \frac{p(x)g_1(x)g_2(x)\cdots g_n(x)}{h_1(x)h_2(x)\cdots h_{n-1}(x)}\Delta x.
\end{aligned}$$

Proof. Taking

$$f_1(x) = \frac{g_1(x)g_2(x)\cdots g_{n-1}(x)}{h_1(x)h_2(x)\cdots h_{n-1}(x)}, \quad k_1(x) = h_{n-1}(x),$$
$$f_2(x) = 1 \quad \text{and} \quad k_2(x) = g_n(x)$$

in Theorem 2.1, (2.1) is reduced to our desired result (2.4). □

The following theorem is a time scales version of Theorem 1 in Beesack and Pečarić [2].

Theorem 2.10. *Let*

$$f_1, f_2, \dots, f_n \in C_{rd}([a,b],[0,\infty) \quad \textit{and} \quad g_1, g_2, \dots, g_n \in C_{rd}([a,b],(0,\infty)).$$

If the functions $f_1, \frac{f_2}{g_1}, \dots, \frac{f_n}{g_{n-1}}$ *are similarly ordered and for each pair* $\frac{f_k}{g_{k-1}}, g_{k-1}$ *is oppositely ordered for* $k = 2, 3, \dots, n$*, then*

$$\int_a^b p(x)f_1(x)\frac{f_2(x)f_3(x)\cdots f_n(x)}{g_1(x)g_2(x)\cdots g_{n-1(x)}}\,\Delta x \geq \frac{\int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x}{\int_a^b p(x)g_1(x)\Delta x \int_a^b p(x)g_2(x)\Delta x \cdots \int_a^b p(x)g_{n-1}(x)\Delta x}. \tag{2.5}$$

Proof. Let $f_1, f_2, \dots, f_n$ be replaced by $f_1, \frac{f_2}{g_1}, \dots, \frac{f_n}{g_{n-1}}$ in Remark 2.7, we obtain

$$\left(\int_a^b p(x)\Delta x\right)^{n-1} \int_a^b p(x)f_1(x)\frac{f_2(x)f_3(x)\cdots f_n(x)}{g_1(x)g_2(x)\cdots g_{n-1}(x)}\Delta x \geq \int_a^b p(x)f_1(x)\Delta x \prod_{k=2}^{n}\int_a^b p(x)\frac{f_k(x)}{g_{k-1}(x)}\Delta x. \tag{2.6}$$

Also, since $\frac{f_k}{g_{k-1}}$ and g_{k-1} are oppositely ordered, it follows from Remark 2.2 that

$$\int_a^b p(x)\Delta x \int_a^b p(x)f_k(x)\Delta x \leq \int_a^b p(x)g_{k-1}(x)\Delta x \int_a^b p(x)\frac{f_k(x)}{g_{k-1}(x)}\Delta x.$$

Thus

$$\int_a^b \frac{p(x)f_k(x)}{g_{k-1}(x)}\Delta x \geq \frac{\int_a^b p(x)\Delta x \int_a^b p(x)f_k(x)\Delta x}{\int_a^b p(x)g_{k-1}(x)\Delta x}.$$

This and (2.6) imply (2.5) holds. □

3. MORE RESULTS

In this section, we generalize some results in Isayama [8].

Theorem 3.1. *Let* $f_1, f_2, \dots, f_n \in C_{rd}([a,b],(0,\infty))$, $k_1, k_2, \dots, k_{n-1} \in C_{rd}([a,b],\mathbb{R})$ *and* $p(x) \in C_{rd}([a,b],[0,\infty))$. *If*

$$\frac{f_1(x)f_2(x)\cdots f_{i-1}(x)}{k_1(x)k_2(x)\cdots k_{i-1}(x)} \quad \textit{and} \quad \frac{k_{i-1}(x)}{f_i(x)}$$

are similarly ordered (or oppositely ordered) for $i = 2, \dots, n$*, then*

$$\begin{aligned} \text{(3.1)} \quad & \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x \\ & \geq (\text{or } \leq) \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)k_2(x)\Delta x \cdots \\ & \cdots \int_a^b p(x)k_{n-1}(x)\Delta x \int_a^b p(x)\frac{f_1(x)f_2(x)\cdots f_n(x)}{k_1(x)k_2(x)\cdots k_{n-1}(x)}\Delta x. \end{aligned}$$

Proof. If $f_1(x), k_1(x), f_2(x)$ and $k_2(x)$ are replaced by $f_1(x), 1, k_1(x)$ and $\frac{f_2(x)}{k_1(x)}$ in Theorem 2.1, then we obtain

$$\int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \geq (\text{or } \leq) \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)\frac{f_1(x)f_2(x)}{k_1(x)}\Delta x.$$

Thus the theorem holds for $n = 2$.

Suppose that the theorem holds for $n - 1$, that is

$$\begin{aligned} \text{(3.2)} \quad & \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_{n-1}(x)\Delta x \\ & \geq (\text{or } \leq) \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)k_2(x)\Delta x \\ & \cdots \int_a^b p(x)k_{n-2}(x)\Delta x \int_a^b p(x)\frac{f_1(x)f_2(x)\cdots f_{n-1}(x)}{k_1(x)k_2(x)\cdots k_{n-2}(x)}\Delta x \end{aligned}$$

if

$$\frac{f_1(x)f_2(x)\cdots f_{i-1}(x)}{k_1(x)k_2(x)\cdots k_{i-1}(x)} \quad \text{and} \quad \frac{k_{i-1}(x)}{f_i(x)}$$

are similarly ordered (or oppositely ordered) for $i = 2, 3, \dots, n-1$. Multiplying the both sides of (3.2) by $\int_a^b p(x)f_n(x)\Delta x$, we see that

$$\begin{aligned} \text{(3.3)} \quad & \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_{n-1}(x)\Delta x \int_a^b p(x)f_n(x)\Delta x \\ & \geq (\text{or } \leq) \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)k_2(x)\Delta x \\ & \cdots \int_a^b p(x)k_{n-2}(x)\Delta x \int_a^b p(x)\frac{f_1(x)f_2(x)\cdots f_{n-1}(x)}{k_1(x)k_2(x)\cdots k_{n-2}(x)}\Delta x \int_a^b p(x)f_n(x)\Delta x. \end{aligned}$$

It follows from Theorem 2.10 that

$$\begin{aligned} & \int_a^b p(x)\frac{f_1(x)f_2(x)\cdots f_{n-1}(x)}{k_1(x)k_2(x)\cdots k_{n-2}(x)}\Delta x \int_a^b p(x)f_n(x)\Delta x \\ & \geq (\text{or } \leq) \int_a^b p(x)\frac{f_1(x)f_2(x)\cdots f_n(x)}{k_1(x)k_2(x)\cdots k_{n-1}(x)}\Delta x \int_a^b p(x)k_{n-1}(x)\Delta x. \end{aligned}$$

This and (3.3) imply

$$\int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x$$
$$\geq (\text{or } \leq) \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)k_2(x)\Delta x$$
$$\cdots \int_a^b p(x)k_{n-1}(x)\Delta x \int_a^b p(x)\frac{f_1(x)f_2(x)\cdots f_n(x)}{k_1(x)k_2(x)\cdots k_{n-1}(x)}\Delta x.$$

By induction, we complete the proof. □

Remark 3.2. Let $k_n \in C_{rd}([a,b],\mathbb{R})$. If $f_1(x), f_2(x), \dots, f_n(x)$, $k_1(x), k_2(x), \dots, k_{n-1}(x)$ are replaced by

$$f_1(x)f_2(x)\cdots f_n(x), k_1(x)k_2(x)\cdots k_n(x), \dots, k_1(x)k_2(x)\cdots k_n(x),$$
$$f_1(x)k_2(x)\cdots k_n(x), k_1(x)f_2(x)k_3(x)\cdots k_n(x), \dots, k_1(x)k_2(x)\cdots k_{n-2}(x)f_{n-1}(x)k_n(x)$$

in Theorem 3.1, respectively, then

$$\int_a^b p(x)f_1(x)f_2(x)\cdots f_n(x)\Delta x \left(\int_a^b p(x)k_1(x)k_2(x)\cdots k_n(x)\Delta x\right)^{n-1}$$
$$\geq \int_a^b p(x)f_1(x)k_2(x)\cdots k_n(x)\Delta x \int_a^b p(x)k_1(x)f_2(x)k_3(x)\cdots k_n(x)\Delta x$$
$$\cdots \int_a^b p(x)k_1(x)k_2(x)\cdots k_{n-1}(x)f_n(x)\Delta x \tag{3.4}$$

if $\frac{f_i(x)}{k_i(x)} > 0$ for $i = 1, 2, \dots, n$ and $k_1(x)k_2(x)\cdots k_n(x) > 0$ on $[a,b]$.

Remark 3.3. Letting $f_1(x) = f_2(x) = \cdots = f_n(x) = f(x)$ and $k_1(x) = k_2(x) = \cdots = k_n(x) = k^{\frac{1}{n-1}}(x)$ in (3.4) with $k(x) > 0$ on $[a,b]$, we obtain the Hölder inequality:

$$\int_a^b p(x)f^n(x)\Delta x \left(\int_a^b p(x)k^{\frac{n}{n-1}}(x)\Delta x\right)^{n-1} \geq \left(\int_a^b p(x)f(x)k(x)\Delta x\right)^n. \tag{3.5}$$

Remark 3.4. Let $p, f, g \in C_{rd}([a,b],[0,\infty))$. Taking

$$f_1(x) = f^n(x)g(x),$$
$$f_2(x) = f_3(x) = \cdots = f_n(x) = g(x) \quad \text{and}$$
$$k_1(x) = k_2(x) = \cdots = k_{n-1}(x) = f(x)g(x),$$

(3.1) is reduced to Jensen's inequality:

$$\int_a^b p(x)f^n(x)g(x)\Delta x \left(\int_a^b p(x)g(x)\Delta x\right)^{n-1} \geq \left(\int_a^b p(x)f(x)g(x)\Delta x\right)^n. \tag{3.6}$$

Remark 3.5. Taking $k_1(x) = k_2(x) = \cdots = k_{n-1}(x) = (f_1(x)f_2(x)\cdots f_n(x))^{\frac{1}{n}}$, (3.1) is reduced to

$$\int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x$$
$$\geq \left[\int_a^b p(x)\left(f_1(x)f_2(x)\cdots f_n(x)\right)^{\frac{1}{n}}\Delta x\right]^n \tag{3.7}$$

if $f_i(x) > 0$ on $[a, b]$ for each $i = 1, 2, \ldots, n$ and $\frac{1}{f_i(x)}[f_1(x)f_2(x)\cdots f_n(x)]^{\frac{1}{n}}$ $(i = 1, 2, \ldots, n)$ are similarly ordered.

Remark 3.6 (see also Remark 2.7)**.** Taking $k_1(x) = k_2(x) = \cdots = k_{n-1}(x) = 1$, (3.1) is reduced to Čebyšev's inequality:

$$\int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x \leq \left(\int_a^b p(x)\Delta x\right)^{n-1} \int_a^b p(x)f_1(x)f_2(x)\cdots f_n(x)\Delta x \tag{3.8}$$

if $f_i(x)$ $(i = 1, 2, \ldots, n)$ are similarly ordered and $f_i(x) \geq 0$ $(i = 1, 2, \ldots, n)$.

Remark 3.7. Taking $f_1(x) = f_2(x) = \cdots = f_n(x) = 1$, then (3.1) is reduced to

$$\left(\int_a^b p(x)\Delta x\right)^n \leq \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)k_2(x)\Delta x \cdots \int_a^b p(x)k_{n-1}(x)\Delta x \int_a^b \frac{p(x)}{k_1(x)k_2(x)\cdots k_{n-1}(x)}\Delta x$$

if $k_i(x) > 0$ are similarly ordered for $i = 1, 2, \ldots, n-1$. Thus, if $f_1(x), \ldots, f_n(x)$ are similarly ordered and $f_i(x) > 0$ on $[a, b]$ $(i = 1, 2, \ldots, n)$, then

$$\frac{\left(\int_a^b p(x)\Delta x\right)^{n+1}}{\int_a^b \frac{p(x)}{f_1(x)f_2(x)\cdots f_n(x)}\Delta x} \leq \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\frac{\left(\int_a^b p(x)\Delta x\right)^{n+1}}{\int_a^b \frac{p(x)}{f_1(x)f_2(x)\cdots f_n(x)}\Delta x} \leq \left(\int_a^b p(x)\Delta x\right)^{n-1} \int_a^b p(x)f_1(x)f_2(x)\cdots f_n(x)\Delta x$$

if $f_1(x), \ldots, f_n(x)$ are similarly ordered.

Remark 3.8. Let $k_1(x) = k_2(x) = \cdots = k_n(x) = 1$. If $f_i(x)$ is replaced by

$$\frac{[f_1(x)f_2(x)\cdots f_n(x)]^{\frac{1}{n}}}{f_i(x)}, \quad n = 1, 2, \ldots, n,$$

then (3.1) is reduced to

$$\prod_{i=1}^n \int_a^b p(x)\frac{\sqrt[n]{f_1(x)f_2(x)\cdots f_n(x)}}{f_i(x)}\Delta x \leq \left(\int_a^b p(x)\Delta x\right)^n$$

if $\frac{\sqrt[n]{f_1(x)f_2(x)\cdots f_n(x)}}{f_i(x)}$ $(i = 1, 2, \ldots, n)$ are similarly ordered.

Remark 3.9. Let $f_1, f_2, \ldots, f_n$; $k_1, k_2, \ldots, k_{n-1}$ be replaced by $f_1f_2, f_3f_4, \ldots, f_{2n-1}f_{2n}$; $f_2f_3, f_4f_5, \ldots, f_{2n-2}f_{2n-1}$, respectively. Then (3.1) is reduced to

$$\int_a^b p(x)f_1(x)f_2(x)\Delta x \int_a^b p(x)f_3(x)f_4(x)\Delta x \cdots \int_a^b p(x)f_{2n-1}(x)f_{2n}(x)\Delta x \geq \int_a^b p(x)f_2(x)f_3(x)\Delta x \int_a^b p(x)f_4(x)f_5(x)\Delta x \cdots \int_a^b p(x)f_{2n-2}(x)f_{2n-1}(x)\Delta x$$

if $\frac{f_i(x)}{f_{i+1}(x)}$ $(i = 1, 2, \ldots, 2n-1)$ are similarly ordered.

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