journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 1, Article 24, 7 pp.

INEQUALITIES FOR THE SMALLEST ZEROS OF LAGUERRE POLYNOMIALS AND THEIR q-ANALOGUES

DHARMA P. GUPTA AND MARTIN E. MULDOON

DEPARTMENT OF MATHEMATICS AND STATISTICS
YORK UNIVERSITY, TORONTO, ONTARIO M3J 1P3
CANADA

muldoon@yorku.ca

Received 03 August, 2006; accepted 28 February, 2007
Communicated by D. Stefanescu

ABSTRACT. We present bounds and approximations for the smallest positive zero of the Laguerre polynomial $L_n^{(\alpha)}(x)$ which are sharp as $\alpha \to -1^+$. We indicate the applicability of the results to more general functions including the q-Laguerre polynomials.

Key words and phrases: Laguerre polynomials, Zeros, q-Laguerre polynomials, Inequalities.

2000 *Mathematics Subject Classification.* 33C45, 33D45.

1. INTRODUCTION

The Laguerre polynomials are given by the explicit formula [13]

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \binom{n+\alpha}{n} \left[1 + \sum_{k=1}^{n} \frac{\binom{n}{k}(-x)^k}{(\alpha+1)_k} \right], \tag{1.1}$$

valid for all $x, \alpha \in \mathbf{C}$ (with the understanding that the second sum is interpreted as a limit when α is a negative integer), where

$$(\alpha+1)_k = (\alpha+1)(\alpha+2)\cdots(\alpha+k).$$

They satisfy the three term recurrence relation

$$xL_n^{(\alpha)}(x) = -(n+1)L_{n+1}^{(\alpha)}(x) + (\alpha+2n+1)L_n^{(\alpha)}(x) - (\alpha+n)L_{n-1}^{(\alpha)}(x), \tag{1.2}$$

with initial conditions $L_{-1}^{(\alpha)}(x) = 0$ and $L_0^{(\alpha)}(x) = 1$ for all complex α and x. When $\alpha > -1$, this recurrence relation is positive definite and the Laguerre polynomials are orthogonal with respect to the weight function $x^\alpha e^{-x}$ on $[0, +\infty)$. From this it follows that the zeros of $L_n^{(\alpha)}(x)$ are positive and simple, that they are increasing functions of α and they interlace with the zeros of $L_{n+1}^{(\alpha)}(x)$ [13]. When $\alpha \le -1$ we no longer have orthogonality with respect to a positive weight function and the zeros can be non-real and non-simple.

210-06

Our purpose here is to present bounds and approximations for the smallest positive zero of $L_n^{(\alpha)}(x),\ \alpha > -1$, which are sharp as $\alpha \to -1^+$. The same kinds of results hold for more general functions including the q-Laguerre polynomials $L_n^{(\alpha)}(x;q),\ 0 < q < 1$ which satisfy $L_n^{(\alpha)}(x(1-q)^{-1};q) \to L_n^{(\alpha)}(x)$ as $q \to 1^-$.

2. Smallest Zeros of Laguerre Polynomials

In the case $\alpha > -1$, successively better upper and lower bounds for the zeros of Laguerre polynomials can be obtained by the method outlined in [7]. They follow from the knowledge of the coefficients in the explicit expression for $L_n^{(\alpha)}(x)$. However, they are obtained more conveniently by noting that $y = L_n^{(\alpha)}(x)$ satisfies the differential equation

$$xy'' + (\alpha+1-x)y' + ny = 0 \tag{2.1}$$

and hence that $u = y'/y$ satisfies the Riccati type equation

$$xu'^2 + (\alpha+1-x)u + n = 0. \tag{2.2}$$

If we write

$$y = \binom{n+\alpha}{n} \prod_{i=1}^{n} \left(1 - \frac{x}{x_i}\right), \tag{2.3}$$

where the zeros x_i satisfy $0 < x_1 < x_2 < \cdots$, then

$$u = \sum_{i=1}^{n} \frac{1}{x - x_i} = -\sum_{k=0}^{\infty} S_{k+1} x^k, \tag{2.4}$$

where

$$S_k = \sum_{i=1}^{n} x_i^{-k}, \qquad k = 1, 2, \dots. \tag{2.5}$$

Substituting in (2.2), we get

$$\sum_{k=1}^{\infty} x^k \left(S_k + \sum_{i=1}^{k} S_i S_{k-i+1}\right) - (\alpha+k+1) \sum_{k=0}^{\infty} S_{k+1} x^k + n = 0, \tag{2.6}$$

from which it follows by comparing coefficients that

$$S_1 = \frac{n}{\alpha+1}, \qquad S_{k+1} = \frac{S_k + \sum_{i=1}^{k} S_i S_{k-i+1}}{\alpha+k+1}, \qquad k = 1, 2 \dots. \tag{2.7}$$

For the case $\alpha > -1$, the zeros are all positive and by the method outlined in [7, §3], we have

$$S_m^{-1/m} < x_1 < S_m/S_{m+1}, \quad m = 1, 2, \dots. \tag{2.8}$$

These upper and lower bounds give successively improving [7, §3] upper and lower bounds for x_1. For example, for $\alpha > -1$, $n \geq 2$, we get, for the smallest zero $x_1(\alpha)$,

$$\frac{1}{n} < \frac{x_1(\alpha)}{\alpha+1} < \frac{(\alpha+2)}{(\alpha+1+n)}, \tag{2.9}$$

$$\left[\frac{\alpha+2}{n(n+\alpha+1)}\right]^{\frac{1}{2}} < \frac{x_1(\alpha)}{\alpha+1} < \frac{(\alpha+3)}{(\alpha+1+2n)}, \tag{2.10}$$

where the upper bound recovers that in [13, (6.31.12)], and

$$\left[\frac{(\alpha+2)(\alpha+3)}{n(n+\alpha+1)(2n+\alpha+1)}\right]^{\frac{1}{3}} < \frac{x_1(\alpha)}{\alpha+1}$$
$$< \frac{(\alpha+2)(\alpha+4)(\alpha+2n+1)}{\alpha^3+4\alpha^2+5\alpha+2+5n\alpha^2+16n\alpha+11n+5n^2\alpha+11n^2}. \tag{2.11}$$

Further such bounds may be found but they become successively more complicated. From the higher estimates we can produce a series expansion valid for $-1 < \alpha < 0$. The first five terms, obtained with the help of MAPLE, are:

$$\begin{aligned} x_1(\alpha) = {} & \frac{\alpha+1}{n} + \frac{n-1}{2}\left(\frac{\alpha+1}{n}\right)^2 - \frac{n^2+3n-4}{12}\left(\frac{\alpha+1}{n}\right)^3 \\ & + \frac{7n^3+6n^2+23n-36}{144}\left(\frac{\alpha+1}{n}\right)^4 \\ & - \frac{293n^4+210n^3+235n^2+990n-1728}{8640}\left(\frac{\alpha+1}{n}\right)^5 + \cdots . \end{aligned} \tag{2.12}$$

It is known [13, Theorem 8.1.3] that

$$\lim_{n\to\infty} n^{-\alpha} L_n^{(\alpha)}\left(\frac{z}{n}\right) = z^{-\alpha/2} J_\alpha(2z^{1/2}), \tag{2.13}$$

and hence that $x_1 \sim j_{\alpha 1}^2/(4n)$ as $n \to \infty$, with the usual notation for zeros of Bessel functions. Hence we get

$$j_{\alpha 1}^2 \sim 4(\alpha+1)\left[1 + \frac{\alpha+1}{2} - \frac{(\alpha+1)^2}{12} + \frac{7(\alpha+1)^3}{144} - \frac{293(\alpha+1)^4}{8640} + \cdots\right], \tag{2.14}$$

which agrees with the expansion of [12] for $j_{\alpha 1}$.

It should be noted that the inequalities obtained here are particularly sharp for α close to -1 but not for large α. Krasikov [10] gives uniform bounds for the extreme zeros of Laguerre and other polynomials.

The series in (2.12) converges for $|\alpha+1| < 1$. This suggests that we consider the case $-2 < \alpha < -1$, when the zeros are still real but $x_1 < 0 < x_2 < x_3 < \cdots$ [13, Theorem 6.73]. In accordance with [7, Lemma 3.3], the inequalities for x_1 are changed, sometimes reversed. For example, we have, for $n \geq 2$,

$$\frac{1}{n} > \frac{x_1(\alpha)}{\alpha+1} > \left[\frac{\alpha+2}{n(n+\alpha+1)}\right]^{\frac{1}{2}}, \qquad -2 < \alpha < -1. \tag{2.15}$$

3. q EXTENSIONS

In extending the previous results, it is natural to consider some of the q-extensions of the Laguerre polynomials. For this purpose we need the standard notations [4, 9] for the basic hypergeometric functions:

$${}_1\phi_1\left(\begin{array}{c} a \\ b \end{array} \middle| \; q\,;\; z\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(b;q)_k} q^{\binom{n}{2}} \frac{(-z)^k}{(q;q)_k},$$

$${}_2\phi_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| \; q; z\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(c;q)_k} \frac{z^k}{(q;q)_k},$$

where $(a;q)_n$ denotes the q-shifted factorial

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

so that $(1-q)^{-k}(q^\alpha;q)_k \to (\alpha)_k$ as $q \to 1^-$.

We seek appropriate q-analogues of the results of Section 2, which will reduce to those results when $q \to 1$. Different q-analogues are possible; we have found that a good approach is through what we now call the *little q-Jacobi polynomials* introduced by W. Hahn [6] (see also [9, (3.12.1), p.192]):

$$p_n(x;a,b;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \;\middle|\; q; xq\right). \tag{3.1}$$

Hahn proved the discrete orthogonality [4, (7.3.4)]

$$\begin{aligned}\sum_{k=0}^{\infty} p_m(q^k;a,b;q)p_n(q^k;a,b;q)\frac{(bq;q)_k}{(q;q)_k}(aq)^k \\ = \frac{(q;q)_n(1-abq)(bq;q)_n(abq^2;q)_\infty(aq)^{-n}}{(abq;q)_n(1-abq^{2n+1})(aq;q)_n(aq;q)_\infty}\delta_{m,n},\end{aligned} \tag{3.2}$$

where $0 < q,\ aq < 1$ and $bq < 1$. In this case the orthogonality measure is positive and the zeros of the polynomials lie in $(0,\infty)$. For a detailed study of the polynomials $p_n(x;a,b;q)$, we refer to the article of Andrews and Askey [2], and the book of Gasper and Rahman [4, §7.3]. In general, the polynomials give a q-analogue of the Jacobi polynomials but, for $b < 0$, they give a q-analogue of the Laguerre polynomials; see (3.6) below.

From (3.1), we get [4, Ex.7.43, p. 210]

$$\begin{aligned}\lim_{b\to\infty} p_n\left(-\frac{(1-q)x}{bq}; q^\alpha, b; q\right) &= {}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \;\middle|\; q;\quad -x(1-q)q^{n+\alpha+1}\right) \\ &= \frac{L_n^{(\alpha)}(x;q)}{L_n^{(\alpha)}(0;q)},\end{aligned} \tag{3.3}$$

with the notation of [11, 8, 4] for the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$. This definition

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \;\middle|\; q;\quad -x(1-q)q^{n+\alpha+1}\right), \tag{3.4}$$

gives [11]

$$\lim_{q\to 1^-} L_n^{(\alpha)}((1-q)^{-1}x;q) = L_n^{(\alpha)}(x). \tag{3.5}$$

(We remark that the definition of $L_n^{(\alpha)}(x;q)$ given in [9, p. 108] has x replaced by $(1-q)^{-1}x$.)

On the other hand, again from (3.1), we have

$$\lim_{q\to 1^-} p_n\left((1-q)x; q^\alpha, -q^\beta; q\right) = \sum_{k=0}^{n} \frac{(-n)_k(2x)^k}{(1+\alpha)_k k!} = \frac{L_n^{(\alpha)}(2x)}{L_n^{(\alpha)}(0)}. \tag{3.6}$$

This is reported in [4, (7.3.9)], but with a small error, $L_n^{(\alpha)}(x)$ rather than $L_n^{(\alpha)}(2x)$ on the right-hand side. The relation (3.6) shows that little q-Jacobi polynomials also provide a q-analogue of the Laguerre polynomials. However, we use the name "q-Laguerre polynomials" only for $L_n^{(\alpha)}(x;q)$, as defined in (3.4).

The Wall, or little q-Laguerre, polynomials $W_n(x;a;q)$ ([3], [9, 3.20.1]) are the particular case $b=0$ of $p_n(x;a,b;q)$:

$$W_n(x;a;q) = p_n(x;a,0;q) = {}_2\phi_1\left(\begin{array}{c} q^{-n},0 \\ a,q \end{array}\middle|\; q\;;\;qx\right), \tag{3.7}$$

where $0<q<1$ and $0<aq<1$. From the Wall polynomials, we can again obtain the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ using [9, p. 108] (changed to our notation):

$$W_n(x;q^{-\alpha}\,\big|q^{-1}) = \frac{(q;q)_n}{(q^{\alpha+1};q)_n} L_n^{(\alpha)}((1-q)^{-1}x;q). \tag{3.8}$$

From the relation (3.6), we have

$$\lim_{q\to 1^{-1}} W_n((1-q)x;q^\alpha;q) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}. \tag{3.9}$$

Here we present in diagrammatic form the relations between the various polynomials considered:

$$\begin{tikzcd}
 & p_n(x;a,b;q) \arrow[d, "(3.3)"] \arrow[dr, "b=0"] \arrow[ddl, "(3.6)"'] & \\
 & L_n^{(\alpha)}(x;q) \arrow[dl, "(3.5)"] & W_n(x;a;q) \arrow[l, "(3.8)"'] \arrow[ddll, phantom] \arrow[dll, "(3.9)"] \\
L_n^{(\alpha)}(x) & &
\end{tikzcd}$$

4. BOUNDS FOR q EXTENSIONS

In finding bounds for the zeros of these polynomials, we no longer have available the differential equations method used in Section 2. However we can still apply the Euler method, described in [7], based on the explicit expressions for the coefficients in the polynomials to obtain bounds for the smallest positive zero of the little q-Jacobi polynomials. We consider the function

$$p_n((1-q)x;a,b;q) = 1 + \sum_{k=1}^{\infty} a_k x^k.$$

where

$$a_k = \frac{(q^{-n};q)_k (abq^{n+1};q)_k}{(q;q)_k (aq;q)_k} q^k (1-q)^k \tag{4.1}$$

We can find $S_1, S_2, \dots$, defined as in (2.5), in terms of $a_1, a_2, \dots$. As in Section 2, as long as $0<q, aq<1,\ b<1$, we have $0<x_1<x_2<\cdots$. Using [7, (3.4),(3.7)], we have $S_1=-a_1$, and

$$S_n = -na_n - \sum_{i=1}^{n-1} a_i S_{n-i}.$$

Using inequalities (2.8) for $m=1$, we obtain the following bounds for the smallest positive zeros $x_1(a,b;q)$ of $p_n(x(1-q);a,b;q)$, where we assume that $0<q,aq<1,\ b<1$:

$$\frac{1}{(1-q^n)(1-abq^{n+1})} < \frac{x_1(a,b;q)}{q^{n-1}(1-aq)} < \frac{(1+q)(1-aq^2)}{(1-q)P}, \tag{4.2}$$

where

$$P = 1 + aq^2 + q^n - 2aq^{n+1} - aq^{n+2} - abq^{n+1} \\ - 2abq^{n+2} + abq^{2n+1} + a^2bq^{n+3} + a^2bq^{2n+3}.$$

For $m = 2$ we get improved lower and upper bounds:

$$\left[\frac{(1+q)(1-aq^2)}{(1-q^n)(1-q)(1-abq^{n+1})P}\right]^{1/2} < \frac{x_1(a,b;q)}{q^{n-1}(1-aq)} \\ < \frac{(1-q^3)(1-aq^3)P}{\sigma_1+\sigma_2+\sigma_3}, \tag{4.3}$$

where

$$\sigma_1 = 3q^3(1-aq)^2(1-q^{n-1})(1-q^{n-2})(1-abq^{n+2})(1-abq^{n+3}), \tag{4.4}$$

$$\sigma_2 = (1-q^3)(1-aq)(1-aq^3)(1-q^n)(1-abq^{n+1})P \tag{4.5}$$

and

$$\sigma_3 = -q(1+q+q^2)(1-aq)(1-aq^3)(1-q^n)(1-q^{n-1})(1-abq^{n+1})(1-abq^{n+2}). \tag{4.6}$$

As observed earlier, with the help of (3.6) we should be able to derive corresponding inequalities for zeros of $L_n^{(\alpha)}(x)$. If we then make the replacements $a \to q^\alpha,\ b \to -q^\beta$ in the modified (4.2) and (4.3) we recover the inequalities (2.9) and (2.10) for $L_n^{(\alpha)}(x)$ by taking limits $q \to 1^-$.

For the case $0 < q, aq < 1$, the bounds for the smallest zero $x_1(a;q)$ of the Wall polynomial

$$W_n((1-q)x;a;q) = {}_2\phi_1(q^{-n},0;aq;q(1-q)x), \tag{4.7}$$

are obtained from (4.2) and (4.3) by substituting $b = 0$.

Finally, we record the bounds for the smallest zero $x_1(\alpha;q)$ for the q-Laguerre polynomial $L_n^{(\alpha)}(x;q)$. This can be done either by a direct calculation from the ${}_1\phi_1$ series in (3.3) or by obtaining them as a limiting case of little q-Jacobi polynomials, employing (3.7), (4.2) and (4.3). We obtain, for $0 < q < 1,\ \alpha > -1$:

$$\frac{1}{1-q^n} < \frac{q^{\alpha+1}x_1(\alpha;q)}{1-q^{\alpha+1}} < \frac{(1+q)(1-q^{\alpha+2})}{(1-q)R}, \tag{4.8}$$

where $R = 1 + 2q - q^{n+\alpha+2} - q^n - q^{\alpha+2}$, and

$$\left[\frac{(1+q)(1-q^{\alpha+2})}{(1-q)(1-q^n)R}\right]^{\frac{1}{2}} < \frac{q^{\alpha+1}x_1(\alpha;q)}{1-q^{\alpha+1}} < \frac{(1-q^{\alpha+3})(1-q)(1+q+q^2)R}{T} \tag{4.9}$$

with

$$T = 3q^6(1-q^{n-1})(1-q^{n-2})(1-q^{\alpha+1})^2 \\ + (1-q^n)(1-q)(1-q^{\alpha+3})(1+q+q^2)R \\ - q^2(1-q^n)(1-q^{n-1})(1-q^{\alpha+1})(1-q^{\alpha+3})(1+q+q^2). \tag{4.10}$$

From (4.8) and (4.9) we can recover the bounds (2.9) and (2.10) for the smallest zero x_1 of Laguerre polynomials $L_n^{(\alpha)}(x)$ by taking limits $q \to 1^-$.

REFERENCES

[1] S. AHMED AND M.E. MULDOON, Reciprocal power sums of differences of zeros of special functions, *SIAM J. Math. Anal.*, **14** (1983), 372–382.

[2] G.E. ANDREWS AND R.A. ASKEY, Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials, *Higher Combinatorics* (M. Aigner, ed.), Reidel, Boston, Mass. (1977), 3–26.

[3] T.S. CHIHARA, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[4] G. GASPER AND M. RAHMAN, *Basic Hypergeometric Series*, 2nd ed., Cambridge University Press, 2004.

[5] D.P. GUPTA AND M.E. MULDOON, Riccati equations and convolution formulae for functions of Rayleigh type, *J. Phys. A: Math. Gen.*, **33** (2000), 1363–1368.

[6] W. HAHN, Über Orthogonalpolynome, die q-Differenzengleichungen genügen, *Math. Nachr.*, **2** (1949), 4–34.

[7] M.E.H. ISMAIL AND M.E. MULDOON, Bounds for the small real and purely imaginary zeros of Bessel and related functions, *Meth. Appl. Anal.*, **2** (1995), 1–21.

[8] M.E.H. ISMAIL AND M. RAHMAN, The q-Laguerre polynomials and related moment problems, *J. Math. Anal. Appl.*, **218** (1998), 155–174.

[9] R. KOEKOEK AND R.F. SWARTTOUW, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, *Faculty of Technical Mathematics and Informatics, Delft University of Technology, Report* 98-17, 1998. http://aw.twi.tudelft.nl/$\sim$koekoek/askey.html

[10] I. KRASIKOV, Bounds for zeros of the Laguerre polynomials, *J. Approx. Theory*, **121** (2003), 287–291.

[11] D.S. MOAK, The q-analogue of the Laguerre polynomials, *J. Math. Anal. Appl.*, **81** (1981), 20–47.

[12] R. PIESSENS, A series expansion for the first positive zero of the Bessel function, *Math. Comp.*, **42** (1984), 195–197.

[13] G. SZEGŐ, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, 4th ed., Amer. Math. Soc., Providence, R.I., 1975.