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A NOTE ON PROBABILITY WEIGHTED MOMENT INEQUALITIES FOR RELIABILITY MEASURES

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ABSTRACT. Weighted distributions in general and length-biased distributions in particular are very useful and convenient for the analysis of lifetime data. These distributions occur naturally for some sampling plans in reliability, ecology, biometry and survival analysis. In this note an increasing failure rate property for lifetime distributions is used to define a natural ordering of the weighted reliability measures. Some useful bounds, probability weighted moment inequalities and variability orderings for weighted and unweighted reliability measures and related functions are presented. Stochastic comparisons and moment inequalities for weighted reliability measures and related functions are given.

Key words and phrases: Stochastic inequalities, Weighted distribution functions, Bounds.

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1. INTRODUCTION

Weighted distributions occur frequently in research related to reliability, bio-medicine, ecology and several other areas. If item lengths are distributed according to the distribution function (df) F, and if the probability of selecting any particular item is proportional to its length, then the lengths of the sampled items have the length-biased distribution. If data is unknowingly sampled from a weighted distribution as opposed to the parent distribution, the reliability function, hazard function and mean residual life function may be over or underestimated, depending on the weight function. For size-biased or length-biased distributions in which the weight function is increasing monotonically, the analyst usually gives an over optimistic estimate of the reliability function and the mean residual life function. A survey of applications of weighted distributions in general and length-biased distributions in particular are given by Patil and Rao [8]. Vardi [10] derived a non-parametric maximum likelihood estimate of a lifetime distribution

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F, on the basis of two independent samples, one sample of size m, from F, and the other a sample of size n, from the length-biased distribution of F, and studied its distributional properties. Gupta and Keating [4] obtained results on relations for reliability measures under length-biased sampling. Oluyede [7] obtained useful results on inequalities and selection of experiments for length-biased distributions, and investigated certain modified length-biased cross-entropy measures. Also, see Daniels [2], Bhattacharyya et al [1], Zelen and Feinleib [11] and references therein for additional results on weighted distributions.

Let X be a non-negative random variable with distribution function F and probability density function (pdf) f. The weighted random variable X_W, has a survival function given by,

$$\overline{G}_W(x) = \frac{\overline{F}(x)(W(x) + T_F(x))}{E_F(W(X))}, \tag{1.1}$$

where $T_F(x) = \int_x^\infty (\overline{F}(u)W'(u)du)/\overline{F}(x)$, and $W'(u) = dW(u)/du$, assuming that $W(x)\overline{F}(x) \to 0$ as $x \to \infty$. The probability density function of the survival function given in (1.1) is referred to as a weighted distribution with weight function $W(x) \geq 0$. If $W(x) = x$ in (1.1), the resulting function is referred to as the length-biased reliability function. The purpose of this note is to establish and compare inequalities for weighted distributions including length-biased distributions. An increasing failure rate property (IFRP) for lifetime distributions is used to define a natural ordering of the weighted reliability measures and some implications are explored. We establish some moment type inequalities for the comparisons of length-biased and weighted distributions with the parent distributions. Some important utility notions, including useful and meaningful inequalities, are presented in Section 2. Section 3 is concerned with orderings, including the increasing failure rate property (IFRP) ordering for weighted reliability measures and related functions. Stochastic comparisons and moment inequalities involving reliability functions, are presented in Section 4.

2. UTILITY NOTIONS AND BASIC RESULTS

In this section, we give some definitions and useful concepts. Let $\mathcal{F}$ be the set of absolutely continuous distribution functions satisfying

$$H(0) = 0, \quad \lim_{x\to\infty} H(x) = 1, \quad \sup\{x : H(x) < 1\} = \infty. \tag{2.1}$$

Note that if the mean of a random variable in $\mathcal{F}$ is finite, it is positive. We begin by presenting some definitions. Let F and G denote the distribution functions of the random variables X and Y respectively. Also, let $\overline{F}(x) = 1 - F(x)$ and $\overline{G}(x) = 1 - G(x)$ be the respective reliability functions. A class of moments, called probability weighted moments (PWM), was introduced by Greenwood et al [3].

Definition 2.1. The PWM is given by

$$M_{l,k,j} = E[X^l F^j \overline{F}^k], \tag{2.2}$$

where l, k, j are real numbers and $\overline{F}(x) = 1 - F(x)$.

Example 2.1 (IRLS and Hazard Functions)**.** In the iterative reweighted least-square (IRLS) algorithm for the maximum likelihood fitting of the binary regression model, the expressions in the algorithm for the weights are given by,

$$W^{-1} = (f(\theta))^{-2} F(\theta)\overline{F}(\theta) \tag{2.3}$$

which can be expressed in terms of the hazard functions $h_F(\theta)$ and $\lambda_F(\theta)$, that is,

$$W = \frac{f(\theta)}{F(\theta)} \frac{f(\theta)}{\overline{F}(\theta)} = h_F(\theta)\lambda_F(\theta), \tag{2.4}$$

where the relationship between the fitted response probability p, and the linear predictor θ is given by $p = F(\theta)$, the derivative $d\theta/dp = 1/f(\theta)$, and $f(\theta) = dF(\theta)/d\theta$ is the probability density function of the predictor. Clearly, the expectation of the expression for the weights in the reweighted least-square (IRLS) algorithm for maximum likelihood fitting of the binary regression model is the probability weighted moment of $(f(\theta))^{-2}$, that is,

$$E[W^{-1}] = E[(f(\theta))^{-2}F(\theta)\overline{F}(\theta)]. \tag{2.5}$$

Following Greenwood et al [3], we define the following probability weighted moment (PWM) order for life distributions F and G.

Definition 2.2. Let F and G be in $\mathcal{F}$, and set $p = l + 1$, then F is said to be larger than G in PWM(l, k, j) ordering, $(F \geq_{PWM(l,k,j)} G)$ if

$$\int_t^\infty x^{p-1}F^j(x)\overline{F}^k(x)dF(x) \geq \int_t^\infty x^{p-1}G^j(x)\overline{G}^k(x)dG(x) \tag{2.6}$$

for all $t \geq 0$.

Next we present some basic definitions for the comparisons of reliability functions of two random variables X and Y. These definitions involve the hazard function $\lambda_F(x) = f(x)/\overline{F}(x)$, and the reverse hazard function $h_F(x) = f(x)/F(x)$, where $f(x) = dF(x)/dx$ is the probability density function (pdf) of the random variable X. Note that the function $h_F(x)$ behaves like the density function $f(x)$ in the upper tail of $F(x)$. See Szekli [9] for details on these and other ageing concepts.

Definition 2.3.

(i) Let X and Y be random variables with distribution functions F and G respectively. We say that X is larger than Y in stochastic ordering $(X \geq_{st} Y)$ if $\overline{F}(t) \geq \overline{G}(t)$ for all $t \geq 0$.

(ii) A random variable X with distribution function F is said to have a decreasing (increasing) hazard rate if and only if $\overline{F}(x+t)/\overline{F}(x)$ is increasing (decreasing) in $x \geq 0$, for every $t \geq 0$.

(iii) Suppose μ_G and μ_H are finite. We say G preceeds H in mean residual life and write $G \leq_{mr} H$ if for every $t \geq 0$,

$$\int_t^\infty \overline{H}(y)dy \geq \int_t^\infty \overline{G}(y)dy. \tag{2.7}$$

The MR ordering is a partial ordering on the class of distributions of non-negative random variables with finite mean. Note that if G_l and H_l are length-biased distribution functions with $F \leq_{st} K$ and $\mu_F = \mu_K$, then $G_l \geq_{mr} H_l$, where $\overline{G}_l(t) = \mu_F^{-1}\int_t^\infty xdF(x)$, and $\overline{H}_l(t) = \mu_K^{-1}\int_t^\infty xdK(x)$ respectively.

Definition 2.4.

(i) If F and G are in $\mathcal{F}$, we say that G has more tail at the origin than F, denoted by $F <_{TOP} G$, if

$$h_G(x) \geq h_F(x), \tag{2.8}$$

where $h_F(x) = f(x)/F(x)$, for all $x \geq 0$, is the reverse hazard function.

(ii) If F and G are in $\mathcal{F}$, we say that G has larger tail than F, denoted by $F <_{LTP} G$, if

$$\frac{\lambda_F(x)}{h_F(x)} \geq \frac{\lambda_G(x)}{h_G(x)}, \tag{2.9}$$

where $\lambda_F(x) = f(x)/\overline{F}(x)$, for all $x \geq 0$.

(iii) A life distribution F is said to have a larger increasing failure rate than the life distribution G, denoted by $F \geq_{IFRP} G$, if and only if

$$\frac{P(X > x_2 | X > y_1)}{P(X > y_2 | X > y_1)} \geq \frac{P(Y > x_2 | X > x_1)}{P(Y > y_2 | X > y_1)} \tag{2.10}$$

for all $x_1 < x_2$, $x_1 < y_1$, $y_1 < y_2$, $x_2 < y_2$.

The next definition is mainly due to Loh [6].

Definition 2.5. An absolutely continuous distribution function F on $(0, \infty)$ for which $\lim_{x \to 0^+} \frac{F(x)}{x}$ exists is:

(i) new better than used in average failure rate (NBAFR) if

$$\begin{aligned} \overline{\lambda}_F(0) &= \lim_{t \to 0^+} t^{-1} \int_0^t \lambda_F(x) dx \\ &\leq t^{-1} \int_0^t \lambda_F(x) dx, \end{aligned} \tag{2.11}$$

for $t > 0$,

(ii) new better than used in failure rate (NBUFR) if there exists a version of λ_F of the failure or hazard rate such that

$$\overline{\lambda}_F(0) \leq \lambda_F(t), \tag{2.12}$$

for all $t > 0$, where $\overline{\lambda}_F(0)$ is given by (2.11). The inequalities are reversed for new better than used in average failure rate (NBAFR) and new worse than used in failure rate (NWUFR) respectively.

The class of NBUFR (NWUFR) enjoys many desirable properties under several reliability operations. This class also includes any life distribution with $\lambda_F(0) = 0$. Clearly, $F \in NBUFR$ if and only if

$$\overline{F}(x + t) \leq \exp\{-\overline{\lambda}_F(0)t\} F(x), \tag{2.13}$$

for all $x, t \geq 0$.

Theorem 2.1. *Let $\overline{G}_l(t)$ and $\overline{H}_l(t)$ be length-biased reliability functions. If there exists $a > 0$ such that for every $v > 0$ and $\beta > 0$, there is a t_0 such that*

$$\lambda_{G_l}(v + \beta t) > \lambda_{H_l}(t) + a,$$

for every $t > t_0$, then $\overline{G}_l(t) > \overline{H}_l(t)$ for all $t > t_0$, and $G_l \geq_{mr} H_l$ for all $t > t_0$.

Proof. For all $t > t_0$ and some $a > 0$, we have for $t_0 < t_1 < t_2$,

$$\int_{t_1}^{t_2} \lambda_{G_l}(t) dt > \int_{t_1}^{t_2} \lambda_{H_l}(t) dt + a(t_2 - t_1). \tag{2.14}$$

This is equivalent to

$$\log \left(\frac{\overline{G}_l(t_1)}{\overline{G}_l(t_2)} \right) > \log \left(\frac{\overline{H}_l(t_1)}{\overline{H}_l(t_2)} \right) + a(t_2 - t_1), \tag{2.15}$$

that is,

$$\frac{\overline{G}_l(t_1)}{\overline{H}_l(t_1)} > \frac{\overline{G}_l(t_2)}{\overline{H}_l(t_2)} e^{a(t_2 - t_1)}.$$

Hence, there exists $t_u > t_0$ such that $\overline{G}_l(t) > \overline{H}_l(t)$, for all $t > t_0$.

Consequently, $G_l \geq_{mr} H_l$ for all $t > t_0$. □

Theorem 2.2. *Let F be absolutely continuous on $[0,\infty)$. Suppose $F(x)/x$ has a limit as $x \to 0$ from above. Then $F \in NWUFR$ implies $G \in NWUFR$, where $\overline{G}_l(t) = \mu_F^{-1}\int_t^\infty x dF(x)$.*

Proof. Note that $F \in NWUFR$ if and only if

$$\overline{F}(x+t) \geq \exp\{-\overline{\lambda}_F(0)t\}F(x), \tag{2.16}$$

for all $x,t \geq 0$. The left side of (2.16) satisfies

$$\overline{G}_l(x+t) \geq \overline{F}(x+t), \tag{2.17}$$

for all $x,t \geq 0$, where $\overline{G}_l(t) = \mu_F^{-1}\int_t^\infty x dF(x)$. Now,

$$\int_0^t \lambda_F(x)dx \leq \int_0^t \lambda_{G_l}(x)dx$$

for all $t \geq 0$, so that

$$\lambda_F(0)t \leq \lambda_{G_l}(0)t, \tag{2.18}$$

for all $t \geq 0$. From (2.17) and (2.18) we have

$$\exp\{-\overline{\lambda}_F(0)t\}F(x) \geq \exp\{-\overline{\lambda}_{G_l}(0)t\}G_l(x), \tag{2.19}$$

for all $x,t \geq 0$.

Consequently,

$$\overline{G}_l(x+t) \geq \overline{F}(x+t) \geq \exp\{-\overline{\lambda}_F(0)t\}F(x) \geq \exp\{-\overline{\lambda}_{G_l}(0)t\}G_l(x), \tag{2.20}$$

for all $x,t \geq 0$. □

In a similar manner, one can show that if G_i, $i = 1,2$, are length-biased distribution functions on $[0,\infty)$ for which $\lim_{x\to 0^+} G_i(x)/x < \infty$ and $\lim_{x\to 0^+} F_i(x)/x < \infty$, $i = 1,2$, then $G_1 * G_2 \in NWUFR$. This follows from the fact that $G_1 * G_2(x) \leq F_1 * F_2(x) \leq F_1(x)F_2(x)$ for every $x \in (0,\infty)$, where $G_1 * G_2$ is the convolution of the distribution functions G_1 and G_2 respectively.

Example 2.2 (Rayleigh Distribution)**.** The Rayleigh distribution plays an important role in applied probability and statistics. The probability density function is given by,

$$f(x;\theta) = 2\pi^{-1/2}\theta^{-1}\exp\left\{-\left(\frac{x}{\theta}\right)^2\right\}, \tag{2.21}$$

for $x > 0$, $\theta > 0$. The corresponding length-biased reliability and hazard functions are given by,

$$\overline{F}(x;\theta) = 2[1-\Phi(\sqrt{2}x/\theta)]$$

and

$$\lambda_F(x;\theta) = \frac{\sqrt{2}\phi(\sqrt{2}x/\theta)}{\theta[1-\Phi(\sqrt{2}x/\theta)]},$$

where Φ and ϕ are the standard normal distribution and density functions, respectively. Let X_W be the corresponding weighted random variable with weight function $W(x) = x$. The probability density function of X_W is given by,

$$g_l(x;\theta) = 2x\theta^{-2}\exp\left\{-\left(\frac{x}{\theta}\right)^2\right\}, \tag{2.22}$$

for $x > 0$, $\theta > 0$. The reliability and hazard functions are $\overline{G}_l(x;\theta) = \exp\{-(x/\theta)^2\}$, $x > 0$, and $\lambda_{G_l}(x;\theta) = 2x/\theta^2$, $x > 0$, respectively. Clearly, $\overline{G}_l(x+t;\theta) \geq \overline{F}(x+t;\theta)$, for all $x,t \geq 0$, and $\overline{\lambda}_{G_l}(0) = 0$. In view of Theorem 2.2, we get that $\overline{G}_l \in NWUFR$.

3. SOME ORDERINGS FOR WEIGHTED RELIABILITY MEASURES

Let X be a non-negative random variable with distribution function F and mean $\mu = E(X)$ and let X_e be a non-negative random variable with distribution function

$$F_e(x) = \frac{\int_0^\infty P(X \geq t)dt}{\mu}, \tag{3.1}$$

$x \geq 0$. The k^{th} moment of the random variable X_e is given by

$$\begin{aligned} E(X_e^k) &= k\int_0^\infty x^{k-1}\overline{F}_e(x)dx \\ &= \frac{\int_0^\infty t^k \overline{F}(t)dt}{\mu_F} \\ &= \frac{E(X^{k+1})}{(k+1)\mu_F} \end{aligned} \tag{3.2}$$

where $\overline{F}(\cdot) = 1 - F(\cdot)$ and $\mu_F = \int_0^\infty \overline{F}(u)du$. The distribution function $F_e(x)$ is called the stationary renewal distribution with mean remaining or residual life given by

$$\mu(t) = \frac{1}{\overline{F}(t)}\int_t^\infty \overline{F}(y)dy, \tag{3.3}$$

$t \geq 0$. The mean remaining life and failure rate function of F_e are given by

$$\mu_{F_e}(t) = \frac{\int_t^\infty \overline{F}(y)\mu(y)dy}{\overline{F}(t)\mu(t)} \tag{3.4}$$

and $\lambda_e(t) = [\mu(t)]^{-1}$ respectively, $t \geq 0$.

If $W(x) = x$ in equation (1.1), then the corresponding probability density function (pdf) is called the length-biased pdf and is given by

$$g_l(x) = \frac{xf(x)}{\mu}, \tag{3.5}$$

$x \geq 0$. The corresponding k^{th} moment is

$$E(X_l^k) = \frac{E(X^{k+1})}{\mu_F}. \tag{3.6}$$

Proposition 3.1. *$E(X_e^k) > E(X_l^k)$ for $k < 0$ and $E(X_e^k) \leq E(X_l^k)$ for $k \geq 0$.*

Proposition 3.2. *Let G_W be a weighted distribution function with increasing weight function $W(x)$, $x \geq 0$, and F the parent distribution function respectively, then $F <_{LTP} G_W$.*

Proof. Let G_W be a weighted distribution function with increasing weight function $W(x)$, $x \geq 0$, then,

$$\overline{G}_W(x) \geq \overline{F}(x), \tag{3.7}$$

for all $x \geq 0$. Equivalently,

$$G_W(x) - F(x)G_W(x) \geq F(x) - F(x)G_W(x), \tag{3.8}$$

for all $x \geq 0$. This is equivalent to

$$\frac{F(x)}{\overline{F}(x)} \geq \frac{G_W(x)}{\overline{G}_W(x)}, \tag{3.9}$$

for all $x \geq 0$, which in turn is seen to be equivalent to

$$\frac{f(x)/\overline{F}(x)}{f(x)/F(x)} \geq \frac{g_W(x)/\overline{G}_W(x)}{g_W(x)/G_W(x)}, \tag{3.10}$$

for all $x \geq 0$. Consequently, $F <_{LTP} G_W$. □

Example 3.1 (Exponential Distribution)**.** The most useful model in reliability studies is the exponential failure model with pdf given by,

$$f(x;\theta) = \theta^{-1} \exp\left\{-\frac{x}{\theta}\right\}, \tag{3.11}$$

for $x > 0$ and $\theta > 0$. It is well known that the reliability and failure rate functions are $\overline{F}(x;\theta) = \exp\{-x/\theta\}$ and $\lambda_F(x;\theta) = \theta^{-1}$ respectively. Let $W(x) = x$, then the reliability and hazard functions are given by, $\overline{G}_W(x;\theta) = \{1 + \frac{x}{\theta}\}\exp\{-x/\theta\}$ and $\lambda_{G_W}(x,\theta) = \frac{x}{\theta(x+\theta)}$ respectively. Clearly, $\overline{G}_W(x;\theta) \geq \overline{F}(x;\theta)$ and $\lambda_{G_W}(x,\theta) \leq \lambda_F(x;\theta)$, for all $x > 0$ and $\theta > 0$. In view of Proposition 3.2, $F <_{LTP} G_W$.

Proposition 3.3. *Let $G_{_W}$ be a weighted distribution function with increasing weight function $W(x)$, $x \geq 0$, and F the parent distribution function respectively. If $W(x) = x$ and $x \geq \mu_{_F} > 0$ then $F <_{TOP} G_{_W}$.*

Proof. Let $W(x) = x$, then the length-biased reliability function is given by

$$\overline{G}_{_W}(x) = \frac{\overline{F}(x)V_F(x)}{\mu_{_F}}, \tag{3.12}$$

where $V_F(x) = E(X|X > x)$ is the vitality function. Clearly, $\overline{G}_{_W}(x) \geq \overline{F}(x)$, for all $x \geq 0$. Now, if $x \geq \mu_F > 0$, then $g_W(x) = \frac{xf(x)}{\mu_F} \geq f(x)$, so that $(G_{_W}(x))^{-1} \geq (F(x))^{-1}$, and

$$h_F(x) = \frac{f(x)}{F(x)} \leq \frac{g_W(x)}{G_W(x)} = h_{G_W}(x). \tag{3.13}$$

Consequently,

$$h_{G_W}(x) \geq h_F(x), \tag{3.14}$$

for all $x \geq \mu_F > 0$. □

Proposition 3.4. *Let $G_{_W}$ be a weighted distribution function with increasing weight function $W(x)$, $x \geq 0$, and F the parent distribution function respectively, then $G_W <_{IFRP} F$.*

Proof. Note that $\lambda_{G_W}(x) \leq \lambda_F(x)$ for all $x \geq 0$, whenever $W(x)$ is an increasing weight function, where $\lambda_F(x) = f(x)/\overline{F}(x)$, $\overline{F}(x) > 0$. However, $\lambda_{G_W}(x) \leq \lambda_F(x)$ implies

$$\int_x^\infty \frac{dG_W(t)}{1 - G_W(t)} \leq \int_x^\infty \frac{dF(t)}{1 - F(t)}, \tag{3.15}$$

that is,

$$\frac{F(y_2) - F(x_2)}{F(y_1) - F(x_1)} \leq \frac{G_W(y_2) - G_W(x_2)}{G_W(y_1) - G_W(x_1)}, \tag{3.16}$$

for all $x_1 < x_2$, $x_1 < y_1$, $y_1 < y_2$, $x_2 < y_2$. Consequently,

$$\frac{\overline{F}(x_2) - \overline{F}(y_2)}{\overline{F}(x_1) - \overline{F}(y_1)} \leq \frac{\overline{G}_W(x_2) - \overline{G}_W(y_2)}{\overline{G}_W(x_1) - \overline{G}_W(y_1)}, \tag{3.17}$$

for all $x_1 < x_2$, $x_1 < y_1$, $y_1 < y_2$, $x_2 < y_2$, and $G_W <_{IFRP} F$. □

Example 3.2 (Pareto Distribution)**.** The Pareto distribution arises in reliability studies as a gamma mixture of the exponential distribution. See Harris [5]. The reliability and failure rate functions are given by, $\overline{F}(x;\alpha,\beta) = (1+\beta x)^{-\alpha}$, and $\lambda_F(x;\alpha,\beta) = \alpha\beta(1+\beta x)^{-1}$, for $x > 0$, $\beta > 0$, and $\alpha > 1$. Let $W(x) = x$, then the corresponding length-biased reliability and hazard functions are

$$\overline{G}_W(x;\alpha,\beta) = (1+\beta x)^{-\alpha}(1+\alpha\beta x) = \overline{F}(x;\alpha,\beta)[1+\alpha\beta x],$$

and

$$\lambda_{G_W}(x;\alpha,\beta) = \alpha\beta(1+\beta x)^{-1}\frac{\beta(\alpha-1)x}{1+\alpha\beta x},$$

for $x > 0$, $\beta > 0$, and $\alpha > 1$. Clearly $\overline{G}_W(x;\beta) \geq \overline{F}(x;\beta)$ and $\lambda_{G_W}(x;\beta) \leq \lambda_F(x;\beta)$. In view of Proposition 3.4, $G_W <_{IFRP} F$.

4. COMPARISONS AND MOMENT INEQUALITIES FOR RELIABILITY MEASURES

Let f_W and g_W be two non-negative integrable functions, possibly weighted probability density functions. A natural and common approach to ordering of distribution functions F and G with probability density functions f and g (pdf) respectively is concerned with the rate at which the density tends to zero at infinity. A pdf f is said to have a lighter tail than a pdf g if $f(x)/g(x) \longrightarrow 0$ as $x \to \infty$. Let g_l and g_W be the length-biased and weighted probability density functions respectively. The length-biased probability density function is a weighted probability density function with weight function $W(x) = x$. The corresponding length-biased reliability function is given by

$$\overline{G}_l(x) = \frac{\overline{F}(x)V_F(x)}{\mu_F}, \tag{4.1}$$

where $V_F(x) = E(X|X > x)$ is the vitality function and the length-biased probability density function (pdf) is given by $g_l(x) = \frac{xf(x)}{\mu_F}$. Note that $f(x)/g_l(x) = \mu_F/x \longrightarrow 0$ as $x \to \infty$, that is, the length-biased distribution has a heavier tail than the original distribution. Indeed,

$$\overline{G}_l(x) \geq \overline{F}(x), \tag{4.2}$$

for all $x \geq 0$.

Theorem 4.1. *Let G_W be a weighted distribution function with weight function $W(x)$, $x \geq 0$, and F the parent distribution function. If $W(x) = x$ and $E_F X^k \leq E_F(X^{k+1})/\mu_F$ for all $k \geq 0$, then*

$$\Psi_{G_W}(t) = \int_0^\infty e^{tx} dG_W(x) \geq \Psi_F(t), \tag{4.3}$$

for all $t \geq 0$.

Proof.

$$\begin{aligned}
\Psi_{G_W}(t) &= \int_0^\infty e^{tx} dG_W(x) = \int_0^\infty \sum_{k=0}^\infty \frac{(tx)^k}{k!} dG_W(x) \\
&= \sum_{k=0}^\infty \frac{t^k}{k!} E_{G_W} X^k = \sum_{k=0}^\infty \frac{t^k}{k!} \frac{E_F X^{k+1}}{\mu_F} \\
&\geq \sum_{k=0}^\infty \frac{t^k}{k!} E_F X^k = \int_0^\infty \sum_{k=0}^\infty \frac{(tx)^k}{k!} dF(x) \\
&= \Psi_F(t), \qquad (4.4)
\end{aligned}$$

for all $t \geq 0$. $\square$

Example 4.1 (Gamma Distribution)**.** The gamma distribution is used to model lifetimes of various practical situations, including lengths of time between catastrophic events, and lengths of time between emergency arrivals at a hospital. The pdf is given by,

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \tag{4.5}$$

for $x > 0$, $\alpha > 0$, and $\beta > 0$. Clearly, $\Psi_F(t) = \left(\frac{\beta}{\beta-t}\right)^{\alpha}$, for $t < \beta$. The length-biased gamma pdf is given by,

$$g_l(x;\alpha,\beta) = \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} x^{\alpha+1-1} e^{-\beta x}, \tag{4.6}$$

for $x > 0$, $\alpha > 0$, and $\beta > 0$. Note that $\Psi_{G_W}(t) = \left(\frac{\beta}{\beta-t}\right)^{\alpha+1}$, for $t < \beta$. In view of Theorem 4.1, we get that $\Psi_{G_W}(t) \geq \Psi_F(t)$ for all $t \geq 0$.

Theorem 4.2. *Let G_W be a weighted distribution function with increasing weight function $W(x)$, $x \geq 0$, and F the parent distribution function. If ψ is a non-negative and non-decreasing function on $(0,\infty)$, then*

$$\int_0^{\infty} \psi^p(x) G_W^k(x)dx \leq \int_0^{\infty} \psi^p(x) F^k(x)dx, \tag{4.7}$$

for all $t \geq 0$ and $p > 0$.

Proof. Note that $G_W(x) \leq F(x)$ for all $x \geq 0$, whenever $W(x)$ is increasing in x. For any integer $k \geq 1$, $G_W^k(x) \leq F^k(x)$. It follows therefore that

$$\int_t^{\infty} \psi^p(x) G_W^k(x)dx \leq \int_t^{\infty} \psi^p(x) F^k(x)dx, \tag{4.8}$$

for all $t \geq 0$ and $p > 0$. The result follows by letting $t \to 0^+$. $\square$

Theorem 4.3. *Let $W(x) = x \geq \mu_F$ and $\overline{G_l}$ the length-biased reliability function, then*

$$\int_0^{\infty} x^{p-1}\overline{G_l}(x)dG_l(x) \geq \int_0^{\infty} x^{p-1}\overline{F}(x)dF(x), \tag{4.9}$$

for all $t \geq 0$.

Proof. Note that for $x \geq \mu_F > 0$, $\overline{G_l}(x) \geq \overline{F}(x)$ and $g_l(x) \geq f(x)$. So that $x^{p-1}\overline{G_l}(x) \geq x^{p-1}\overline{F}(x)$, and

$$\int_t^{\infty} x^{p-1}\overline{G_l}(x)dG_l(x) \geq \int_t^{\infty} x^{p-1}\overline{F}(x)dF(x), \tag{4.10}$$

Consequently,

$$\int_0^{\infty} x^{p-1}\overline{G_l}(x)dG_l(x) \geq \int_0^{\infty} x^{p-1}\overline{F}(x)dF(x), \tag{4.11}$$

by letting $t \to 0^+$. $\square$

Corollary 4.4. *Let G_l be the length-biased distribution function, then under the conditions of Theorem 4.2,*

$$\int_0^{\infty} f^2(x)dx \leq \int_0^{\infty} g_l^2(x)dx, \tag{4.12}$$

where $g_l(x) = \frac{xf(x)}{\mu_F}$ is the length-biased probability density function, and $0 < \mu_F < \infty$.

Example 4.2 (Log-logistic Distribution)**.** The log-logistic distribution is a useful model that provides a good fit to a wide variety of data, including mortality, precipitation, and stream flow. The log-logistic distribution is mathematically tractable and provides a reasonably good alternative to the Weibull distribution. The reliability function is given by,

$$\overline{F}(x;\alpha,\beta) = \frac{1}{1+\left(\frac{x}{\alpha}\right)^{\beta}},$$

for $x > 0$, $\alpha > 0$, and $\beta > 1$. The hazard function is

$$\lambda_F(x;\alpha,\beta) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{1+\left(\frac{x}{\alpha}\right)^{\beta}},$$

for $x > 0$, $\alpha > 0$, and $\beta > 1$. The PWM with $l = 1$, $j = s$, and $k = 0$ is

$$\begin{aligned} E[X(F(X))^s(\overline{F}(X))^0] &= \int_0^\infty x \frac{\left(\frac{x}{\alpha}\right)^{\beta s}}{[1+\left(\frac{x}{\alpha}\right)^{\beta}]^s} \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{[1+\left(\frac{x}{\alpha}\right)^{\beta}]^2} dx \\ &= \frac{\alpha\Gamma\left(s+1+\frac{1}{\beta}\right)\Gamma\left(1-\frac{1}{\beta}\right)}{\Gamma(s+2)}, \end{aligned} \tag{4.13}$$

for $s = 0, 1, 2, \dots$, and $\beta > 1$. Applying Definition 2.2, and Theorem 4.2, for fixed α, we have $F \geq_{PWM(1,s,0)} G$, if and only if

$$\Gamma\left(s+1+\frac{1}{\beta_1}\right)\Gamma\left(1-\frac{1}{\beta_1}\right) \geq \Gamma\left(s+1+\frac{1}{\beta_2}\right)\Gamma\left(1-\frac{1}{\beta_2}\right).$$

In particular, $F \geq_{PWM(1,1,0)} G$, if and only if

$$\frac{1+\beta_1}{2\beta_1}\mu_F \geq \frac{1+\beta_2}{2\beta_2}\mu_G,$$

where

$$\mu_F = \int_0^\infty \overline{F}(y:\alpha,\beta)dy = \alpha\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1-\frac{1}{\beta}\right) = \frac{\alpha\pi}{\beta}\left(\sin\left(\frac{\pi}{\beta}\right)\right)^{-1}.$$

For fixed β, $F \geq_{PWM(1,s,0)} G$, if and only if $\alpha_1 \geq \alpha_2$.

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