ON AN INTEGRATION-BY-PARTS FORMULA FOR MEASURES

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Abstract: An integration-by-parts formula, involving finite Borel measures supported by intervals on real line, is proved. Some applications to Ostrowski-type and Grüss-type inequalities are presented.

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1. Introduction

In the paper [4], S.S. Dragomir introduced the notion of a w_0-Appell type sequence of functions as a sequence $w_0, w_1, \ldots, w_n$, for $n \geq 1$, of real absolutely continuous functions defined on $[a,b]$, such that

$$w_k' = w_{k-1}, \text{ a.e. on } [a,b], \qquad k = 1, \ldots, n.$$

For such a sequence the author proved a generalisation of Mitrinović-Pečarić integration-by-parts formula

$$\int_a^b w_0(t) g(t) dt = A_n + B_n, \tag{1.1}$$

where

$$A_n = \sum_{k=1}^{n} (-1)^{k-1} \left[w_k(b) g^{(k-1)}(b) - w_k(a) g^{(k-1)}(a) \right]$$

and

$$B_n = (-1)^n \int_a^b w_n(t) g^{(n)}(t) dt,$$

for every $g : [a,b] \to \mathbb{R}$ such that $g^{(n-1)}$ is absolutely continuous on $[a,b]$ and $w_n g^{(n)} \in L_1[a,b]$. Using identity (1.1) the author proved the following inequality

$$\left| \int_a^b w_0(t) g(t) dt - A_n \right| \leq \|w_n\|_p \, \|g^{(n)}\|_q, \tag{1.2}$$

for $w_n \in L_p[a,b]$, $g^{(n)} \in L_p[a,b]$, where $p, q \in [1, \infty]$ and $1/p + 1/q = 1$, giving explicitly some interesting special cases. For some similar inequalities, see also [5],

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[6] and [7]. The aim of this paper is to give a generalization of the integration-by-parts formula (1.1), by replacing the w_0-Appell type sequence of functions by a more general sequence of functions, and to generalize inequality (1.2), as well as to prove some related inequalities.

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2. Integration-by-parts Formula for Measures

For $a, b \in \mathbb{R}$, $a < b$, let $C[a,b]$ be the Banach space of all continuous functions $f : [a,b] \to \mathbb{R}$ with the max norm, and $M[a,b]$ the Banach space of all real Borel measures on $[a,b]$ with the total variation norm. For $\mu \in M[a,b]$ define the function $\check{\mu}_n : [a,b] \to \mathbb{R}$, $n \geq 1$, by

$$\check{\mu}_n(t) = \frac{1}{(n-1)!} \int_{[a,t]} (t-s)^{n-1} d\mu(s).$$

Note that

$$\check{\mu}_n(t) = \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} \check{\mu}_1(s) ds, \qquad n \geq 2$$

and

$$|\check{\mu}_n(t)| \leq \frac{(t-a)^{n-1}}{(n-1)!} \|\mu\|, \qquad t \in [a,b],\ n \geq 1.$$

The function $\check{\mu}_n$ is differentiable, $\check{\mu}_n'(t) = \check{\mu}_{n-1}(t)$ and $\check{\mu}_n(a) = 0$, for every $n \geq 2$, while for $n = 1$

$$\check{\mu}_1(t) = \int_{[a,t]} d\mu(s) = \mu([a,t]),$$

which means that $\check{\mu}_1(t)$ is equal to the distribution function of μ. A sequence of functions $P_n : [a,b] \to \mathbb{R}$, $n \geq 1$, is called a μ-harmonic sequence of functions on $[a,b]$ if

$$P_n'(t) = P_{n-1}(t),\ n \geq 2; \quad P_1(t) = c + \check{\mu}_1(t), \quad t \in [a,b],$$

for some $c \in \mathbb{R}$. The sequence $(\check{\mu}_n, n \geq 1)$ is an example of a μ-harmonic sequence of functions on $[a,b]$. The notion of a μ-harmonic sequence of functions has been introduced in [2]. See also [1].

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Remark 1. Let $w_0 : [a,b] \to \mathbb{R}$ be an absolutely integrable function and let $\mu \in M[a,b]$ be defined by

$$d\mu(t) = w_0(t)dt.$$

If $(P_n, n \geq 1)$ is a μ-harmonic sequence of functions on $[a,b]$, then $w_0, P_1, \ldots, P_n$ is a w_0-Appell type sequence of functions on $[a,b]$.

For $\mu \in M[a,b]$ let $\mu = \mu_+ - \mu_-$ be the Jordan-Hahn decomposition of μ, where μ_+ and μ_- are orthogonal and positive measures. Then we have $|\mu| = \mu_+ + \mu_-$ and

$$\|\mu\| = |\mu|([a,b]) = \|\mu_+\| + \|\mu_-\| = \mu_+([a,b]) + \mu_-([a,b]).$$

The measure $\mu \in M[a,b]$ is said to be balanced if $\mu([a,b]) = 0$. This is equivalent to

$$\|\mu_+\| = \|\mu_-\| = \frac{1}{2}\|\mu\|.$$

Measure $\mu \in M[a,b]$ is called n-balanced if $\check{\mu}_n(b) = 0$. We see that a 1-balanced measure is the same as a balanced measure. We also write

$$m_k(\mu) = \int_{[a,b]} t^k d\mu(t), \qquad k \geq 0$$

for the k-th moment of μ.

Lemma 2.1. *For every $f \in C[a,b]$ and $\mu \in M[a,b]$ we have*

$$\int_{[a,b]} f(t)d\check{\mu}_1(t) = \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a).$$

Proof. Define $I, J : C[a,b] \times M[a,b] \to \mathbb{R}$ by

$$I(f,\mu) = \int_{[a,b]} f(t)d\check{\mu}_1(t)$$

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and

$$J(f,\mu) = \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a).$$

Then I and J are continuous bilinear functionals, since

$$|I(f,\mu)| \le \|f\|\,\|\mu\|\,, \qquad |J(f,\mu)| \le 2\,\|f\|\,\|\mu\|\,.$$

Let us prove that $I(f,\mu) = J(f,\mu)$ for every $f \in C[a,b]$ and every discrete measure $\mu \in M[a,b]$.

For $x \in [a,b]$ let $\mu = \delta_x$ be the Dirac measure at x, i.e. the measure defined by

$$\int\limits_{[a,b]} f(t)\mathrm{d}\delta_x(t) = f(x).$$

If $a < x \le b$, then

$$\check{\mu}_1(t) = \delta_x([a,t]) = \begin{cases} 0, & a \le t < x \\ 1, & x \le t \le b \end{cases}$$

and by a simple calculation we have

$$\begin{aligned} I(f,\delta_x) &= \int_{[a,b]} f(t)d\check{\mu}_1(t) = f(x) = \int\limits_{[a,b]} f(t)\mathrm{d}\delta_x(t) - 0 \\ &= \int\limits_{[a,b]} f(t)\mathrm{d}\delta_x(t) - \delta_x(\{a\})f(a) = J(f,\delta_x). \end{aligned}$$

Similarly, if $x = a$, then

$$\check{\mu}_1(t) = \delta_a([a,t]) = 1, \qquad a \le t \le b$$

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and by a similar calculation we have

$$I(f,\delta_a) = \int_{[a,b]} f(t)d\check{\mu}_1(t) \;= 0 = f(a) - f(a)$$
$$= \int_{[a,b]} f(t)\mathrm{d}\delta_a(t) - \delta_a(\{a\})f(a) = J(f,\delta_x).$$

Therefore, for every $f \in C[a,b]$ and every $x \in [a,b]$ we have $I(f,\delta_x) = J(f,\delta_x)$. Every discrete measure $\mu \in M[a,b]$ has the form

$$\mu = \sum_{k\geq 1} c_k \delta_{x_k},$$

where $(c_k, k \geq 1)$ is a sequence in $\mathbb{R}$ such that

$$\sum_{k\geq 1} |c_k| < \infty,$$

and $\{x_k; k \geq 1\}$ is a subset of $[a,b]$.

By using the continuity of I and J, for every $f \in C[a,b]$ and every discrete measure $\mu \in M[a,b]$ we have

$$I(f,\mu) = I\left(f, \sum_{k\geq 1} c_k \delta_{x_k}\right) = \sum_{k\geq 1} c_k I(f,\delta_{x_k})$$
$$= \sum_{k\geq 1} c_k J(f,\delta_{x_k}) = J\left(f, \sum_{k\geq 1} c_k \delta_{x_k}\right)$$
$$= J(f,\mu).$$

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Since the Banach subspace $M[a,b]_d$ of all discrete measures is weakly* dense in $M[a,b]$ and the functionals $I(f,\cdot)$ and $J(f,\cdot)$ are also weakly* continuous we conclude that $I(f,\mu)=J(f,\mu)$ for every $f\in C[a,b]$ and $\mu\in M[a,b]$. □

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ has bounded variation for some $n\geq 1$. Then for every μ-harmonic sequence $(P_n, n\geq 1)$ we have*

$$\int_{[a,b]} f(t)d\mu(t)=\mu(\{a\})f(a)+S_n+R_n, \tag{2.1}$$

where

$$S_n=\sum_{k=1}^{n}(-1)^{k-1}\left[P_k(b)f^{(k-1)}(b)-P_k(a)f^{(k-1)}(a)\right] \tag{2.2}$$

and

$$R_n=(-1)^n\int_{[a,b]} P_n(t)df^{(n-1)}(t). \tag{2.3}$$

Proof. By partial integration, for $n\geq 2$, we have

$$\begin{aligned}R_n&=(-1)^n\int_{[a,b]} P_n(t)df^{(n-1)}(t)\\&=(-1)^n\left[P_n(b)f^{(n-1)}(b)-P_n(a)f^{(n-1)}(a)\right]\\&\qquad-(-1)^n\int_{[a,b]} P_{n-1}(t)f^{(n-1)}(t)dt\\&=(-1)^n\left[P_n(b)f^{(n-1)}(b)-P_n(a)f^{(n-1)}(a)\right]+R_{n-1}.\end{aligned}$$

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By Lemma 2.1 we have

$$\begin{aligned} R_1 &= -\int_{[a,b]} P_1(t)df(t) \\ &= -\left[P_1(b)f(b) - P_n(a)f(a)\right] + \int_{[a,b]} f(t)dP_1(t) \\ &= -\left[P_1(b)f(b) - P_n(a)f(a)\right] + \int_{[a,b]} f(t)d\check{\mu}_1(t) \\ &= -\left[P_1(b)f(b) - P_n(a)f(a)\right] + \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a). \end{aligned}$$

Therefore, by iteration, we have

$$R_n = \sum_{k=1}^{n} (-1)^k \left[P_k(b)f^{(k-1)}(b) - P_k(a)f^{(k-1)}(a)\right] + \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a),$$

which proves our assertion. □

Remark 2. By Remark 1 we see that identity (2.1) is a generalization of the integration-by-parts formula (1.1).

Corollary 2.3. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *has bounded variation for some* $n \geq 1$. *Then for every* $\mu \in M[a,b]$ *we have*

$$\int_{[a,b]} f(t)d\mu(t) = \check{S}_n + \check{R}_n,$$

where

$$\check{S}_n = \sum_{k=1}^{n} (-1)^{k-1} \check{\mu}_k(b) f^{(k-1)}(b)$$

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and

$$\check{R}_n = (-1)^n \int_{[a,b]} \check{\mu}_n(t) df^{(n-1)}(t).$$

Proof. Apply the theorem above for the μ-harmonic sequence $(\check{\mu}_n, n \geq 1)$ and note that $\check{\mu}_n(a) = 0$, for $n \geq 2$. □

Corollary 2.4. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ has bounded variation for some $n \geq 1$. Then for every $x \in [a,b]$ we have*

$$f(x) = \sum_{k=1}^{n} \frac{(x-b)^{k-1}}{(k-1)!} f^{(k-1)}(b) + R_n(x),$$

where

$$R_n(x) = \frac{(-1)^n}{(n-1)!} \int_{[x,b]} (t-x)^{n-1} df^{(n-1)}(t).$$

Proof. Apply Corollary 2.3 for $\mu = \delta_x$ and note that in this case

$$\check{\mu}_k(t) = \frac{(t-x)^{k-1}}{(k-1)!}, \quad x \leq t \leq b, \quad \text{and} \quad \check{\mu}_k(t) = 0, \quad a \leq t < x,$$

for $k \geq 1$. □

Corollary 2.5. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ has bounded variation for some $n \geq 1$. Further, let $(c_m, m \geq 1)$ be a sequence in $\mathbb{R}$ such that*

$$\sum_{m \geq 1} |c_m| < \infty$$

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and let $\{x_m; m \geq 1\} \subset [a,b]$. *Then*

$$\sum_{m\geq 1} c_m f(x_m) = \sum_{m\geq 1}\sum_{k=1}^{n} c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) + \sum_{m\geq 1} c_m R_n(x_m),$$

where $R_n(x_m)$ *is from Corollary 2.4.*

Proof. Apply Corollary 2.3 for the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

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3. Some Ostrowski-type Inequalities

In this section we shall use the same notations as above.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian for some $n \geq 1$. Then for every μ-harmonic sequence $(P_n, n \geq 1)$ we have*

$$\left| \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) - S_n \right| \leq L \int_a^b |P_n(t)|\, dt, \tag{3.1}$$

where S_n is defined by (2.2).

Proof. By Theorem 2.2 we have

$$|R_n| = \left| \int_{[a,b]} P_n(t) df^{(n-1)}(t) \right| \leq L \int_a^b |P_n(t)|\, dt,$$

which proves our assertion. □

Corollary 3.2. *If f is L-Lipschitzian, then for every $c \in \mathbb{R}$ and $\mu \in M[a,b]$ we have*

$$\left| \int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(b) - c\,[f(b) - f(a)] \right| \leq L \int_a^b |c + \check{\mu}_1(t)|\, dt.$$

Proof. Put $n = 1$ in the theorem above and note that $P_1(t) = c + \check{\mu}_1(t)$, for some $c \in \mathbb{R}$. □

Corollary 3.3. *If f is L-Lipschitzian, then for every $c \geq 0$ and $\mu \geq 0$ we have*

$$\begin{aligned} &\left| \int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(b) - c\,[f(b) - f(a)] \right| \\ &\leq L\,[c(b-a) + \check{\mu}_2(b)] \\ &\leq L(b-a)(c + \|\mu\|). \end{aligned}$$

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Proof. Apply Corollary 3.2 and note that in this case

$$\begin{aligned}\int_a^b |c+\check{\mu}_1(t)|\,dt &= \int_a^b [c+\check{\mu}_1(t)]\,dt \\ &= c(b-a)+\check{\mu}_2(b) \\ &\le c(b-a)+(b-a)\,\|\mu\| \\ &= (b-a)(c+\|\mu\|).\end{aligned}$$

□

Corollary 3.4. *Let f be L-Lipschitzian, $(c_m, m \ge 1)$ a sequence in $[0,\infty)$ such that*

$$\sum_{m\ge 1} c_m < \infty,$$

and let $\{x_m; m \ge 1\} \subset [a,b]$. Then for every $c \ge 0$ we have

$$\begin{aligned}\left|\sum_{m\ge 1} c_m\,[f(b)-f(x_m)] + c\,[f(b)-f(a)]\right| &\le L\left[c(b-a)+\sum_{m\ge 1} c_m(b-x_m)\right] \\ &\le L(b-a)\left[c+\sum_{m\ge 1} c_m\right].\end{aligned}$$

Proof. Apply Corollary 3.3 for the discrete measure $\mu = \sum_{m\ge 1} c_m \delta_{x_m}$. □

Corollary 3.5. *If f is L-Lipschitzian and $\mu \ge 0$, then*

$$\begin{aligned}&\left|\int_{[a,b]} f(t)d\mu(t) - \mu([a,x])f(a) - \mu((x,b])f(b)\right| \\ &\qquad\le L\left[(2x-a-b)\check{\mu}_1(x) - 2\check{\mu}_2(x) + \check{\mu}_2(b)\right],\end{aligned}$$

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for every $x \in [a,b]$.

Proof. Apply Corollary 3.2 for $c = -\breve{\mu}_1(x)$. Then

$$c + \breve{\mu}_1(b) = \mu((x,b]), \qquad \breve{\mu}_1(x) = \mu([a,x])$$

and

$$\int_a^b |-\breve{\mu}_1(x) + \breve{\mu}_1(t)|\, dt = \int_a^x (\breve{\mu}_1(x) - \breve{\mu}_1(t))\, dt + \int_x^b (\breve{\mu}_1(t) - \breve{\mu}_1(x))\, dt$$
$$= (2x - a - b)\breve{\mu}_1(x) - 2\breve{\mu}_2(x) + \breve{\mu}_2(b).$$

□

Corollary 3.6. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is* L*-Lipschitzian for some* $n \geq 1$. *Then for every* $\mu \in M[a,b]$ *we have*

$$\left| \int_{[a,b]} f(t) d\mu(t) - \check{S}_n \right| \leq L \int_a^b |\breve{\mu}_n(t)|\, dt \leq \frac{(b-a)^n}{n!} L \|\mu\|,$$

where $\check{S}_n$ *is from Corollary 2.3.*

Proof. Apply the theorem above for the μ-harmonic sequence $(\breve{\mu}_n, n \geq 1)$. □

Corollary 3.7. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is* L*-Lipschitzian for some* $n \geq 1$. *Then for every* $x \in [a,b]$ *we have*

$$\left| f(x) - \sum_{k=1}^{n} \frac{(x-b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \leq \frac{(b-x)^n}{n!} L.$$

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Proof. Apply Corollary 3.6 for $\mu = \delta_x$ and note that in this case

$$\check{\mu}_k(t) = \frac{(t-x)^{k-1}}{(k-1)!}, \quad x \le t \le b, \qquad \text{and} \qquad \check{\mu}_k(t) = 0, \quad a \le t < x,$$

for $k \ge 1$. □

Corollary 3.8. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian, for some $n \ge 1$. Further, let $(c_m, m \ge 1)$ be a sequence in $\mathbb{R}$ such that*

$$\sum_{m\ge1} |c_m| < \infty$$

and let $\{x_m; m \ge 1\} \subset [a,b]$. Then

$$\begin{aligned}
&\left| \sum_{m\ge1} c_m f(x_m) - \sum_{m\ge1} \sum_{k=1}^{n} c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \\
&\le \frac{L}{n!} \sum_{m\ge1} |c_m| (b - x_m)^n \\
&\le \frac{L}{n!} (b-a)^n \sum_{m\ge1} |c_m| .
\end{aligned}$$

Proof. Apply Corollary 3.6 for the discrete measure $\mu = \sum_{m\ge1} c_m \delta_{x_m}$. □

Theorem 3.9. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ has bounded variation for some $n \ge 1$. Then for every μ-harmonic sequence $(P_n, n \ge 1)$ we have*

$$\left| \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) - S_n \right| \le \max_{t\in[a,b]} |P_n(t)| \bigvee_a^b (f^{(n-1)}),$$

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where $\bigvee_a^b(f^{(n-1)})$ *is the total variation of* $f^{(n-1)}$ *on* $[a,b]$.

Proof. By Theorem 2.2 we have

$$|R_n| = \left|\int_{[a,b]} P_n(t)df^{(n-1)}(t)\right| \leq \max_{t\in[a,b]} |P_n(t)| \bigvee_a^b (f^{(n-1)}),$$

which proves our assertion. □

Corollary 3.10. *If* f *is a function of bounded variation, then for every* $c \in \mathbb{R}$ *and* $\mu \in M[a,b]$ *we have*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(b) - c\,[f(b) - f(a)]\right| \leq \max_{t\in[a,b]} |c + \check{\mu}_1(t)| \bigvee_a^b (f).$$

Proof. Put $n = 1$ in the theorem above. □

Corollary 3.11. *If* f *is a function of bounded variation, then for every* $c \geq 0$ *and* $\mu \geq 0$ *we have*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(b) - c\,[f(b) - f(a)]\right| \leq [c + \|\mu\|] \bigvee_a^b (f).$$

Proof. In this case we have

$$\max_{t\in[a,b]} |c + \check{\mu}_1(t)| = c + \check{\mu}_1(b) = c + \|\mu\|.$$

□

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Corollary 3.12. *Let f be a function of bounded variation, $(c_m, m \geq 1)$ a sequence in $[0, \infty)$ such that*

$$\sum_{m \geq 1} c_m < \infty$$

and let $\{x_m; m \geq 1\} \subset [a, b]$. Then for every $c \geq 0$ we have

$$\left| \sum_{m \geq 1} c_m \left[f(b) - f(x_m) \right] + c \left[f(b) - f(a) \right] \right| \leq \left[c + \sum_{m \geq 1} c_m \right] \bigvee_a^b (f).$$

Proof. Apply Corollary 3.11 for the discrete measure $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$. □

Corollary 3.13. *If f is a function of bounded variation and $\mu \geq 0$, then we have*

$$\begin{aligned} &\left| \int_{[a,b]} f(t) d\mu(t) - \mu([a, x]) f(a) - \mu((x, b]) f(b) \right| \\ &\qquad \leq \frac{1}{2} \left[\check{\mu}_1(b) - \check{\mu}_1(a) + \left| \check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(x) \right| \right] \bigvee_a^b (f). \end{aligned}$$

Proof. Apply Corollary 3.11 for $c = -\check{\mu}_1(x)$. Then

$$\begin{aligned} \max_{t \in [a,b]} \left| c + \check{\mu}_1(t) \right| &= \max_{t \in [a,b]} \left| \check{\mu}_1(t) - \check{\mu}_1(x) \right| \\ &= \max\{ \check{\mu}_1(x) - \check{\mu}_1(a), \check{\mu}_1(b) - \check{\mu}_1(x) \} \\ &= \frac{1}{2} \left[\check{\mu}_1(b) - \check{\mu}_1(a) + \left| \check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(x) \right| \right]. \end{aligned}$$

□

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Corollary 3.14. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ has bounded variation for some $n \geq 1$. Then for every $\mu \in M[a,b]$ we have*

$$\left| \int_{[a,b]} f(t)d\mu(t) - \check{S}_n \right| \leq \max_{t\in[a,b]} |\check{\mu}_n(t)| \bigvee_a^b (f^{(n-1)})$$
$$\leq \frac{(b-a)^{n-1}}{(n-1)!} \|\mu\| \bigvee_a^b (f^{(n-1)}),$$

where $\check{S}_n$ is from Corollary 2.3.

Proof. Apply the theorem above for the μ-harmonic sequence $(\check{\mu}_n, n \geq 1)$. □

Corollary 3.15. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ has bounded variation for some $n \geq 1$. Then for every $x \in [a,b]$ we have*

$$\left| f(x) - \sum_{k=1}^{n} \frac{(x-b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \leq \frac{(b-x)^{n-1}}{(n-1)!} \bigvee_a^b (f^{(n-1)}).$$

Proof. Apply Corollary 3.14 for $\mu = \delta_x$ and note that in this case

$$\max_{t\in[a,b]} |\check{\mu}_n(t)| = \frac{(b-x)^{n-1}}{(n-1)!}.$$

□

Corollary 3.16. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ has bounded variation for some $n \geq 1$. Further, let $(c_m, m \geq 1)$ be a sequence in $\mathbb{R}$ such that*

$$\sum_{m\geq 1} |c_m| < \infty$$

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and let $\{x_m; m \geq 1\} \subset [a,b]$. *Then*

$$\left|\sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1}\sum_{k=1}^{n} c_m \frac{(x_m-b)^{k-1}}{(k-1)!} f^{(k-1)}(b)\right|$$

$$\leq \frac{1}{(n-1)!} \bigvee_a^b (f^{(n-1)}) \sum_{m\geq 1} |c_m| (b-x_m)^{n-1}$$

$$\leq \frac{(b-a)^{n-1}}{(n-1)!} \bigvee_a^b (f^{(n-1)}) \sum_{m\geq 1} |c_m|$$

Proof. Apply Corollary 3.14 for the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

Theorem 3.17. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n)} \in L_p[a,b]$ *for some* $n \geq 1$. *Then for every* μ*-harmonic sequence* $(P_n, n \geq 1)$ *we have*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) - S_n\right| \leq \|P_n\|_q \, \|f^{(n)}\|_p,$$

where $p, q \in [1,\infty]$ *and* $1/p + 1/q = 1$.

Proof. By Theorem 2.2 and the Hölder inequality we have

$$|R_n| = \left|\int_{[a,b]} P_n(t)df^{(n-1)}(t)\right| = \left|\int_{[a,b]} P_n(t)f^{(n)}(t)dt\right|$$

$$\leq \left(\int_a^b |P_n(t)|^q \, dt\right)^{\frac{1}{q}} \left(\int_a^b \left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}$$

$$= \|P_n\|_q \, \|f^{(n)}\|_p.$$

□

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Remark 3. We see that the inequality of the theorem above is a generalization of inequality (1.2).

Corollary 3.18. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$ for some $n \geq 1$, and $\mu \in M[a,b]$. Then*

$$\left| \int_{[a,b]} f(t) d\mu(t) - \check{S}_n \right| \leq \|\check{\mu}_n\|_q \, \|f^{(n)}\|_p$$

$$\leq \frac{(b-a)^{n-1+1/q}}{(n-1)! \left[(n-1)q+1\right]^{1/q}} \|\mu\| \, \|f^{(n)}\|_p,$$

where $p, q \in [1, \infty]$ and $1/p + 1/q = 1$.

Proof. Apply the theorem above for the μ-harmonic sequence $(\check{\mu}_n, n \geq 1)$. □

Corollary 3.19. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$, for some $n \geq 1$. Further, let $(c_m, m \geq 1)$ be a sequence in $\mathbb{R}$ such that*

$$\sum_{m \geq 1} |c_m| < \infty$$

and let $\{x_m; m \geq 1\} \subset [a,b]$. Then

$$\left| \sum_{m\geq 1} c_m f(x_m) - \sum_{m \geq 1} \sum_{k=1}^{n} c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right|$$

$$\leq \frac{\|f^{(n)}\|_p}{(n-1)! \left[(n-1)q+1\right]^{1/q}} \sum_{m \geq 1} |c_m| \, (b - x_m)^{n-1+1/q}$$

$$\leq \frac{(b-a)^{n-1+1/q} \|f^{(n)}\|_p}{(n-1)! \left[(n-1)q+1\right]^{1/q}} \sum_{m \geq 1} |c_m| \, ,$$

where $p, q \in [1, \infty]$ and $1/p + 1/q = 1$.

Proof. Apply the theorem above for the discrete measure $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$. □

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4. Some Grüss-type Inequalities

Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}\in L_\infty[a,b]$, for some $n\geq 1$. Then

$$m_n \leq f^{(n)}(t) \leq M_n, \qquad t\in[a,b], \text{ a.e.}$$

for some real constants m_n and M_n.

Theorem 4.1. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}\in L_\infty[a,b]$, for some $n\geq 1$. Further, let $(P_k, k\geq 1)$ be a μ-harmonic sequence such that*

$$P_{n+1}(a) = P_{n+1}(b),$$

for that particular n. Then

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) - S_n\right| \leq \frac{M_n - m_n}{2}\int_a^b |P_n(t)|\, dt.$$

Proof. Apply Theorem 2.2 for the special case when $f^{(n-1)}$ is absolutely continuous and its derivative $f^{(n)}$, existing $a.e.$, is bounded $a.e.$ Define the measure ν_n by

$$d\nu_n(t) = -P_n(t)\, dt.$$

Then

$$\nu_n([a,b]) = -\int_a^b P_n(t)\, dt = P_{n+1}(a) - P_{n+1}(b) = 0,$$

which means that ν_n is balanced. Further,

$$\|\nu_n\| = \int_a^b |P_n(t)|\, dt$$

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and by [1, Theorem 2]

$$
\begin{aligned}
|R_n| &= \left| \int_a^b P_n(t)\, f^{(n)}(t) dt \right| \\
&\le \frac{M_n - m_n}{2} \|\nu_n\| \\
&= \frac{M_n - m_n}{2} \int_a^b |P_n(t)|\, dt,
\end{aligned}
$$

which proves our assertion. □

Corollary 4.2. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a,b]$, for some $n \ge 1$. Then for every $(n+1)$-balanced measure $\mu \in M[a,b]$ we have*

$$
\begin{aligned}
\left| \int_{[a,b]} f(t) d\mu(t) - \check{S}_n \right| &\le \frac{M_n - m_n}{2} \int_a^b |\check{\mu}_n(t)|\, dt \\
&\le \frac{M_n - m_n}{2} \frac{(b-a)^n}{n!} \|\mu\| ,
\end{aligned}
$$

where $\check{S}_n$ is from Corollary 2.3.

Proof. Apply Theorem 4.1 for the μ-harmonic sequence $(\check{\mu}_k, k \ge 1)$ and note that the condition $P_{n+1}(a) = P_{n+1}(b)$ reduces to $\check{\mu}_{n+1}(b) = 0$, which means that μ is $(n+1)$-balanced. □

Corollary 4.3. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a,b]$ for some $n \ge 1$. Further, let $(c_m, m \ge 1)$ be a sequence in $\mathbb{R}$ such that*

$$
\sum_{m \ge 1} |c_m| < \infty
$$

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and let $\{x_m; m \geq 1\} \subset [a,b]$ *satisfy the condition*

$$\sum_{m\geq 1} c_m (b - x_m)^n = 0.$$

Then

$$\begin{aligned}
&\left| \sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1} \sum_{k=1}^{n} c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \\
&\leq \frac{M_n - m_n}{2n!} \sum_{m\geq 1} |c_m| \, (b - x_m)^n \\
&\leq \frac{M_n - m_n}{2n!} (b-a)^n \sum_{m\geq 1} |c_m| \, .
\end{aligned}$$

Proof. Apply Corollary 4.2 for the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

Corollary 4.4. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n)} \in L_\infty[a,b]$ *for some* $n \geq 1$. *Then for every* $\mu \in M[a,b]$, *such that all* k*-moments of* μ *are zero for* $k = 0, \ldots, n$, *we have*

$$\begin{aligned}
\left| \int_{[a,b]} f(t) d\mu(t) \right| &\leq \frac{M_n - m_n}{2} \int_a^b |\check{\mu}_n(t)| \, dt \\
&\leq \frac{M_n - m_n}{2} \frac{(b-a)^n}{n!} \|\mu\| \, .
\end{aligned}$$

Proof. By [1, Theorem 5], the condition $m_k(\mu) = 0$, $k = 0, \ldots, n$ is equivalent to $\check{\mu}_k(b) = 0$, $k = 1, \ldots, n+1$. Apply Corollary 4.2 and note that in this case $\check{S}_n = 0$. □

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Corollary 4.5. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n)} \in L_\infty[a,b]$ *for some* $n \geq 1$. *Further, let* $(c_m, m \geq 1)$ *be a sequence in* $\mathbb{R}$ *such that*

$$\sum_{m\geq 1} |c_m| < \infty$$

and let $\{x_m; m \geq 1\} \subset [a,b]$. *If*

$$\sum_{m\geq 1} c_m = \sum_{m\geq 1} c_m x_m = \cdots = \sum_{m\geq 1} c_m x_m^n = 0,$$

then

$$\left| \sum_{m\geq 1} c_m f(x_m) \right| \leq \frac{M_n - m_n}{2n!} \sum_{m\geq 1} |c_m| (b - x_m)^n$$
$$\leq \frac{M_n - m_n}{2n!} (b-a)^n \sum_{m\geq 1} |c_m| .$$

Proof. Apply Corollary 4.4 for the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

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