



# ON AN INTEGRATION-BY-PARTS FORMULA FOR MEASURES

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*Key words:* Integration-by-parts formula, Harmonic sequences, Inequalities.

*Abstract:* An integration-by-parts formula, involving finite Borel measures supported by intervals on real line, is proved. Some applications to Ostrowski-type and Grüss-type inequalities are presented.

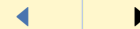
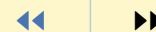
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**Integration-by-parts Formula**  
A. Čivljak, Lj. Dedić and M. Matić  
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## Integration-by-parts Formula

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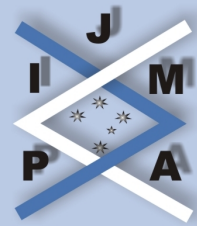
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## 1. Introduction

In the paper [4], S.S. Dragomir introduced the notion of a  $w_0$ -Appell type sequence of functions as a sequence  $w_0, w_1, \dots, w_n$ , for  $n \geq 1$ , of real absolutely continuous functions defined on  $[a, b]$ , such that

$$w'_k = w_{k-1}, \text{ a.e. on } [a, b], \quad k = 1, \dots, n.$$

For such a sequence the author proved a generalisation of Mitrinović-Pečarić integration-by-parts formula

$$(1.1) \quad \int_a^b w_0(t)g(t)dt = A_n + B_n,$$

where

$$A_n = \sum_{k=1}^n (-1)^{k-1} [w_k(b)g^{(k-1)}(b) - w_k(a)g^{(k-1)}(a)]$$

and

$$B_n = (-1)^n \int_a^b w_n(t)g^{(n)}(t)dt,$$

for every  $g : [a, b] \rightarrow \mathbb{R}$  such that  $g^{(n-1)}$  is absolutely continuous on  $[a, b]$  and  $w_n g^{(n)} \in L_1[a, b]$ . Using identity (1.1) the author proved the following inequality

$$(1.2) \quad \left| \int_a^b w_0(t)g(t)dt - A_n \right| \leq \|w_n\|_p \|g^{(n)}\|_q,$$

for  $w_n \in L_p[a, b]$ ,  $g^{(n)} \in L_q[a, b]$ , where  $p, q \in [1, \infty]$  and  $1/p + 1/q = 1$ , giving explicitly some interesting special cases. For some similar inequalities, see also [5],

[6] and [7]. The aim of this paper is to give a generalization of the integration-by-parts formula (1.1), by replacing the  $w_0$ -Appell type sequence of functions by a more general sequence of functions, and to generalize inequality (1.2), as well as to prove some related inequalities.



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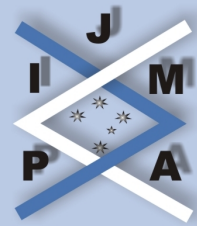
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## 2. Integration-by-parts Formula for Measures

For  $a, b \in \mathbb{R}$ ,  $a < b$ , let  $C[a, b]$  be the Banach space of all continuous functions  $f : [a, b] \rightarrow \mathbb{R}$  with the max norm, and  $M[a, b]$  the Banach space of all real Borel measures on  $[a, b]$  with the total variation norm. For  $\mu \in M[a, b]$  define the function  $\check{\mu}_n : [a, b] \rightarrow \mathbb{R}$ ,  $n \geq 1$ , by

$$\check{\mu}_n(t) = \frac{1}{(n-1)!} \int_{[a,t]} (t-s)^{n-1} d\mu(s).$$

Note that

$$\check{\mu}_n(t) = \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} \check{\mu}_1(s) ds, \quad n \geq 2$$

and

$$|\check{\mu}_n(t)| \leq \frac{(t-a)^{n-1}}{(n-1)!} \|\mu\|, \quad t \in [a, b], \quad n \geq 1.$$

The function  $\check{\mu}_n$  is differentiable,  $\check{\mu}'_n(t) = \check{\mu}_{n-1}(t)$  and  $\check{\mu}_n(a) = 0$ , for every  $n \geq 2$ , while for  $n = 1$

$$\check{\mu}_1(t) = \int_{[a,t]} d\mu(s) = \mu([a, t]),$$

which means that  $\check{\mu}_1(t)$  is equal to the distribution function of  $\mu$ . A sequence of functions  $P_n : [a, b] \rightarrow \mathbb{R}$ ,  $n \geq 1$ , is called a  $\mu$ -harmonic sequence of functions on  $[a, b]$  if

$$P'_n(t) = P_{n-1}(t), \quad n \geq 2; \quad P_1(t) = c + \check{\mu}_1(t), \quad t \in [a, b],$$

for some  $c \in \mathbb{R}$ . The sequence  $(\check{\mu}_n, n \geq 1)$  is an example of a  $\mu$ -harmonic sequence of functions on  $[a, b]$ . The notion of a  $\mu$ -harmonic sequence of functions has been introduced in [2]. See also [1].

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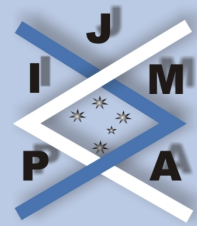
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*Remark 1.* Let  $w_0 : [a, b] \rightarrow \mathbb{R}$  be an absolutely integrable function and let  $\mu \in M[a, b]$  be defined by

$$d\mu(t) = w_0(t)dt.$$

If  $(P_n, n \geq 1)$  is a  $\mu$ -harmonic sequence of functions on  $[a, b]$ , then  $w_0, P_1, \dots, P_n$  is a  $w_0$ -Appell type sequence of functions on  $[a, b]$ .

For  $\mu \in M[a, b]$  let  $\mu = \mu_+ - \mu_-$  be the Jordan-Hahn decomposition of  $\mu$ , where  $\mu_+$  and  $\mu_-$  are orthogonal and positive measures. Then we have  $|\mu| = \mu_+ + \mu_-$  and

$$\|\mu\| = |\mu|([a, b]) = \|\mu_+\| + \|\mu_-\| = \mu_+([a, b]) + \mu_-([a, b]).$$

The measure  $\mu \in M[a, b]$  is said to be balanced if  $\mu([a, b]) = 0$ . This is equivalent to

$$\|\mu_+\| = \|\mu_-\| = \frac{1}{2} \|\mu\|.$$

Measure  $\mu \in M[a, b]$  is called  $n$ -balanced if  $\tilde{\mu}_n(b) = 0$ . We see that a 1-balanced measure is the same as a balanced measure. We also write

$$m_k(\mu) = \int_{[a,b]} t^k d\mu(t), \quad k \geq 0$$

for the  $k$ -th moment of  $\mu$ .

**Lemma 2.1.** For every  $f \in C[a, b]$  and  $\mu \in M[a, b]$  we have

$$\int_{[a,b]} f(t) d\tilde{\mu}_1(t) = \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\})f(a).$$

*Proof.* Define  $I, J : C[a, b] \times M[a, b] \rightarrow \mathbb{R}$  by

$$I(f, \mu) = \int_{[a,b]} f(t) d\tilde{\mu}_1(t)$$

and

$$J(f, \mu) = \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\})f(a).$$

Then  $I$  and  $J$  are continuous bilinear functionals, since

$$|I(f, \mu)| \leq \|f\| \|\mu\|, \quad |J(f, \mu)| \leq 2 \|f\| \|\mu\|.$$

Let us prove that  $I(f, \mu) = J(f, \mu)$  for every  $f \in C[a, b]$  and every discrete measure  $\mu \in M[a, b]$ .

For  $x \in [a, b]$  let  $\mu = \delta_x$  be the Dirac measure at  $x$ , i.e. the measure defined by

$$\int_{[a,b]} f(t) d\delta_x(t) = f(x).$$

If  $a < x \leq b$ , then

$$\check{\mu}_1(t) = \delta_x([a, t]) = \begin{cases} 0, & a \leq t < x \\ 1, & x \leq t \leq b \end{cases}$$

and by a simple calculation we have

$$\begin{aligned} I(f, \delta_x) &= \int_{[a,b]} f(t) d\check{\mu}_1(t) = f(x) = \int_{[a,b]} f(t) d\delta_x(t) - 0 \\ &= \int_{[a,b]} f(t) d\delta_x(t) - \delta_x(\{a\})f(a) = J(f, \delta_x). \end{aligned}$$

Similarly, if  $x = a$ , then

$$\check{\mu}_1(t) = \delta_a([a, t]) = 1, \quad a \leq t \leq b$$



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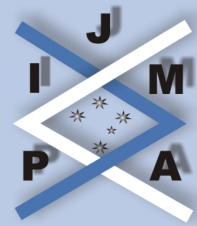


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and by a similar calculation we have

$$\begin{aligned} I(f, \delta_a) &= \int_{[a,b]} f(t) d\tilde{\mu}_1(t) = 0 = f(a) - f(a) \\ &= \int_{[a,b]} f(t) d\delta_a(t) - \delta_a(\{a\})f(a) = J(f, \delta_x). \end{aligned}$$

Therefore, for every  $f \in C[a, b]$  and every  $x \in [a, b]$  we have  $I(f, \delta_x) = J(f, \delta_x)$ .  
Every discrete measure  $\mu \in M[a, b]$  has the form

$$\mu = \sum_{k \geq 1} c_k \delta_{x_k},$$

where  $(c_k, k \geq 1)$  is a sequence in  $\mathbb{R}$  such that

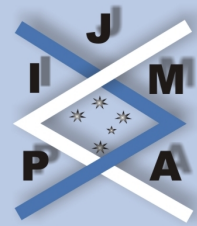
$$\sum_{k \geq 1} |c_k| < \infty,$$

and  $\{x_k; k \geq 1\}$  is a subset of  $[a, b]$ .

By using the continuity of  $I$  and  $J$ , for every  $f \in C[a, b]$  and every discrete measure  $\mu \in M[a, b]$  we have

$$\begin{aligned} I(f, \mu) &= I\left(f, \sum_{k \geq 1} c_k \delta_{x_k}\right) = \sum_{k \geq 1} c_k I(f, \delta_{x_k}) \\ &= \sum_{k \geq 1} c_k J(f, \delta_{x_k}) = J\left(f, \sum_{k \geq 1} c_k \delta_{x_k}\right) \\ &= J(f, \mu). \end{aligned}$$





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Since the Banach subspace  $M[a, b]_d$  of all discrete measures is weakly\* dense in  $M[a, b]$  and the functionals  $I(f, \cdot)$  and  $J(f, \cdot)$  are also weakly\* continuous we conclude that  $I(f, \mu) = J(f, \mu)$  for every  $f \in C[a, b]$  and  $\mu \in M[a, b]$ .  $\square$

**Theorem 2.2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  has bounded variation for some  $n \geq 1$ . Then for every  $\mu$ -harmonic sequence  $(P_n, n \geq 1)$  we have

$$(2.1) \quad \int_{[a,b]} f(t) d\mu(t) = \mu(\{a\})f(a) + S_n + R_n,$$

where

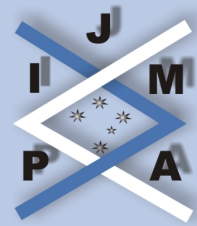
$$(2.2) \quad S_n = \sum_{k=1}^n (-1)^{k-1} [P_k(b)f^{(k-1)}(b) - P_k(a)f^{(k-1)}(a)]$$

and

$$(2.3) \quad R_n = (-1)^n \int_{[a,b]} P_n(t) df^{(n-1)}(t).$$

*Proof.* By partial integration, for  $n \geq 2$ , we have

$$\begin{aligned} R_n &= (-1)^n \int_{[a,b]} P_n(t) df^{(n-1)}(t) \\ &= (-1)^n [P_n(b)f^{(n-1)}(b) - P_n(a)f^{(n-1)}(a)] \\ &\quad - (-1)^n \int_{[a,b]} P_{n-1}(t) f^{(n-1)}(t) dt \\ &= (-1)^n [P_n(b)f^{(n-1)}(b) - P_n(a)f^{(n-1)}(a)] + R_{n-1}. \end{aligned}$$



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By Lemma 2.1 we have

$$\begin{aligned} R_1 &= - \int_{[a,b]} P_1(t)df(t) \\ &= - [P_1(b)f(b) - P_n(a)f(a)] + \int_{[a,b]} f(t)dP_1(t) \\ &= - [P_1(b)f(b) - P_n(a)f(a)] + \int_{[a,b]} f(t)d\check{\mu}_1(t) \\ &= - [P_1(b)f(b) - P_n(a)f(a)] + \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a). \end{aligned}$$

Therefore, by iteration, we have

$$R_n = \sum_{k=1}^n (-1)^k [P_k(b)f^{(k-1)}(b) - P_k(a)f^{(k-1)}(a)] + \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a),$$

which proves our assertion.  $\square$

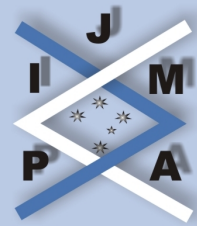
*Remark 2.* By Remark 1 we see that identity (2.1) is a generalization of the integration-by-parts formula (1.1).

**Corollary 2.3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  has bounded variation for some  $n \geq 1$ . Then for every  $\mu \in M[a, b]$  we have

$$\int_{[a,b]} f(t)d\mu(t) = \check{S}_n + \check{R}_n,$$

where

$$\check{S}_n = \sum_{k=1}^n (-1)^{k-1} \check{\mu}_k(b) f^{(k-1)}(b)$$



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and

$$\check{R}_n = (-1)^n \int_{[a,b]} \check{\mu}_n(t) df^{(n-1)}(t).$$

*Proof.* Apply the theorem above for the  $\mu$ -harmonic sequence  $(\check{\mu}_n, n \geq 1)$  and note that  $\check{\mu}_n(a) = 0$ , for  $n \geq 2$ .  $\square$

**Corollary 2.4.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  has bounded variation for some  $n \geq 1$ . Then for every  $x \in [a, b]$  we have

$$f(x) = \sum_{k=1}^n \frac{(x-b)^{k-1}}{(k-1)!} f^{(k-1)}(b) + R_n(x),$$

where

$$R_n(x) = \frac{(-1)^n}{(n-1)!} \int_{[x,b]} (t-x)^{n-1} df^{(n-1)}(t).$$

*Proof.* Apply Corollary 2.3 for  $\mu = \delta_x$  and note that in this case

$$\check{\mu}_k(t) = \frac{(t-x)^{k-1}}{(k-1)!}, \quad x \leq t \leq b, \quad \text{and} \quad \check{\mu}_k(t) = 0, \quad a \leq t < x,$$

for  $k \geq 1$ .  $\square$

**Corollary 2.5.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  has bounded variation for some  $n \geq 1$ . Further, let  $(c_m, m \geq 1)$  be a sequence in  $\mathbb{R}$  such that

$$\sum_{m \geq 1} |c_m| < \infty$$

and let  $\{x_m; m \geq 1\} \subset [a, b]$ . Then

$$\sum_{m \geq 1} c_m f(x_m) = \sum_{m \geq 1} \sum_{k=1}^n c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) + \sum_{m \geq 1} c_m R_n(x_m),$$

where  $R_n(x_m)$  is from Corollary 2.4.

*Proof.* Apply Corollary 2.3 for the discrete measure  $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$ . □



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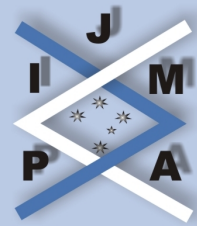
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### 3. Some Ostrowski-type Inequalities

In this section we shall use the same notations as above.

**Theorem 3.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  is  $L$ -Lipschitzian for some  $n \geq 1$ . Then for every  $\mu$ -harmonic sequence  $(P_n, n \geq 1)$  we have*

$$(3.1) \quad \left| \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\})f(a) - S_n \right| \leq L \int_a^b |P_n(t)| dt,$$

where  $S_n$  is defined by (2.2).

*Proof.* By Theorem 2.2 we have

$$|R_n| = \left| \int_{[a,b]} P_n(t) df^{(n-1)}(t) \right| \leq L \int_a^b |P_n(t)| dt,$$

which proves our assertion.  $\square$

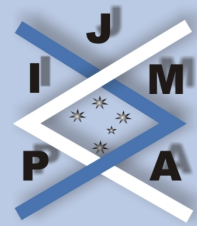
**Corollary 3.2.** *If  $f$  is  $L$ -Lipschitzian, then for every  $c \in \mathbb{R}$  and  $\mu \in M[a, b]$  we have*

$$\left| \int_{[a,b]} f(t) d\mu(t) - \mu([a, b])f(b) - c[f(b) - f(a)] \right| \leq L \int_a^b |c + \check{\mu}_1(t)| dt.$$

*Proof.* Put  $n = 1$  in the theorem above and note that  $P_1(t) = c + \check{\mu}_1(t)$ , for some  $c \in \mathbb{R}$ .  $\square$

**Corollary 3.3.** *If  $f$  is  $L$ -Lipschitzian, then for every  $c \geq 0$  and  $\mu \geq 0$  we have*

$$\begin{aligned} & \left| \int_{[a,b]} f(t) d\mu(t) - \mu([a, b])f(b) - c[f(b) - f(a)] \right| \\ & \leq L [c(b-a) + \check{\mu}_2(b)] \\ & \leq L(b-a)(c + \|\mu\|). \end{aligned}$$



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*Proof.* Apply Corollary 3.2 and note that in this case

$$\begin{aligned}\int_a^b |c + \check{\mu}_1(t)| dt &= \int_a^b [c + \check{\mu}_1(t)] dt \\ &= c(b - a) + \check{\mu}_2(b) \\ &\leq c(b - a) + (b - a) \|\mu\| \\ &= (b - a)(c + \|\mu\|).\end{aligned}$$

□

**Corollary 3.4.** Let  $f$  be  $L$ -Lipschitzian,  $(c_m, m \geq 1)$  a sequence in  $[0, \infty)$  such that

$$\sum_{m \geq 1} c_m < \infty,$$

and let  $\{x_m; m \geq 1\} \subset [a, b]$ . Then for every  $c \geq 0$  we have

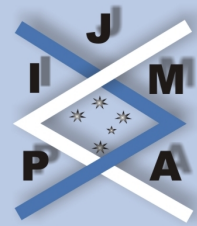
$$\begin{aligned}\left| \sum_{m \geq 1} c_m [f(b) - f(x_m)] + c [f(b) - f(a)] \right| &\leq L \left[ c(b - a) + \sum_{m \geq 1} c_m (b - x_m) \right] \\ &\leq L(b - a) \left[ c + \sum_{m \geq 1} c_m \right].\end{aligned}$$

*Proof.* Apply Corollary 3.3 for the discrete measure  $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$ .

□

**Corollary 3.5.** If  $f$  is  $L$ -Lipschitzian and  $\mu \geq 0$ , then

$$\begin{aligned}\left| \int_{[a,b]} f(t) d\mu(t) - \mu([a, x])f(a) - \mu((x, b])f(b) \right| \\ \leq L [(2x - a - b)\check{\mu}_1(x) - 2\check{\mu}_2(x) + \check{\mu}_2(b)],\end{aligned}$$



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for every  $x \in [a, b]$ .

*Proof.* Apply Corollary 3.2 for  $c = -\check{\mu}_1(x)$ . Then

$$c + \check{\mu}_1(b) = \mu((x, b]), \quad \check{\mu}_1(x) = \mu([a, x])$$

and

$$\begin{aligned} \int_a^b |-\check{\mu}_1(x) + \check{\mu}_1(t)| dt &= \int_a^x (\check{\mu}_1(x) - \check{\mu}_1(t)) dt + \int_x^b (\check{\mu}_1(t) - \check{\mu}_1(x)) dt \\ &= (2x - a - b)\check{\mu}_1(x) - 2\check{\mu}_2(x) + \check{\mu}_2(b). \end{aligned}$$

□

**Corollary 3.6.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  is  $L$ -Lipschitzian for some  $n \geq 1$ . Then for every  $\mu \in M[a, b]$  we have

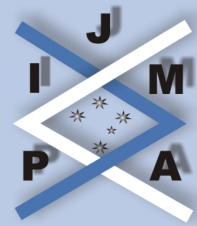
$$\left| \int_{[a,b]} f(t) d\mu(t) - \check{S}_n \right| \leq L \int_a^b |\check{\mu}_n(t)| dt \leq \frac{(b-a)^n}{n!} L \|\mu\|,$$

where  $\check{S}_n$  is from Corollary 2.3.

*Proof.* Apply the theorem above for the  $\mu$ -harmonic sequence  $(\check{\mu}_n, n \geq 1)$ . □

**Corollary 3.7.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  is  $L$ -Lipschitzian for some  $n \geq 1$ . Then for every  $x \in [a, b]$  we have

$$\left| f(x) - \sum_{k=1}^n \frac{(x-b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \leq \frac{(b-x)^n}{n!} L.$$



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*Proof.* Apply Corollary 3.6 for  $\mu = \delta_x$  and note that in this case

$$\check{\mu}_k(t) = \frac{(t-x)^{k-1}}{(k-1)!}, \quad x \leq t \leq b, \quad \text{and} \quad \check{\mu}_k(t) = 0, \quad a \leq t < x,$$

for  $k \geq 1$ . □

**Corollary 3.8.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  is  $L$ -Lipschitzian, for some  $n \geq 1$ . Further, let  $(c_m, m \geq 1)$  be a sequence in  $\mathbb{R}$  such that

$$\sum_{m \geq 1} |c_m| < \infty$$

and let  $\{x_m; m \geq 1\} \subset [a, b]$ . Then

$$\begin{aligned} & \left| \sum_{m \geq 1} c_m f(x_m) - \sum_{m \geq 1} \sum_{k=1}^n c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \\ & \leq \frac{L}{n!} \sum_{m \geq 1} |c_m| (b - x_m)^n \\ & \leq \frac{L}{n!} (b - a)^n \sum_{m \geq 1} |c_m|. \end{aligned}$$

*Proof.* Apply Corollary 3.6 for the discrete measure  $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$ . □

**Theorem 3.9.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  has bounded variation for some  $n \geq 1$ . Then for every  $\mu$ -harmonic sequence  $(P_n, n \geq 1)$  we have

$$\left| \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\}) f(a) - S_n \right| \leq \max_{t \in [a,b]} |P_n(t)| \bigvee_a^b (f^{(n-1)}),$$



where  $\bigvee_a^b(f^{(n-1)})$  is the total variation of  $f^{(n-1)}$  on  $[a, b]$ .

*Proof.* By Theorem 2.2 we have

$$|R_n| = \left| \int_{[a,b]} P_n(t) df^{(n-1)}(t) \right| \leq \max_{t \in [a,b]} |P_n(t)| \bigvee_a^b(f^{(n-1)}),$$

which proves our assertion. □

**Corollary 3.10.** *If  $f$  is a function of bounded variation, then for every  $c \in \mathbb{R}$  and  $\mu \in M[a, b]$  we have*

$$\left| \int_{[a,b]} f(t) d\mu(t) - \mu([a, b])f(b) - c[f(b) - f(a)] \right| \leq \max_{t \in [a,b]} |c + \check{\mu}_1(t)| \bigvee_a^b(f).$$

*Proof.* Put  $n = 1$  in the theorem above. □

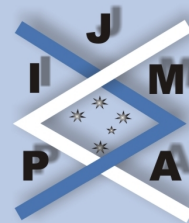
**Corollary 3.11.** *If  $f$  is a function of bounded variation, then for every  $c \geq 0$  and  $\mu \geq 0$  we have*

$$\left| \int_{[a,b]} f(t) d\mu(t) - \mu([a, b])f(b) - c[f(b) - f(a)] \right| \leq [c + \|\mu\|] \bigvee_a^b(f).$$

*Proof.* In this case we have

$$\max_{t \in [a,b]} |c + \check{\mu}_1(t)| = c + \check{\mu}_1(b) = c + \|\mu\|.$$

□



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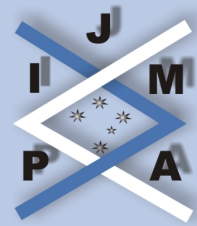
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**Corollary 3.12.** Let  $f$  be a function of bounded variation,  $(c_m, m \geq 1)$  a sequence in  $[0, \infty)$  such that

$$\sum_{m \geq 1} c_m < \infty$$

and let  $\{x_m; m \geq 1\} \subset [a, b]$ . Then for every  $c \geq 0$  we have

$$\left| \sum_{m \geq 1} c_m [f(b) - f(x_m)] + c [f(b) - f(a)] \right| \leq \left[ c + \sum_{m \geq 1} c_m \right] \bigvee_a^b(f).$$

*Proof.* Apply Corollary 3.11 for the discrete measure  $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$ . □

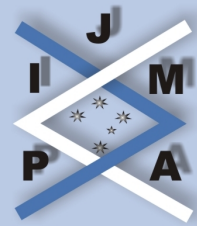
**Corollary 3.13.** If  $f$  is a function of bounded variation and  $\mu \geq 0$ , then we have

$$\begin{aligned} \left| \int_{[a,b]} f(t) d\mu(t) - \mu([a, x])f(a) - \mu((x, b])f(b) \right| \\ \leq \frac{1}{2} [\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(x)|] \bigvee_a^b(f). \end{aligned}$$

*Proof.* Apply Corollary 3.11 for  $c = -\check{\mu}_1(x)$ . Then

$$\begin{aligned} \max_{t \in [a,b]} |c + \check{\mu}_1(t)| &= \max_{t \in [a,b]} |\check{\mu}_1(t) - \check{\mu}_1(x)| \\ &= \max\{\check{\mu}_1(x) - \check{\mu}_1(a), \check{\mu}_1(b) - \check{\mu}_1(x)\} \\ &= \frac{1}{2} [\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(x)|]. \end{aligned}$$

□



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**Corollary 3.14.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  has bounded variation for some  $n \geq 1$ . Then for every  $\mu \in M[a, b]$  we have

$$\left| \int_{[a,b]} f(t) d\mu(t) - \check{S}_n \right| \leq \max_{t \in [a,b]} |\check{\mu}_n(t)| \bigvee_a^b (f^{(n-1)})$$

$$\leq \frac{(b-a)^{n-1}}{(n-1)!} \|\mu\| \bigvee_a^b (f^{(n-1)}),$$

where  $\check{S}_n$  is from Corollary 2.3.

*Proof.* Apply the theorem above for the  $\mu$ -harmonic sequence  $(\check{\mu}_n, n \geq 1)$ . □

**Corollary 3.15.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  has bounded variation for some  $n \geq 1$ . Then for every  $x \in [a, b]$  we have

$$\left| f(x) - \sum_{k=1}^n \frac{(x-b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \leq \frac{(b-x)^{n-1}}{(n-1)!} \bigvee_a^b (f^{(n-1)}).$$

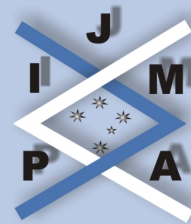
*Proof.* Apply Corollary 3.14 for  $\mu = \delta_x$  and note that in this case

$$\max_{t \in [a,b]} |\check{\mu}_n(t)| = \frac{(b-x)^{n-1}}{(n-1)!}.$$

□

**Corollary 3.16.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  has bounded variation for some  $n \geq 1$ . Further, let  $(c_m, m \geq 1)$  be a sequence in  $\mathbb{R}$  such that

$$\sum_{m \geq 1} |c_m| < \infty$$



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and let  $\{x_m; m \geq 1\} \subset [a, b]$ . Then

$$\begin{aligned} & \left| \sum_{m \geq 1} c_m f(x_m) - \sum_{m \geq 1} \sum_{k=1}^n c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \\ & \leq \frac{1}{(n-1)!} \int_a^b (f^{(n-1)}) \sum_{m \geq 1} |c_m| (b - x_m)^{n-1} \\ & \leq \frac{(b-a)^{n-1}}{(n-1)!} \int_a^b (f^{(n-1)}) \sum_{m \geq 1} |c_m| \end{aligned}$$

*Proof.* Apply Corollary 3.14 for the discrete measure  $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$ . □

**Theorem 3.17.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n)} \in L_p[a, b]$  for some  $n \geq 1$ . Then for every  $\mu$ -harmonic sequence  $(P_n, n \geq 1)$  we have

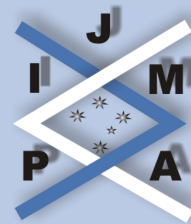
$$\left| \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\})f(a) - S_n \right| \leq \|P_n\|_q \|f^{(n)}\|_p,$$

where  $p, q \in [1, \infty]$  and  $1/p + 1/q = 1$ .

*Proof.* By Theorem 2.2 and the Hölder inequality we have

$$\begin{aligned} |R_n| &= \left| \int_{[a,b]} P_n(t) df^{(n-1)}(t) \right| = \left| \int_{[a,b]} P_n(t) f^{(n)}(t) dt \right| \\ &\leq \left( \int_a^b |P_n(t)|^q dt \right)^{\frac{1}{q}} \left( \int_a^b |f^{(n)}(t)|^p dt \right)^{\frac{1}{p}} \\ &= \|P_n\|_q \|f^{(n)}\|_p. \end{aligned}$$

□



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*Remark 3.* We see that the inequality of the theorem above is a generalization of inequality (1.2).

**Corollary 3.18.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n)} \in L_p[a, b]$  for some  $n \geq 1$ , and  $\mu \in M[a, b]$ . Then

$$\left| \int_{[a,b]} f(t) d\mu(t) - \check{S}_n \right| \leq \|\check{\mu}_n\|_q \|f^{(n)}\|_p$$

$$\leq \frac{(b-a)^{n-1+1/q}}{(n-1)! [(n-1)q+1]^{1/q}} \|\mu\| \|f^{(n)}\|_p,$$

where  $p, q \in [1, \infty]$  and  $1/p + 1/q = 1$ .

*Proof.* Apply the theorem above for the  $\mu$ -harmonic sequence  $(\check{\mu}_n, n \geq 1)$ . □

**Corollary 3.19.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n)} \in L_p[a, b]$ , for some  $n \geq 1$ . Further, let  $(c_m, m \geq 1)$  be a sequence in  $\mathbb{R}$  such that

$$\sum_{m \geq 1} |c_m| < \infty$$

and let  $\{x_m; m \geq 1\} \subset [a, b]$ . Then

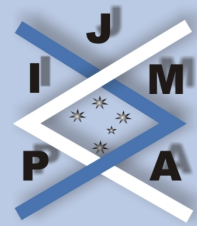
$$\left| \sum_{m \geq 1} c_m f(x_m) - \sum_{m \geq 1} \sum_{k=1}^n c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right|$$

$$\leq \frac{\|f^{(n)}\|_p}{(n-1)! [(n-1)q+1]^{1/q}} \sum_{m \geq 1} |c_m| (b - x_m)^{n-1+1/q}$$

$$\leq \frac{(b-a)^{n-1+1/q} \|f^{(n)}\|_p}{(n-1)! [(n-1)q+1]^{1/q}} \sum_{m \geq 1} |c_m|,$$

where  $p, q \in [1, \infty]$  and  $1/p + 1/q = 1$ .

*Proof.* Apply the theorem above for the discrete measure  $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$ . □



## 4. Some Grüss-type Inequalities

Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n)} \in L_\infty[a, b]$ , for some  $n \geq 1$ . Then

$$m_n \leq f^{(n)}(t) \leq M_n, \quad t \in [a, b], \text{ a.e.}$$

for some real constants  $m_n$  and  $M_n$ .

**Theorem 4.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n)} \in L_\infty[a, b]$ , for some  $n \geq 1$ . Further, let  $(P_k, k \geq 1)$  be a  $\mu$ -harmonic sequence such that*

$$P_{n+1}(a) = P_{n+1}(b),$$

for that particular  $n$ . Then

$$\left| \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\})f(a) - S_n \right| \leq \frac{M_n - m_n}{2} \int_a^b |P_n(t)| dt.$$

*Proof.* Apply Theorem 2.2 for the special case when  $f^{(n-1)}$  is absolutely continuous and its derivative  $f^{(n)}$ , existing a.e., is bounded a.e. Define the measure  $\nu_n$  by

$$d\nu_n(t) = -P_n(t) dt.$$

Then

$$\nu_n([a, b]) = - \int_a^b P_n(t) dt = P_{n+1}(a) - P_{n+1}(b) = 0,$$

which means that  $\nu_n$  is balanced. Further,

$$\|\nu_n\| = \int_a^b |P_n(t)| dt$$

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and by [1, Theorem 2]

$$\begin{aligned} |R_n| &= \left| \int_a^b P_n(t) f^{(n)}(t) dt \right| \\ &\leq \frac{M_n - m_n}{2} \|\nu_n\| \\ &= \frac{M_n - m_n}{2} \int_a^b |P_n(t)| dt, \end{aligned}$$

which proves our assertion.  $\square$

**Corollary 4.2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n)} \in L_\infty[a, b]$ , for some  $n \geq 1$ . Then for every  $(n + 1)$ -balanced measure  $\mu \in M[a, b]$  we have*

$$\begin{aligned} \left| \int_{[a,b]} f(t) d\mu(t) - \check{S}_n \right| &\leq \frac{M_n - m_n}{2} \int_a^b |\check{\mu}_n(t)| dt \\ &\leq \frac{M_n - m_n}{2} \frac{(b - a)^n}{n!} \|\mu\|, \end{aligned}$$

where  $\check{S}_n$  is from Corollary 2.3.

*Proof.* Apply Theorem 4.1 for the  $\mu$ -harmonic sequence  $(\check{\mu}_k, k \geq 1)$  and note that the condition  $P_{n+1}(a) = P_{n+1}(b)$  reduces to  $\check{\mu}_{n+1}(b) = 0$ , which means that  $\mu$  is  $(n + 1)$ -balanced.  $\square$

**Corollary 4.3.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n)} \in L_\infty[a, b]$  for some  $n \geq 1$ . Further, let  $(c_m, m \geq 1)$  be a sequence in  $\mathbb{R}$  such that*

$$\sum_{m \geq 1} |c_m| < \infty$$



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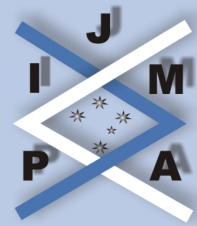
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and let  $\{x_m; m \geq 1\} \subset [a, b]$  satisfy the condition

$$\sum_{m \geq 1} c_m (b - x_m)^n = 0.$$

Then

$$\begin{aligned} & \left| \sum_{m \geq 1} c_m f(x_m) - \sum_{m \geq 1} \sum_{k=1}^n c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \\ & \leq \frac{M_n - m_n}{2n!} \sum_{m \geq 1} |c_m| (b - x_m)^n \\ & \leq \frac{M_n - m_n}{2n!} (b - a)^n \sum_{m \geq 1} |c_m|. \end{aligned}$$

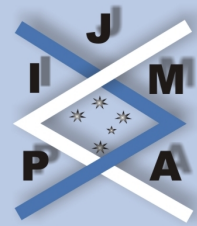
*Proof.* Apply Corollary 4.2 for the discrete measure  $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$ .  $\square$

**Corollary 4.4.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n)} \in L_\infty[a, b]$  for some  $n \geq 1$ . Then for every  $\mu \in M[a, b]$ , such that all  $k$ -moments of  $\mu$  are zero for  $k = 0, \dots, n$ , we have

$$\begin{aligned} \left| \int_{[a,b]} f(t) d\mu(t) \right| & \leq \frac{M_n - m_n}{2} \int_a^b |\check{\mu}_n(t)| dt \\ & \leq \frac{M_n - m_n}{2} \frac{(b - a)^n}{n!} \|\mu\|. \end{aligned}$$

*Proof.* By [1, Theorem 5], the condition  $m_k(\mu) = 0$ ,  $k = 0, \dots, n$  is equivalent to  $\check{\mu}_k(b) = 0$ ,  $k = 1, \dots, n + 1$ . Apply Corollary 4.2 and note that in this case  $\check{S}_n = 0$ .  $\square$





**Corollary 4.5.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n)} \in L_\infty[a, b]$  for some  $n \geq 1$ . Further, let  $(c_m, m \geq 1)$  be a sequence in  $\mathbb{R}$  such that

$$\sum_{m \geq 1} |c_m| < \infty$$

and let  $\{x_m; m \geq 1\} \subset [a, b]$ . If

$$\sum_{m \geq 1} c_m = \sum_{m \geq 1} c_m x_m = \cdots = \sum_{m \geq 1} c_m x_m^n = 0,$$

then

$$\begin{aligned} \left| \sum_{m \geq 1} c_m f(x_m) \right| &\leq \frac{M_n - m_n}{2n!} \sum_{m \geq 1} |c_m| (b - x_m)^n \\ &\leq \frac{M_n - m_n}{2n!} (b - a)^n \sum_{m \geq 1} |c_m|. \end{aligned}$$

*Proof.* Apply Corollary 4.4 for the discrete measure  $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$ . □

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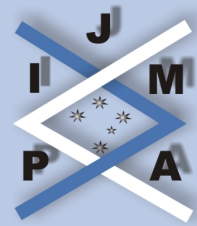
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## References

- [1] A. ČIVLJAK, LJ. DEDIĆ AND M. MATIĆ, Euler-Grüss type inequalities involving measures, submitted.
- [2] A.ČIVLJAK, LJ. DEDIĆ AND M. MATIĆ, Euler harmonic identities for measures, *Nonlinear Functional Anal. & Applics.*, **12**(1) (2007).
- [3] Lj. DEDIĆ, M. MATIĆ, J. PEČARIĆ AND A. AGLIĆ ALJINOVIĆ, On weighted Euler harmonic identities with applications, *Math. Inequal. & Appl.*, **8**(2), (2005), 237–257.
- [4] S.S. DRAGOMIR, The generalised integration by parts formula for Appell sequences and related results, *RGMIA Res. Rep. Coll.*, **5**(E) (2002), Art. 18. [ONLINE: [http://rgmia.vu.edu.au/v5\(E\).html](http://rgmia.vu.edu.au/v5(E).html)].
- [5] P. CERONE, Generalised Taylor's formula with estimates of the remainder, *RGMIA Res. Rep. Coll.*, **5**(2) (2002), Art. 8. [ONLINE: <http://rgmia.vu.edu.au/v5n2.html>].
- [6] P. CERONE, Perturbated generalised Taylor's formula with sharp bounds, *RGMIA Res. Rep. Coll.*, **5**(2) (2002), Art. 6. [ONLINE: <http://rgmia.vu.edu.au/v5n2.html>].
- [7] S.S. DRAGOMIR AND A. SOFO, A perturbed version of the generalised Taylor's formula and applications, *RGMIA Res. Rep. Coll.*, **5**(2) (2002), Art. 16. [ONLINE: <http://rgmia.vu.edu.au/v5n2.html>].

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