

ČEBYŠEV'S INEQUALITY ON TIME SCALES

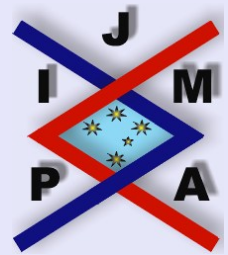
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Abstract

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Abstract

In this paper we establish some Čebyšev's inequalities on time scales under suitable conditions.

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1. Introduction

The purpose of this paper is to establish the well-known Čebyšev's inequality on time scales. To do this, we simply introduce the time scales calculus as follows:

In 1988, Hilger [7] introduced the time scales theory to unify continuous and discrete analysis. A time scale \mathbb{T} is a closed subset of the set \mathbb{R} of the real numbers. We assume that any time scale has the topology that it inherits from the standard topology on \mathbb{R} . Since a time scale may or may not be connected, we need the concept of jump operators.

Definition 1.1. Let $t \in \mathbb{T}$, where \mathbb{T} is a time scale. Then the two mappings

$$\sigma, \rho : \mathbb{T} \rightarrow \mathbb{R}$$

satisfying

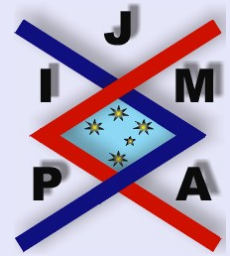
$$\begin{aligned}\sigma(t) &= \inf\{\gamma > t \mid \gamma \in \mathbb{T}\}, \\ \rho(t) &= \sup\{\gamma < t \mid \gamma \in \mathbb{T}\}\end{aligned}$$

are called the jump operators on \mathbb{T} .

These jump operators classify the points $\{t\}$ of a time scale \mathbb{T} as right-dense, right-scattered, left-dense and left-scattered according to whether $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ or $\rho(t) < t$, respectively, for $t \in \mathbb{T}$.

Let t be the maximum element of a time scale \mathbb{T} . If t is left-scattered, then t is called a generate point of \mathbb{T} . Let \mathbb{T}^∇ denote the set of all non-degenerate points of \mathbb{T} . Throughout this paper, we suppose that

- (a) \mathbb{T} is a time scale;



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- (b) an interval means the intersection of a real interval with the given time scale;
- (c) $\mathbb{R} = (-\infty, \infty)$.

Definition 1.2. Let \mathbb{T} be a time scale. Then the mapping $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous if

- (a) f is continuous at each right-dense or maximal point of \mathbb{T} ;
- (b) $\lim_{s \rightarrow t^-} f(s) = f(t^-)$ exists for each left-dense point $t \in \mathbb{T}$.

Let $C_{rd}[\mathbb{T}, \mathbb{R}]$ denote the set of all rd-continuous mappings from \mathbb{T} to \mathbb{R} .

Definition 1.3. Let $f : \mathbb{T} \rightarrow \mathbb{R}$, $t \in \mathbb{T}^\nabla$. Then we say that f has the (delta) derivative $f^\Delta(t) \in \mathbb{R}$ at t if for each $\epsilon > 0$ there exists a neighborhood U of t such that for all $s \in U$

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s|.$$

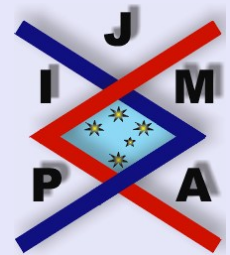
In this case, we say that f is (delta) differentiable at t .

Clearly, f^Δ is the usual derivative if $\mathbb{T} = \mathbb{R}$, and is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$ (the set of all integers).

Definition 1.4. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ if $F^\Delta(t) = f(t)$ for each $t \in \mathbb{T}^\nabla$. In this case, we define the (Cauchy) integral of f by

$$\int_s^t f(\gamma) \Delta\gamma = F(t) - F(s)$$

for all $s, t \in \mathbb{T}$.



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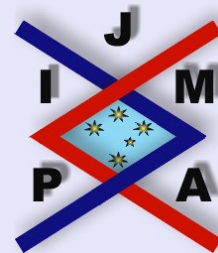
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It follows from Theorem 1.74 of Bohner and Peterson [3] that every rd-continuous function has an antiderivative. For further results on time scales calculus, we refer to [3, 9].

The purpose of this paper is to establish the well-known Čebyšev inequality [1, 5, 6, 8, 11] on time scales. For other related results, we refer to [4, 10, 12, 13].



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2. Main Results

We first establish some Čebyšev inequalities which generalize some results of Audréief [1], Beesack and Pečarić [2], Dunkel [4], Fujimara [5, 6], Isayama [8], and Winckler [13]. For other related results, we refer to the book of Mitrinović [10].

Theorem 2.1. *Suppose that $p \in C_{rd}([a, b]; [0, \infty))$. Let $f_1, f_2, k_1, k_2 \in C_{rd}([a, b]; \mathbb{R})$ satisfy the following two conditions:*

$$(C_1) \quad f_2(x)k_2(x) > 0 \text{ on } [a, b];$$

(C₂) $\frac{f_1(x)}{f_2(x)}$ and $\frac{k_1(x)}{k_2(x)}$ are similarly ordered (or oppositely ordered), that is, for all $x, y \in [a, b]$,

$$\left(\frac{f_1(x)}{f_2(x)} - \frac{f_1(y)}{f_2(y)} \right) \left(\frac{k_1(x)}{k_2(x)} - \frac{k_1(y)}{k_2(y)} \right) \geq 0 \quad (\text{or } \leq 0),$$

then

$$(2.1) \quad \frac{1}{2!} \int_a^b \int_a^b p(x)p(y) \left| \begin{array}{cc} f_1(x) & f_1(y) \\ f_2(x) & f_2(y) \end{array} \right| \left| \begin{array}{cc} k_1(x) & k_1(y) \\ k_2(x) & k_2(y) \end{array} \right| \Delta x \Delta y$$

$$= \left| \begin{array}{cc} \int_a^b p(x)f_1(x)k_1(x)\Delta x & \int_a^b p(x)f_1(x)k_2(x)\Delta x \\ \int_a^b p(x)f_2(x)k_1(x)\Delta x & \int_a^b p(x)f_2(x)k_2(x)\Delta x \end{array} \right| \geq 0 \quad (\text{or } \leq 0)$$



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Proof. Let $x, y \in [a, b]$. Then it follows from (C_1) , (C_2) and the identity

$$p(x)p(y) \begin{vmatrix} f_1(x) & f_1(y) \\ f_2(x) & f_2(y) \end{vmatrix} \begin{vmatrix} k_1(x) & k_1(y) \\ k_2(x) & k_2(y) \end{vmatrix} \\ = p(x)p(y)f_2(x)f_2(y)k_2(x)k_2(y) \left(\frac{f_1(x)}{f_2(x)} - \frac{f_1(y)}{f_2(y)} \right) \left(\frac{k_1(x)}{k_2(x)} - \frac{k_1(y)}{k_2(y)} \right)$$

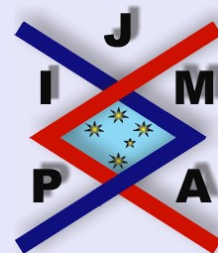
that (2.1) holds. \square

Remark 1. Suppose that $p, f, g \in C_{rad}([a, b]; \mathbb{R})$ with $p(x) \geq 0$ on $[a, b]$. Let f and g be similarly ordered (or oppositely ordered). Taking $f_1(x) = f(x)$, $k_1(x) = g(x)$ and $f_2(x) = k_2(x) = 1$, (2.1) is reduced to the generalized Čebyšev inequality:

$$(2.2) \quad \int_a^b p(x)f(x)g(x)\Delta x \int_a^b p(x)\Delta x \\ \geq (\text{or } \leq) \int_a^b p(x)f(x)\Delta x \int_a^b p(x)g(x)\Delta x,$$

which generalizes a Winckler's result in [13] if $a = 0$ and $b = x$. Let $\mathbb{T} = \mathbb{Z}$, if $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ are similarly ordered (or oppositely ordered), and if $p = (p_1, p_2, \dots, p_n)$ is a nonnegative sequence, then (2.2) is reduced to

$$\sum_{i=1}^n p_i \sum_{i=1}^n p_i a_i b_i \geq (\text{or } \leq) \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i.$$



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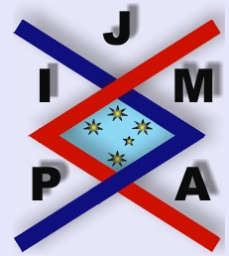


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If $\mathbb{T} = \mathbb{R}$, then (2.2) is reduced to

$$\int_a^b p(x)f(x)g(x) dx \int_a^b p(x) dx \geq (\text{or } \leq) \int_a^b p(x)f(x) dx \int_a^b p(x)g(x) dx.$$

Remark 2. Taking $f(x) = \frac{f_1(x)}{f_2(x)}$, $g(x) = \frac{g_1(x)}{g_2(x)}$ and $p(x) = f_2(x)g_2(x)$, inequality (2.2) is reduced to

$$(2.3) \quad \int_a^b f_1(x)g_1(x)\Delta x \int_a^b f_2(x)g_2(x)\Delta x \geq (\text{or } \leq) \int_a^b f_1(x)g_2\Delta x \int_a^b f_2(x)g_1\Delta x,$$

if $f_2(x)g_2(x) \geq 0$ on $[a, b]$, $\frac{f_1(x)}{f_2(x)}$ and $\frac{g_1(x)}{g_2(x)}$ are both increasing or both decreasing (or one of the functions $\frac{f_1(x)}{f_2(x)}$ or $\frac{g_1(x)}{g_2(x)}$ is nonincreasing and the other non-decreasing). Here $f_1, f_2, g_1, g_2 \in C_{rd}([a, b], \mathbb{R})$ with $f_2(x)g_2(x) \neq 0$ on $[a, b]$. Conversely, if $f_1(x) = f(x)f_2(x)$, $g_1(x) = g(x)g_2(x)$ and $p(x) = f_2(x)g_2(x)$, then inequality (2.3) is reduced to inequality (2.2).

Theorem 2.2. Let $f \in C_{rd}([a, b], [0, \infty))$ be decreasing (or increasing) with $\int_a^b xp(x)f(x)\Delta x > 0$ and $\int_a^b p(x)f(x)\Delta x > 0$. Then

$$\frac{\int_a^b xp(x)f^2(x)\Delta x}{\int_a^b xp(x)f(x)\Delta x} \leq (\geq) \frac{\int_a^b p(x)f^2(x)\Delta x}{\int_a^b p(x)f(x)\Delta x}.$$

Proof. Clearly, for any $x, y \in [a, b]$,

$$\int_a^b \int_a^b f(x)f(y)p(x)p(y)(y-x)(f(x) - f(y))\Delta x\Delta y \geq (\leq) 0,$$

which implies that the desired result holds. \square

Remark 3. Let $f \in C_{rd}([a, b], (0, \infty))$ and n be a positive integer. If p and g are replaced by $\frac{p}{f}$ and f^n respectively, then Čebyšev's inequality (2.2) is reduced to

$$\int_a^b p(x)f^n(x)\Delta x \int_a^b \frac{p(x)}{f(x)}\Delta x \geq \int_a^b p(x)\Delta x \int_a^b p(x)[f(x)]^{n-1}\Delta x,$$

which implies

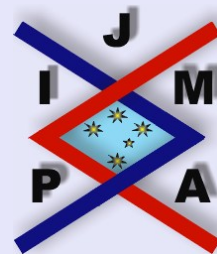
$$\begin{aligned} & \int_a^b p(x)f^n(x)\Delta x \left(\int_a^b \frac{p(x)}{f(x)}\Delta x \right)^2 \\ & \geq \int_a^b p(x)\Delta x \int_a^b p(x)[f(x)]^{n-1}\Delta x \int_a^b \frac{p(x)}{f(x)}\Delta x \\ & \geq \left(\int_a^b p(x)\Delta x \right)^2 \int_a^b p(x)[f(x)]^{n-2}\Delta x. \end{aligned}$$

provided f and f^n are similarly ordered. Continuing in this way, we get

$$\left(\int_a^b \frac{p(x)}{f(x)}\Delta x \right)^n \int_a^b p(x)[f(x)]^n\Delta x \geq \left(\int_a^b p(x)\Delta x \right)^{n+1},$$

which extends a result in Dunkel [4].

Remark 4. Let $\nu, p \in C_{rd}([a, b], [0, \infty))$. If f and g are similarly ordered (or



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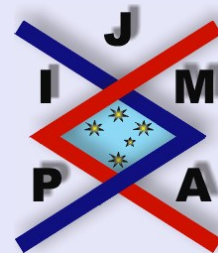
oppositely ordered), then it follows from Remark 1 that

$$\int_a^b p(t)f(\nu(t))g(\nu(t))\Delta t \int_a^b p(t)\Delta t \geq (\text{or } \leq) \int_a^b p(t)f(\nu(t))\Delta t \int_a^b p(t)g(\nu(t))\Delta t,$$

which is a generalization of a result given in Stein [12].

Remark 5. Let $p, f_i \in C_{rd}([a, b], \mathbb{R})$ for each $i = 1, 2, \dots, n$. Suppose that f_1, f_2, \dots, f_n are similarly ordered and $p(x) \geq 0$ on $[a, b]$, then it follows from Remark 1 that

$$\begin{aligned} & \left(\int_a^b p(x)\Delta x \right)^{n-1} \left(\int_a^b p(x)f_1(x)f_2(x)\cdots f_n(x)\Delta x \right) \\ &= \left(\int_a^b p(x)\Delta x \right)^{n-2} \left(\int_a^b p(x)\Delta x \right) \left(\int_a^b p(x)f_1(x)f_2(x)\cdots f_n(x)\Delta x \right) \\ &\geq \left(\int_a^b p(x)\Delta x \right)^{n-2} \left(\int_a^b p(x)f_1(x)\Delta x \right) \left(\int_a^b p(x)f_2(x)\cdots f_n(x)\Delta x \right) \\ &\geq \left(\int_a^b p(x)f_1(x)\Delta x \right) \left(\int_a^b p(x)\Delta x \right)^{n-3} \\ &\quad \times \left(\int_a^b p(x)f_2(x)\Delta x \right) \left(\int_a^b p(x)f_3(x)\cdots f_n(x)\Delta x \right) \\ &\geq \dots \end{aligned}$$



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$$\geq \left(\int_a^b p(x) f_1(x) \Delta x \right) \left(\int_a^b p(x) f_2(x) \Delta x \right) \cdots \left(\int_a^b p(x) f_n(x) \Delta x \right),$$

which is a generalization of a result in Dunkel [4].

In particular, if $f_1(x) = f_2(x) = \cdots = f_n(x) = f(x)$, then

$$\left(\int_a^b p(x) \Delta x \right)^{n-1} \left(\int_a^b p(x) f^n(x) \Delta x \right) \geq \left(\int_a^b p(x) f(x) \Delta x \right)^n.$$

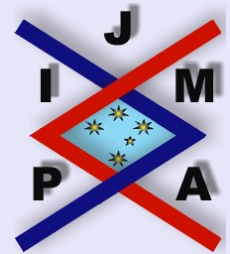
Theorem 2.3. If $p(x), f(x) \in C_{rd}([a, b], [0, \infty))$ with $f(x) > 0$ on $[a, b]$ and n is a positive integer, then

$$\left(\int_a^b \frac{p(x)}{f(x)} \Delta x \right)^n \left(\int_a^b p(x) f^n(x) \Delta x \right) \geq \left(\int_a^b p(x) \Delta x \right)^n.$$

Proof. It follows from $f(x) > 0$ on $[a, b]$ that $f^n(x)$ and $\frac{1}{f(x)}$ are oppositely ordered on $[a, b]$. Hence by (2.2),

$$\begin{aligned} & \int_a^b p(x) f^n(x) \Delta x \left(\int_a^b \frac{p(x)}{f(x)} \Delta x \right)^n \\ & \geq \int_a^b p(x) \Delta x \left(\int_a^b \frac{p(x)}{f(x)} \Delta x \right)^{n-1} \int_a^b p(x) f^{n-1}(x) \Delta x \\ & \geq \left(\int_a^b p(x) \Delta x \right)^2 \left(\int_a^b \frac{p(x)}{f(x)} \Delta x \right)^{n-2} \int_a^b p(x) f^{n-2}(x) \Delta x \\ & \geq \cdots \geq \left(\int_a^b p(x) \Delta x \right)^n. \end{aligned}$$

□



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Theorem 2.4. Let $g_1, g_2, \dots, g_n \in C_{rd}([a, b], \mathfrak{R})$ and $p, h_1, h_2, \dots, h_{n-1} \in C_{rd}([a, b], (0, \infty))$ with $g_n(x) > 0$ on $[a, b]$. If

$$\frac{g_1(x)g_2(x) \cdots g_{n-1}(x)}{h_1(x)h_2(x) \cdots h_{n-1}(x)} \quad \text{and} \quad \frac{h_{n-1}(x)}{g_n(x)}$$

are similarly ordered (or oppositely ordered), then

$$(2.4) \quad \int_a^b p(x)g_n(x)\Delta x \int_a^b \frac{p(x)g_1(x)g_2(x) \cdots g_{n-1}(x)}{h_1(x)h_2(x) \cdots h_{n-2}(x)} \Delta x \\ \geq (\text{or } \leq) \int_a^b p(x)h_{n-1}(x)\Delta x \int_a^b \frac{p(x)g_1(x)g_2(x) \cdots g_n(x)}{h_1(x)h_2(x) \cdots h_{n-1}(x)} \Delta x.$$

Proof. Taking

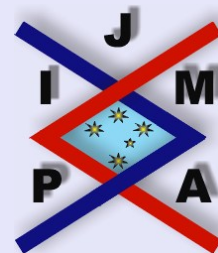
$$f_1(x) = \frac{g_1(x)g_2(x) \cdots g_{n-1}(x)}{h_1(x)h_2(x) \cdots h_{n-1}(x)}, \quad k_1(x) = h_{n-1}(x), \\ f_2(x) = 1 \quad \text{and} \quad k_2(x) = g_n(x)$$

in Theorem 2.1, (2.1) is reduced to our desired result (2.4). □

The following theorem is a time scales version of Theorem 1 in Beesack and Pečarić [2].

Theorem 2.5. Let

$$f_1, f_2, \dots, f_n \in C_{rd}([a, b], [0, \infty)) \quad \text{and} \quad g_1, g_2, \dots, g_n \in C_{rd}([a, b], (0, \infty)).$$



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If the functions $f_1, \frac{f_2}{g_1}, \dots, \frac{f_n}{g_{n-1}}$ are similarly ordered and for each pair $\frac{f_k}{g_{k-1}}, g_{k-1}$ is oppositely ordered for $k = 2, 3, \dots, n$, then

$$(2.5) \quad \int_a^b p(x) f_1(x) \frac{f_2(x) f_3(x) \cdots f_n(x)}{g_1(x) g_2(x) \cdots g_{n-1}(x)} \Delta x \\ \geq \frac{\int_a^b p(x) f_1(x) \Delta x \int_a^b p(x) f_2(x) \Delta x \cdots \int_a^b p(x) f_n(x) \Delta x}{\int_a^b p(x) g_1(x) \Delta x \int_a^b p(x) g_2(x) \Delta x \cdots \int_a^b p(x) g_{n-1}(x) \Delta x}.$$

Proof. Let f_1, f_2, \dots, f_n be replaced by $f_1, \frac{f_2}{g_1}, \dots, \frac{f_n}{g_{n-1}}$ in Remark 5, we obtain

$$(2.6) \quad \left(\int_a^b p(x) \Delta x \right)^{n-1} \int_a^b p(x) f_1(x) \frac{f_2(x) f_3(x) \cdots f_n(x)}{g_1(x) g_2(x) \cdots g_{n-1}(x)} \Delta x \\ \geq \int_a^b p(x) f_1(x) \Delta x \prod_{k=2}^n \int_a^b p(x) \frac{f_k(x)}{g_{k-1}(x)} \Delta x.$$

Also, since $\frac{f_k}{g_{k-1}}$ and g_{k-1} are oppositely ordered, it follows from Remark 1 that

$$\int_a^b p(x) \Delta x \int_a^b p(x) f_k(x) \Delta x \leq \int_a^b p(x) g_{k-1}(x) \Delta x \int_a^b p(x) \frac{f_k(x)}{g_{k-1}(x)} \Delta x.$$

Thus

$$\int_a^b \frac{p(x) f_k(x)}{g_{k-1}(x)} \Delta x \geq \frac{\int_a^b p(x) \Delta x \int_a^b p(x) f_k(x) \Delta x}{\int_a^b p(x) g_{k-1}(x) \Delta x}.$$

This and (2.6) imply (2.5) holds. \square



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3. More Results

In this section, we generalize some results in Isayama [8].

Theorem 3.1. Let $f_1, f_2, \dots, f_n \in C_{rd}([a, b], (0, \infty))$, $k_1, k_2, \dots, k_{n-1} \in C_{rd}([a, b], \mathbb{R})$ and $p(x) \in C_{rd}([a, b], [0, \infty))$. If

$$\frac{f_1(x)f_2(x) \cdots f_{i-1}(x)}{k_1(x)k_2(x) \cdots k_{i-1}(x)} \quad \text{and} \quad \frac{k_{i-1}(x)}{f_i(x)}$$

are similarly ordered (or oppositely ordered) for $i = 2, \dots, n$, then

$$(3.1) \quad \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x \\ \geq (\text{or } \leq) \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)k_2(x)\Delta x \cdots \\ \cdots \int_a^b p(x)k_{n-1}(x)\Delta x \int_a^b p(x) \frac{f_1(x)f_2(x) \cdots f_n(x)}{k_1(x)k_2(x) \cdots k_{n-1}(x)} \Delta x.$$

Proof. If $f_1(x), k_1(x), f_2(x)$ and $k_2(x)$ are replaced by $f_1(x), 1, k_1(x)$ and $\frac{f_2(x)}{k_1(x)}$ in Theorem 2.1, then we obtain

$$\int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \\ \geq (\text{or } \leq) \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x) \frac{f_1(x)f_2(x)}{k_1(x)} \Delta x.$$



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Thus the theorem holds for $n = 2$.

Suppose that the theorem holds for $n - 1$, that is

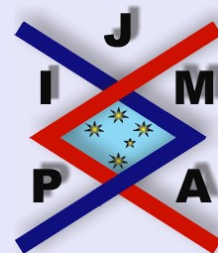
$$\begin{aligned}
 (3.2) \quad & \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_{n-1}(x)\Delta x \\
 & \geq (\text{or } \leq) \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)k_2(x)\Delta x \\
 & \quad \cdots \int_a^b p(x)k_{n-2}(x)\Delta x \int_a^b p(x) \frac{f_1(x)f_2(x) \cdots f_{n-1}(x)}{k_1(x)k_2(x) \cdots k_{n-2}(x)} \Delta x
 \end{aligned}$$

if

$$\frac{f_1(x)f_2(x) \cdots f_{i-1}(x)}{k_1(x)k_2(x) \cdots k_{i-1}(x)} \quad \text{and} \quad \frac{k_{i-1}(x)}{f_i(x)}$$

are similarly ordered (or oppositely ordered) for $i = 2, 3, \dots, n-1$. Multiplying the both sides of (3.2) by $\int_a^b p(x)f_n(x)\Delta x$, we see that

$$\begin{aligned}
 (3.3) \quad & \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_{n-1}(x)\Delta x \int_a^b p(x)f_n(x)\Delta x \\
 & \geq (\text{or } \leq) \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)k_2(x)\Delta x \\
 & \quad \cdots \int_a^b p(x)k_{n-2}(x)\Delta x \int_a^b p(x) \frac{f_1(x)f_2(x) \cdots f_{n-1}(x)}{k_1(x)k_2(x) \cdots k_{n-2}(x)} \Delta x \int_a^b p(x)f_n(x)\Delta x.
 \end{aligned}$$



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It follows from Theorem 2.5 that

$$\int_a^b p(x) \frac{f_1(x)f_2(x) \cdots f_{n-1}(x)}{k_1(x)k_2(x) \cdots k_{n-2}(x)} \Delta x \int_a^b p(x) f_n(x) \Delta x$$

$$\geq (\text{or } \leq) \int_a^b p(x) \frac{f_1(x)f_2(x) \cdots f_n(x)}{k_1(x)k_2(x) \cdots k_{n-1}(x)} \Delta x \int_a^b p(x) k_{n-1}(x) \Delta x.$$

This and (3.3) imply

$$\int_a^b p(x) f_1(x) \Delta x \int_a^b p(x) f_2(x) \Delta x \cdots \int_a^b p(x) f_n(x) \Delta x$$

$$\geq (\text{or } \leq) \int_a^b p(x) k_1(x) \Delta x \int_a^b p(x) k_2(x) \Delta x$$

$$\cdots \int_a^b p(x) k_{n-1}(x) \Delta x \int_a^b p(x) \frac{f_1(x)f_2(x) \cdots f_n(x)}{k_1(x)k_2(x) \cdots k_{n-1}(x)} \Delta x.$$

By induction, we complete the proof. \square

Remark 6. Let $k_n \in C_{rd}([a, b], \mathbb{R})$. If $f_1(x), f_2(x), \dots, f_n(x), k_1(x), k_2(x), \dots, k_{n-1}(x)$ are replaced by

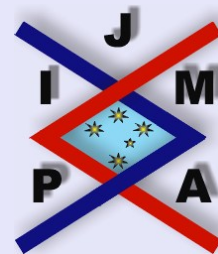
$$f_1(x) f_2(x) \cdots f_n(x), \quad k_1(x) k_2(x) \cdots k_n(x),$$

$\dots,$

$$k_1(x) k_2(x) \cdots k_n(x), \quad f_1(x) k_2(x) \cdots k_n(x), \quad k_1(x) f_2(x) k_3(x) \cdots k_n(x),$$

$\dots,$

$$k_1(x) k_2(x) \cdots k_{n-2}(x) f_{n-1}(x) k_n(x)$$



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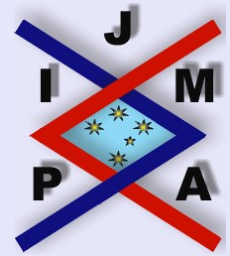
in Theorem 3.1, respectively, then

$$\begin{aligned}
 (3.4) \quad & \int_a^b p(x) f_1(x) f_2(x) \cdots f_n(x) \Delta x \\
 & \times \left(\int_a^b p(x) k_1(x) k_2(x) \cdots k_n(x) \Delta x \right)^{n-1} \\
 & \geq \int_a^b p(x) f_1(x) k_2(x) \cdots k_n(x) \Delta x \\
 & \quad \times \int_a^b p(x) k_1(x) f_2(x) k_3(x) \cdots k_n(x) \Delta x \\
 & \quad \cdots \int_a^b p(x) k_1(x) k_2(x) \cdots k_{n-1}(x) f_n(x) \Delta x
 \end{aligned}$$

if $\frac{f_i(x)}{k_i(x)} > 0$ for $i = 1, 2, \dots, n$ and $k_1(x)k_2(x) \cdots k_n(x) > 0$ on $[a, b]$.

Remark 7. Letting $f_1(x) = f_2(x) = \cdots = f_n(x) = f(x)$ and $k_1(x) = k_2(x) = \cdots = k_n(x) = k^{\frac{1}{n-1}}(x)$ in (3.4) with $k(x) > 0$ on $[a, b]$, we obtain the Hölder inequality:

$$\begin{aligned}
 (3.5) \quad & \int_a^b p(x) f^n(x) \Delta x \left(\int_a^b p(x) k^{\frac{n}{n-1}}(x) \Delta x \right)^{n-1} \\
 & \geq \left(\int_a^b p(x) f(x) k(x) \Delta x \right)^n.
 \end{aligned}$$



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Remark 8. Let $p, f, g \in C_{rd}([a, b], [0, \infty))$. Taking

$$\begin{aligned} f_1(x) &= f^n(x)g(x), \\ f_2(x) &= f_3(x) = \cdots = f_n(x) = g(x) \quad \text{and} \\ k_1(x) &= k_2(x) = \cdots = k_{n-1}(x) = f(x)g(x), \end{aligned}$$

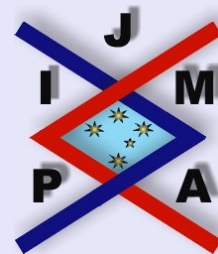
(3.1) is reduced to Jensen's inequality:

$$(3.6) \quad \int_a^b p(x)f^n(x)g(x)\Delta x \left(\int_a^b p(x)g(x)\Delta x \right)^{n-1} \geq \left(\int_a^b p(x)f(x)g(x)\Delta x \right)^n.$$

Remark 9. Taking $k_1(x) = k_2(x) = \cdots = k_{n-1}(x) = (f_1(x)f_2(x) \cdots f_n(x))^{\frac{1}{n}}$, (3.1) is reduced to

$$(3.7) \quad \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x \geq \left[\int_a^b p(x) (f_1(x)f_2(x) \cdots f_n(x))^{\frac{1}{n}} \Delta x \right]^n$$

if $f_i(x) > 0$ on $[a, b]$ for each $i = 1, 2, \dots, n$ and $\frac{1}{f_i(x)}[f_1(x)f_2(x) \cdots f_n(x)]^{\frac{1}{n}}$ ($i = 1, 2, \dots, n$) are similarly ordered.



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Remark 10 (see also Remark 5). Taking $k_1(x) = k_2(x) = \dots = k_{n-1}(x) = 1$, (3.1) is reduced to Čebyšev's inequality:

$$(3.8) \quad \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x \\ \leq \left(\int_a^b p(x)\Delta x \right)^{n-1} \int_a^b p(x)f_1(x)f_2(x)\cdots f_n(x)\Delta x$$

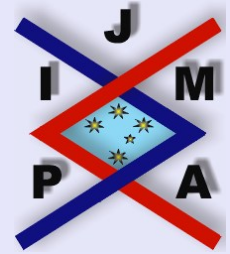
if $f_i(x)$ ($i = 1, 2, \dots, n$) are similarly ordered and $f_i(x) \geq 0$ ($i = 1, 2, \dots, n$).

Remark 11. Taking $f_1(x) = f_2(x) = \dots = f_n(x) = 1$, then (3.1) is reduced to

$$\left(\int_a^b p(x)\Delta x \right)^n \leq \int_a^b p(x)k_1(x)\Delta x \int_a^b p(x)k_2(x)\Delta x \\ \cdots \int_a^b p(x)k_{n-1}(x)\Delta x \int_a^b \frac{p(x)}{k_1(x)k_2(x)\cdots k_{n-1}(x)}\Delta x$$

if $k_i(x) > 0$ are similarly ordered for $i = 1, 2, \dots, n-1$. Thus, if $f_1(x), \dots, f_n(x)$ are similarly ordered and $f_i(x) > 0$ on $[a, b]$ ($i = 1, 2, \dots, n$), then

$$(3.9) \quad \frac{\left(\int_a^b p(x)\Delta x \right)^{n+1}}{\int_a^b \frac{p(x)}{f_1(x)f_2(x)\cdots f_n(x)}\Delta x} \\ \leq \int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \cdots \int_a^b p(x)f_n(x)\Delta x.$$



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It follows from (3.8) and (3.9) that

$$\frac{\left(\int_a^b p(x)\Delta x\right)^{n+1}}{\int_a^b \frac{p(x)}{f_1(x)f_2(x)\cdots f_n(x)}\Delta x} \leq \left(\int_a^b p(x)\Delta x\right)^{n-1} \int_a^b p(x)f_1(x)f_2(x)\cdots f_n(x)\Delta x$$

if $f_1(x), \dots, f_n(x)$ are similarly ordered.

Remark 12. Let $k_1(x) = k_2(x) = \dots = k_n(x) = 1$. If $f_i(x)$ is replaced by

$$\frac{[f_1(x)f_2(x)\cdots f_n(x)]^{\frac{1}{n}}}{f_i(x)}, \quad n = 1, 2, \dots, n,$$

then (3.1) is reduced to

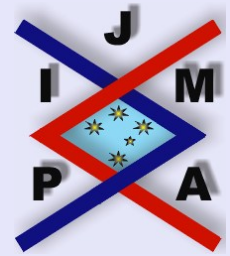
$$\prod_{i=1}^n \int_a^b p(x) \frac{\sqrt[n]{f_1(x)f_2(x)\cdots f_n(x)}}{f_i(x)} \Delta x \leq \left(\int_a^b p(x)\Delta x\right)^n$$

if $\frac{\sqrt[n]{f_1(x)f_2(x)\cdots f_n(x)}}{f_i(x)}$ ($i = 1, 2, \dots, n$) are similarly ordered.

Remark 13. Let $f_1, f_2, \dots, f_n; k_1, k_2, \dots, k_{n-1}$ be replaced by $f_1 f_2, f_3 f_4, \dots, f_{2n-1} f_{2n}; f_2 f_3, f_4 f_5, \dots, f_{2n-2} f_{2n-1}$, respectively. Then (3.1) is reduced to

$$\begin{aligned} & \int_a^b p(x)f_1(x)f_2(x)\Delta x \int_a^b p(x)f_3(x)f_4(x)\Delta x \cdots \int_a^b p(x)f_{2n-1}(x)f_{2n}(x)\Delta x \\ & \geq \int_a^b p(x)f_2(x)f_3(x)\Delta x \int_a^b p(x)f_4(x)f_5(x)\Delta x \cdots \int_a^b p(x)f_{2n-2}(x)f_{2n-1}(x)\Delta x \end{aligned}$$

if $\frac{f_i(x)}{f_{i+1}(x)}$ ($i = 1, 2, \dots, 2n - 1$) are similarly ordered.



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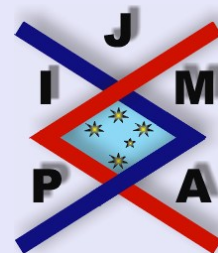
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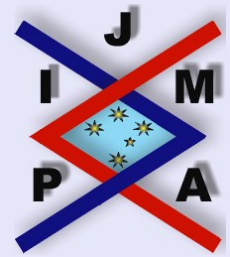
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