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A NOTE ON A PAPER OF H. ALZER AND S. KOUMANDOS

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Abstract

In the paper “Sharp inequalities for trigonometric sums in two variables,” (*Illinois Journal of Mathematics*, Vol. 48, No.3, (2004), 887–907) Alzer and Koumandos investigated some special trigonometric sums. One of them is the sum

$$A_n^*(x, y) := \sum_{k=1}^n \frac{\cos(k - \frac{1}{2})x \sin(k - \frac{1}{2})y}{k - \frac{1}{2}}.$$

In the present note we show that the results of [1] can be easily obtained by a very simple elementary argument. And the results we obtained are more exact.

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Key words: Inequalities, Trigonometric sums.

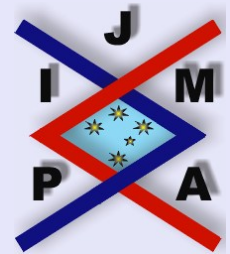
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In a recent long paper [1], Alzer and Koumandos investigated the trigonometric sums:

$$A_n(x, y) = \sum_{k=1}^n \frac{\cos kx \sin ky}{k}, \quad A_n^*(x, y) = \sum_{k=1}^n \frac{\cos(k - \frac{1}{2})x \sin(k - \frac{1}{2})y}{k - \frac{1}{2}},$$

$$B_n(x, y) = \sum_{k=1}^n \frac{\sin kx \sin ky}{k}.$$

Their results can be restated as follows:



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$$(A) |A_n(x, y)| < \sup\{A_n(x, y) : x, y \in [0, \pi], n \in \mathbb{N}\} = \int_0^\pi \frac{\sin t}{t} dt;$$

$$(B) \min\{B_n(x, y) : x, y \in [0, \pi], n \in \mathbb{N}\} = -\frac{1}{8},$$

$$B_n(x, y) = -\frac{1}{8} \iff n = 2 \text{ and } (x, y) = \left(\frac{5\pi}{6}, \frac{\pi}{6}\right) \text{ or } (x, y) = \left(\frac{\pi}{6}, \frac{5\pi}{6}\right);$$

$$(C) -\frac{2}{3}(\sqrt{2} - 1) \leq A_n^*(x, y) \leq 2,$$

$$A_n^*(x, y) = -\frac{2}{3}(\sqrt{2} - 1) \iff n = 2, (x, y) = \left(\frac{3\pi}{4}, \frac{\pi}{4}\right),$$

$$A_n^*(x, y) = 2 \iff n = 1, (x, y) = (0, \pi).$$

The purpose of the present note is to give more exact results by very much simpler proof.

For a continuous function f on $D := [0, \pi] \times [0, \pi]$ we define

$$\min(f) = \min\{f(x, y) : (x, y) \in D\}, \quad \max(f) = \max\{f(x, y) : (x, y) \in D\}.$$

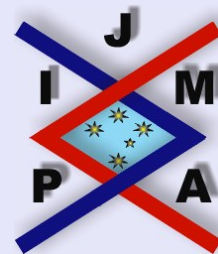
Our results are

(A')

$$\max(A_n) = A_n \left(0, \frac{\pi}{n+1}\right) = \int_0^{\frac{\pi}{2(n+1)}} \frac{2 \cos(n+1)t \sin nt}{\sin t} dt,$$

$$\min(A_n) = -\max(A_n) = A_n \left(\pi, \pi - \frac{\pi}{n+1}\right),$$

$$\max(A_n) < \lim_{n \rightarrow \infty} \max(A_n) = \int_0^\pi \frac{\sin t}{t} dt.$$



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(B') For $n \geq 2$

$$\begin{aligned} \min(B_n) &= B_n \left(\frac{(2n+1)\pi}{n(n+1)}, \frac{\pi}{n(n+1)} \right) \\ &= \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{\sin(n+1)t \sin nt}{\sin t} dt < \min(B_{n+1}), \\ \lim_{n \rightarrow \infty} \min(B_n) &= 0. \end{aligned}$$

(C') For all n

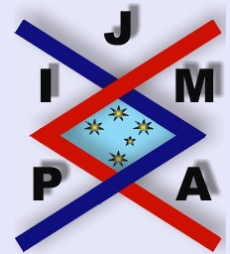
$$\begin{aligned} \max(A_n^*) &= A_n^* \left(0, \frac{\pi}{n} \right) \\ &= \int_0^{\frac{\pi}{2n}} \frac{\sin 2nt}{\sin t} dt \\ &> \max(A_{n+1}^*) \rightarrow \int_0^{\pi} \frac{\sin t}{t} dt, \quad (n \rightarrow \infty), \end{aligned}$$

and for $n \geq 2$

$$\min(A_n^*) = A_n^* \left(\frac{3\pi}{2n}, \frac{\pi}{2n} \right) = \int_{\frac{\pi}{2n}}^{\frac{3\pi}{2n}} \frac{\sin 2nt}{2 \sin t} dt < \min(A_{n+1}^*) \rightarrow 0 \quad (n \rightarrow \infty).$$

In particular, $\min(A_2^*) = \frac{2}{3}(1 - \sqrt{2})$.

The results (A), (B), (C) are easy consequences of (A'), (B') and (C') respectively.



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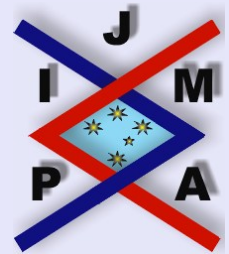
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Proof of (A'). We have

$$\begin{aligned}
 A_n(x, y) &= \sum_{k=1}^n \frac{\sin k(x+y) - \sin k(x-y)}{2k} \\
 &= \sum_{k=1}^n \frac{1}{2k} \int_{k(x-y)}^{k(x+y)} \cos t \, dt \\
 &= \sum_{k=1}^n \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \cos 2kt \, dt \\
 &= \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \sum_{k=1}^n \frac{2 \cos 2kt \sin t}{2 \sin t} \\
 &= \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \frac{\sin(2n+1)t - \sin t}{2 \sin t} \, dt = \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \frac{\cos(n+1)t \sin nt}{\sin t} \, dt.
 \end{aligned}$$

Then we get

$$\begin{aligned}
 \max(A_n) &= A_n\left(0, \frac{\pi}{n+1}\right) \\
 &= \int_{\frac{-\pi}{2(n+1)}}^{\frac{\pi}{2(n+1)}} \frac{\cos(n+1)t \sin nt}{\sin t} \, dt \\
 &= \int_0^{\frac{\pi}{2(n+1)}} \frac{2 \cos(n+1)t \sin nt}{\sin t} \, dt \\
 &< \int_0^{\frac{\pi}{2(n+1)}} \frac{2 \cos(n+1)t \sin(n+1)t}{t} \, dt = \int_0^{\pi} \frac{\sin t}{t} \, dt;
 \end{aligned}$$



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$$\lim_{n \rightarrow \infty} \max(A_n) = \int_0^\pi \frac{\sin t}{t} dt;$$

$$\begin{aligned} \min(A_n) &= \min\{A_n(\pi - x, \pi - y) : (x, y) \in D\} \\ &= -\max(A_n) = A_n\left(\pi, \pi - \frac{\pi}{n+1}\right). \end{aligned}$$

□

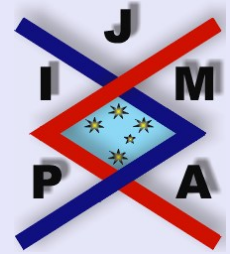
Proof of (B') and (C'). We have

$$\begin{aligned} B_n(x, y) &= \sum_{k=1}^n \frac{\cos k|x-y| - \cos k(x+y)}{2k} \\ &= \sum_{k=1}^n \int_{\frac{|x-y|}{2}}^{\frac{x+y}{2}} \sin 2kt \, dt \\ &= \int_{\frac{|x-y|}{2}}^{\frac{x+y}{2}} \frac{\cos t - \cos(2n+1)t}{2 \sin t} \, dt \\ &= \int_{\frac{|x-y|}{2}}^{\frac{x+y}{2}} \frac{\sin(n+1)t \sin nt}{\sin t} \, dt, \end{aligned}$$

$$A_n^*(x, y) = \frac{1}{2} \sum_{k=1}^n \int_{x-y}^{x+y} \cos\left(k - \frac{1}{2}\right)t \, dt = \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \frac{\sin 2nt}{2 \sin t} \, dt.$$

Then we get (B') and (C').

□



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