A NOTE ON THE MODULUS OF U-CONVEXITY AND MODULUS OF W^*-CONVEXITY

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Abstract: We present some sufficient conditions for which a Banach space X has normal structure in term of the modulus of U-convexity, modulus of W*-convexity and the coefficient of weak orthogonality. Some known results are improved.

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1. Introduction

We assume that X and X^* stand for a Banach space and its dual space, respectively. By S_X and B_X we denote the unit sphere and the unit ball of a Banach space X, respectively. Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be nonexpansive provided the inequality

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for every $x, y \in C$. A Banach space X is said to have the fixed point property if every nonexpansive mapping $T : C \to C$ has a fixed point, where C is a nonempty bounded closed convex subset of a Banach space X.

Recall that a Banach space X is said to be uniformly non-square if there exists $\delta > 0$ such that $\|x + y\|/2 \leq 1 - \delta$ or $\|x - y\|/2 \leq 1 - \delta$ whenever $x, y \in S_X$. A bounded convex subset K of a Banach space X is said to have normal structure if for every convex subset H of K that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

A Banach space X is said to have weak normal structure if every weakly compact convex subset of X that contains more than one point has normal structure. In reflexive spaces, both notions coincide. A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \sup\{\|x - y\| : x, y \in K\}.$$

It was proved by W.A. Kirk that every reflexive Banach space with normal structure has the fixed point property (see [9]).

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The WORTH property was introduced by B. Sims in [15] as follows: a Banach space X has the WORTH property if

$$\lim_{n\to\infty}\Big|\|x_n+x\|-\|x_n-x\|\Big|=0$$

for all $x\in X$ and all weakly null sequences $\{x_n\}$. In [16], Sims introduced the following geometric constant

$$\omega(X)=\sup\left\{\lambda>0:\lambda\cdot\liminf_{n\to\infty}\|x_n+x\|\le\liminf_{n\to\infty}\|x_n-x\|\right\},$$

where the supremum is taken over all the weakly null sequences $\{x_n\}$ in X and all elements x of X. It was proved that $\frac{1}{3}\le\omega(X)\le 1$. It is known that X has the WORTH property if and only if $\omega(X)=1$. We also note here that $\omega(X)=\omega(X^*)$ in a reflexive Banach space (see [7]).

In [1] and [2], Gao introduced the modulus of U-convexity and modulus of W^*-convexity of a Banach space X, respectively, as follows:

$$U_X(\epsilon):=\inf\left\{1-\frac{1}{2}\|x+y\|:x,y\in S_X,f(x-y)\ge\epsilon\text{ for some }f\in\nabla_x\right\},$$

$$W_X^*(\epsilon):=\inf\left\{\frac{1}{2}f(x-y):x,y\in S_X,\|x-y\|\ge\epsilon\text{ for some }f\in\nabla_x\right\}.$$

Here $\nabla_x:=\{f\in S_{X^*}:f(x)=\|x\|\}$. S. Saejung (see [11], [12]) studied the above modulus extensively, and obtained some useful results as follows :

(1) If $U_X(\epsilon)>0$ or $W^*(\epsilon)>0$ for some $\epsilon\in(0,2)$, then X is uniformly non-square.

(2) If $U_X(\epsilon)>\frac{1}{2}\max\{0,\epsilon-1\}$ for some $\epsilon\in(0,2)$, then X has uniform normal structure. Further, if $U_X(\epsilon)>\max\{0,\epsilon-1\}$ for some $\epsilon\in(0,2)$, then X and X^* has uniform normal structure.

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(3) If $W_X^*(\epsilon) > \frac{1}{2}\max\{0, \epsilon - 1\}$ for some $\epsilon \in (0,2)$, then X and X^* has uniform normal structure.

In a recent paper [4], Gao introduced the following quadratic parameter, which is defined as

$$E(X) = \sup\left\{\|x+y\|^2 + \|x-y\|^2 : x, y \in S_X\right\}.$$

The constant is also a significant tool in the geometric theory of Banach spaces. Furthermore, Gao obtained the values of $E(X)$ for some classical Banach spaces. In terms of the constant, he obtained some sufficient conditions for a Banach space X to have uniform normal structure, which plays an important role in fixed point theory.

In this paper, we will show that a Banach space X has uniform normal structure whenever

$$U_X(1+\omega(X)) > \frac{1-\omega(X)}{2} \quad \text{or} \quad W_X^*(1+\omega(X)) > \frac{1-\omega(X)}{2}.$$

These results improve S. Saejung's and Gao's results. Furthermore, sufficient conditions for uniform normal structure in terms of $E(X)$ and $\omega(X)$ have been obtained which improve the results in [3].

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2. Uniform Normal Structure

As our proof uses the ultraproduct technique, we start by making some basic definitions. Let $\mathcal{U}$ be a filter on I. Then, $\{x_i\}$ is said to be convergent to x with respect to $\mathcal{U}$, denoted by $\lim_{\mathcal{U}} x_i = x$, if for each neighborhood V of x, $\{i \in I : x_i \in V\} \in \mathcal{U}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the ordering of set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathcal{U}$ is an ultrafilter, then

(1) for any $A \subseteq I$, either $A \in \mathcal{U}$ or I $A \in \mathcal{U}$;

(2) if $\{x_i\}$ has a cluster point x, then $\lim_{\mathcal{U}} x_i$ exists and equals x.

Let $\{X_i\}$ be a family of Banach spaces and $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$. Let $\mathcal{U}$ be an ultrafilter on I and $N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}$. The ultraproduct of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. We will use $\widetilde{(x_i)}_{\mathcal{U}}$ to denote the element of the ultraproduct. In the following, we will restrict our set I to be $\mathbb{N}$ (the set of $\mathcal{U}$ natural numbers), and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $\widetilde{X}_{\mathcal{U}}$ to denote the ultraproduct. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}_{\mathcal{U}}$ isometrically.

Lemma 2.1 (see [5]). *Let X be a Banach space without weak normal structure, then there exists a weakly null sequence $\{x_n\}_{n=1}^{\infty} \subseteq S_X$ such that*

$$\lim_n \|x_n - x\| = 1 \ \text{ for all } \ x \in co\{x_n\}_{n=1}^{\infty}$$

Theorem 2.2. *If $U_X(1 + \omega(X)) > \frac{1-\omega(X)}{2}$, then X has uniform normal structure.*

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Proof. It suffices to prove that X has weak normal structure whenever

$$U_X(1+\omega(X)) > \frac{1-\omega(X)}{2}.$$

In fact, since $\frac{1}{3} \le \omega(X) \le 1$, we have

$$U_X(\epsilon) > \frac{1-\omega(X)}{2} \ge 0$$

for some $\epsilon \in (0,2)$. This implies that X is super-reflexive, and then $U_X(\epsilon) = U_{\tilde{X}}(\epsilon)$ (see [11]). Now suppose that X fails to have weak normal structure. Then, by the Lemma 2.1, there exists a weakly null sequence $\{x_n\}_{n=1}^{\infty}$ in S_X such that

$$\lim_n \|x_n - x\| = 1 \text{ for all } x \in co\{x_n\}_{n=1}^{\infty}.$$

Take $\{f_n\} \subset S_{X^*}$ such that $f_n \in \nabla_{x_n}$ for all $n \in \mathbb{N}$. By the reflexivity of X^*, without loss of generality we may assume that $f_n \rightharpoonup f$ for some $f \in B_{X^*}$ (where $\rightharpoonup$ denotes weak star convergence). We now choose a subsequence of $\{x_n\}_{n=1}^{\infty}$, denoted again by $\{x_n\}_{n=1}^{\infty}$, such that

$$\lim_n \|x_{n+1} - x_n\| = 1, \quad |(f_{n+1} - f)(x_n)| < \frac{1}{n}, \quad f_n(x_{n+1}) < \frac{1}{n}$$

for all $n \in \mathbb{N}$. It follows that

$$\lim_n f_{n+1}(x_n) = \lim_n (f_{n+1} - f)(x_n) + f(x_n) = 0.$$

Put $\tilde{x} = (x_{n+1} - x_n)_{\mathcal{U}}$, $\tilde{y} = [\omega(X)(x_{n+1} + x_n)]_{\mathcal{U}}$, and $\tilde{f} = (-f_n)_{\mathcal{U}}$. By the definition of $\omega(X)$ and Lemma 2.1, then

$$\|\tilde{f}\| = \tilde{f}(\tilde{x}) = \|\tilde{x}\| = 1$$

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and

$$\|\tilde{y}\| = \|[\omega(X)(x_{n+1} + x_n)]_{\mathcal{U}}\| \leq \|x_{n+1} - x_n\| = 1.$$

Furthermore, we have

$$\begin{aligned}\tilde{f}(\tilde{x} - \tilde{y}) &= \lim_{\mathcal{U}}(-f_n)\Big((1 - \omega(X))x_{n+1} - (1 + \omega(X))x_n\Big)\\ &= 1 + \omega(X),\end{aligned}$$

$$\begin{aligned}\|\tilde{x} + \tilde{y}\| &= \lim_{\mathcal{U}}\|(1 + \omega(X))x_{n+1} - (1 - \omega(X))x_n\|\\ &\geq \lim_{\mathcal{U}}(f_{n+1})\Big((1 + \omega(X))x_{n+1} - (1 - \omega(X))x_n\Big)\\ &= 1 + \omega(X).\end{aligned}$$

From the definition of $U_X(\epsilon)$, we have

$$U_X(1 + \omega(X)) = U_{\tilde{X}}(1 + \omega(X)) \leq \frac{1 - \omega(X)}{2},$$

which is a contradiction. Therefore

$$U_X(1 + \omega(X)) > \frac{1 - \omega(X)}{2}$$

implies that X has uniform normal structure. $\square$

Remark 1. Compare to the result of S. Saejung (2). Let $\epsilon = 1 + \omega(X)$. Then $U_X(\epsilon) > \frac{2-\epsilon}{2}$ implies that X has uniform normal structure from Theorem 2.2. It is well known that $\frac{1}{3} \leq \omega(X) \leq 1$, therefore $\frac{\epsilon-1}{2} > \frac{2-\epsilon}{2}$ whenever $\omega(X) > \frac{1}{2}$, therefore Theorem 2.2 strengthens the result of S. Saejung (2).

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The modulus of convexity of X is the function $\delta_X(\epsilon) : [0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \|x-y\| = \epsilon\right\}$$
$$= \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}.$$

The function $\delta_X(\epsilon)$ is strictly increasing on $[\epsilon_0(X), 2]$. Here $\epsilon_0(X) = \sup\{\epsilon : \delta_X(\epsilon) = 0\}$ is the characteristic of convexity of X. Also, X is uniformly non-square provided $\epsilon_0(X) < 2$. Some sufficient conditions for which a Banach space X has uniform normal structure in terms of the modulus of convexity have been widely studied in [3], [5], [13], [18]. It is easy to prove that $U_X(\epsilon) \ge \delta_X(\epsilon)$, therefore we have the following corollary which strengthens Theorem 6 of Gao [3].

Corollary 2.3. *If $\delta_X((1+\omega(X)) > \frac{1-\omega(X)}{2}$, then X has uniform normal structure.*

Remark 2. In fact, it is well known that $J(X) < \epsilon$ if and only if $\delta_X(\epsilon) > 1 - \frac{\epsilon}{2}$ (see [6]). Therefore Corollary 2.3 is equivalent to $J(X) < 1 + \omega(X)$ implies that X has uniform normal structure (see [7, Theorem 2]). Moreover, if X is the Bynum space $b_{2,\infty}$, then X does not have normal structure and $\delta_X((1+\omega(X)) = \frac{1-\omega(X)}{2}$. Hence Theorem 2.2 and Corollary 2.3 are sharp.

It is well known that $\epsilon_0(X) = 2\rho'_{X^*}(0)$. Here, $\rho'_X(0) = \lim_{t\to 0} \frac{\rho_X(t)}{t}$, where $\rho_X(t)$ is the modulus of smoothness defined as

$$\rho_X(t) = \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in S_X\right\}.$$

Therefore we have the following corollary.

Corollary 2.4. *If $\delta_X(2\omega(X)) > \frac{1-\omega(X)}{2}$, then X and X^* have uniform normal structure.*

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Proof. From $2\omega(X) \le 1+\omega(X)$ and the monotonicity of $\delta_X(\epsilon)$, we have that X has uniform normal structure from Corollary 2.3. It is well known that $\omega(X) = \omega(X^*)$ in a reflexive Banach space. So the inequality $\rho'_{X^*}(0) < \omega(X)$, or, equivalently, $\epsilon_0(X) < 2\omega(X)$ imply X^* has uniform normal structure (see [10], [13]). From the definition of $\epsilon_0(X)$, obviously the condition $\delta_X(2\omega(X)) > \frac{1-\omega(X)}{2}$ implies that $\epsilon_0(X) < 2\omega(X)$. So X^* have uniform normal structure. □

Theorem 2.5. *If $W^*_X(1+\omega(X)) > \frac{1-\omega(X)}{2}$, then X has uniform normal structure.*

Proof. It suffices to prove that X has weak normal structure whenever $W^*_X(1+\omega(X)) > \frac{1-\omega(X)}{2}$. In fact, since $\frac{1}{3} \le \omega(X) \le 1$, we have $W^*_X(2\epsilon) > \frac{1-\omega(X)}{2} \ge 0$ for some $\epsilon \in (0,2)$. This implies that X is super-reflexive, and $W^*_X(\epsilon) = W^*_{\tilde{X}}(\epsilon)$ (see [12]). Repeating the arguments in the proof of Theorem 2.2, and $\tilde{x} = (x_n - x_{n+1})_{\mathcal{U}}$, $\tilde{y} = [\omega(X)(x_{n+1} + x_n)]_{\mathcal{U}}$, and $\tilde{f} = (f_n)_{\mathcal{U}}$. Then

$$f(\tilde{x}) = \|\tilde{x}\| = 1 \quad \text{and} \quad \|\tilde{y}\| \le 1.$$

Furthermore, we have

$$\begin{aligned}\|\tilde{x}-\tilde{y}\| &= \lim_{\mathcal{U}} \|(1+\omega(X))x_{n+1} - (1-\omega(X))x_n\| \\ &\ge \lim_{\mathcal{U}} (f_{n+1})\Big((1+\omega(X))x_{n+1} - (1-\omega(X))x_n\Big) = 1+\omega(X),\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\tilde{f}(\tilde{x}-\tilde{y}) &= \frac{1}{2}\lim_{\mathcal{U}}(f_n)\Big((1-\omega(X))x_n - (1+\omega(X))x_{n+1}\Big) \\ &= \frac{1-\omega(X)}{2}.\end{aligned}$$

However, this implies

$$W^*_X(1+\omega(X)) = W^*_{\tilde{X}}(1+\omega(X)) \le \frac{1-\omega(X)}{2}$$

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which is a contradiction. Therefore

$$W_X^*(1+\omega(X)) > \frac{1-\omega(X)}{2}$$

implies that X has uniform normal structure. $\square$

Remark 3. Similarly, the above theorem strengthens the result of S. Saejung (3), whenever $\omega(X) > \frac{1}{2}$. Since $W_X^*(\epsilon) \geq \delta_X(\epsilon)$, therefore we also obtain Corollary 2.3 from Theorem 2.5.

The following theorem can be found in [14].

Theorem 2.6. *Let X be a Banach space, we have*

$$E(X) = \sup\{\epsilon^2 + 4(1-\delta_X(\epsilon))^2 : \epsilon \in (0,2]\}$$

Remark 4. Letting $\epsilon \to 2^-$ in Theorem 2.6, we obtain the following inequality

$$E(X) \geq 4 + [\epsilon_0(X)]^2.$$

Corollary 2.7. *If $E(X) < 2(1+\omega(X))^2$, then X and X^* have uniform normal structure.*

Proof. From Theorem 2.6, $E(X) < 2(1+\omega(X))^2$ implies that $\delta_X((1+\omega(X)) > \frac{1-\omega(X)}{2}$, so X has uniform normal structure from Corollary 2.3. It is well known that $\epsilon_0(X) < 2\omega(X)$ implies that X^* have uniform normal structure. Therefore, from Remark 4, $E(X) < 4(1+\omega(X)^2)$ implies that X^* have uniform normal structure. Obviously

$$E(X) < 2(1+\omega(X))^2 \leq 4(1+\omega(X)^2)$$

implies X^* have uniform normal structure. $\square$

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Remark 5. In [3], Gao obtained that if $E(X) < 1 + 2\omega(X) + 5(\omega(X))^2$, then X has uniform normal structure. Comparing the result of Gao and Corollary 2.7, we have the following equality

$$2(1+\omega(X))^2 - 1 - 2\omega(X) - 5(\omega(X))^2 = (1-\omega(X))(3\omega(X)+1).$$

It is well known that $\frac{1}{3} \leq \omega(X) \leq 1$, so when $\omega(X) < 1$, we have

$$(1-\omega(X))(3\omega(X)+1) > 0.$$

Therefore Corollary 2.7 is strict generalization of Gao's result. Moreover this is extended to conclude uniform normal structure for X^*. In fact repeating the arguments in [7], we have that $E(b_{2,\,\infty}) = 3+2\sqrt{2}$, where $b_{2,\,\infty}$ is the Bynum space which does not have normal structure and $E(X) = 2(1+\omega(X))^2$ (note that $\omega(b_{2,\,\infty}) = \frac{\sqrt{2}}{2}$). Therefore Corollary 2.7 is sharp.

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