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# ON PERTURBED TRAPEZOID INEQUALITIES

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ABSTRACT. A method for obtaining large numbers of perturbed trapezoid inequalities is derived.

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## 1. INTRODUCTION

Considerable attention has been given recently to extensions of the trapezoid inequality which reads as follows:

If $f \in C^{(2)}[a,b]$ is a real-valued function with $\left|f^{(2)}(x)\right| \le M_2$ then

$$\left|\int_a^b f(x)dx - \frac{1}{2}(b-a)[f(a)+f(b)]\right| \le \frac{1}{12}M_2(b-a)^3. \tag{1.1}$$

Another form of this inequality, applicable when $f \in C^{(1)}[a,b]$, with $\gamma_1 \le f^{(1)}(x) \le \Gamma_1$ is:

$$\left|\int_a^b f(x)dx - \frac{1}{2}(b-a)[f(a)+f(b)]\right| \le \frac{1}{8}(\Gamma_1-\gamma_1)(b-a)^2. \tag{1.2}$$

Extensions of these are called *perturbed* or *corrected* trapezoid inequalities. Many of this type have been studied lately and, for example, we quote three inequalities from [1]: Writing

$$L_1 \equiv \left|\int_a^b f(x)dx - \frac{1}{2}(b-a)[f(a)+f(b)] + \frac{1}{12}(b-a)^2[f^{(1)}(b)-f^{(1)}(a)]\right|$$

these are

$$L_1 \le \frac{1}{36\sqrt{3}}(\Gamma_2-\gamma_2)(b-a)^3, \tag{1.3}$$

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$$L_1 \leq \frac{1}{384}(\Gamma_3 - \gamma_3)(b-a)^4, \tag{1.4}$$

$$L_1 \leq \frac{1}{720} M_4 (b-a)^5, \tag{1.5}$$

and two others, from [2] are

$$L_1 \leq \frac{1}{18\sqrt{3}} M_2 (b-a)^3 \tag{1.6}$$

and

$$L_1 \leq \frac{1}{192} M_3 (b-a)^4 \tag{1.7}$$

In each of these, the function $f : [a,b] \to \mathbb{R}$ is supposed to have a continuous derivative of order the same as the suffix appearing on the right hand side and

$$\Gamma_\nu = \sup f^{(\nu)}(x), \qquad \gamma_\nu = \inf f^{(\nu)}(x)$$
$$\text{and} \quad M_\nu = \sup \left| f^{(\nu)}(x) \right| \quad \text{over the interval.}$$

We note, in passing, that the obvious relationship between the coefficients in (1.3) and (1.6) and between those in (1.4) and (1.7) is not accidental, but can be explained by a fact pointed out in [3], namely:

**Lemma 1.1.** *If $F, g \in C[a,b]$ and $\int_a^b g(x)dx = 0$ and if $\gamma \leq F(x) \leq \Gamma$ and $|F(x)| \leq M$, then both*

$$\left| \int_a^b F(x)g(x)dx \right| \leq M \int_a^b |g(x)|\, dx$$

*and*

$$\left| \int_a^b F(x)g(x)dx \right| \leq \frac{1}{2}(\Gamma - \gamma) \int_a^b |g(x)|\, dx$$

*are true. Of course, the stronger one is the latter.*

This phenomenon occurs in connection with the pairs of inequalities mentioned above and explains the relationships of the pairs of coefficients.

The methods of deriving these inequalities, of which we are aware, become more complicated as the order increases and the techniques used to derive them appear to be special for each case. So the derivatives of $f$ appearing on the left hand side are usually no higher than of the first order.

It is the purpose of this note to give a general formula for such perturbed inequalities merely by applying continued integration-by-parts to a certain integral. This formula will allow derivatives of any order to appear on the left and several types of dominating terms to appear on the right hand side. Throughout we shall deal with functions defined over the interval $x \in [-1,1]$. There is clearly no loss of generality here since, after the inequality is obtained, one can, if one wishes, return to the interval $u \in [a,b]$ using the transformation

$$2u = a(1-x) + b(1+x).$$

Moreover, the interval $[-1,1]$ is a natural one in the sense that it allows the Legendre polynomials to enter the analysis and this simplifies matters considerably. Indeed, it is the use of this interval rather than $[a,b]$, which makes the whole matter more transparent.

## 2. DERIVATION

Rather than state a theorem here we shall proceed directly with our analysis and then various inequalities can be given later. It will be assumed throughout that the function $f$ possesses continuous derivatives of all the orders which appear, throughout the closed interval $[-1, 1]$.

We consider the integral

$$I_n \equiv \int_{-1}^{1} f^{(2n)}(x)(x^2-1)^n dx.$$

Due to the fact that the factor $(x^2-1)^n$ vanishes $n$ times at $-1$ and at $+1$, we find, on integrating by parts $k$ times, that

$$I_n = I_{n+k} = (-1)^k \int_{-1}^{1} f^{(2n-k)}(x) D^k[(x^2-1)^n] dx \quad \text{for} \quad k = 0, 1, 2, \dots, n, \tag{2.1}$$

where $D$ denotes differentiation with respect to $x$.

We note here, concerning the case $k = n$, that Rodrigues formula is:

$$D^n[(x^2-1)^n] = 2^n n! P_n(x),$$

where $P_n(x)$ is the Legendre polynomial of degree $n$.

So, from (2.1), the integral in that particular case takes the form:

$$\begin{aligned} I_{2n} &= (-1)^n \int_{-1}^{1} f^{(n)}(x) D^n[(x^2-1)^n] dx \\ &= (-1)^n 2^n n! \int_{-1}^{1} f^{(n)}(x) P_n(x) dx \\ &= (-1)^n 2^n n! Q_n \quad \text{(say)}. \end{aligned} \tag{2.2}$$

Continuing with integration by parts, we have:

$$\begin{aligned} Q_n &= \left[f^{(n-1)}(x)P_n(x)\right]_{-1}^{+1} - \int_{-1}^{1} f^{(n-1)}(x) P_n^{(1)}(x) dx \\ &= \left[f^{(n-1)}(x)P_n(x)\right]_{-1}^{+1} - [f^{(n-2)}(x)P_n^{(1)}(x)]_{-1}^{+1} + \int_{-1}^{1} f^{(n-2)}(x) P_n^{(2)}(x) dx \\ &\dots \\ &= \sum_{p=0}^{n-1} (-1)^p \left[f^{(n-1-p}(x)P_n^{(p)}(x)\right]_{-1}^{+1} + (-1)^n \int_{-1}^{1} f(x) P_n^{(n)}(x) dx \\ &= \sum_{p=0}^{n-1} (-1)^p \left\{[f^{(n-1-p)}(1) + (-1)^{n+p+1} f^{(n-1-p)}(-1)] P_n^{(p)}(1)\right\} \\ &\qquad + (-1)^n \int_{-1}^{1} f(x) P_n^{(n)}(x) dx. \end{aligned} \tag{2.3}$$

Here we have used the fact that $P_n^{(k)}(-x) = (-1)^{n+k} P_n^{(k)}(x)$ since $P_n(x)$ contains only even (odd) powers of $x$ according as $n$ is even (odd).

One also has

$$P_n^{(n)}(x) = \frac{(2n)!}{2^n n!} \tag{2.4}$$

and so, collecting the results from (2.1), (2.2), (2.3) and (2.4) we get :

$$\text{(2.5)}\quad \int_{-1}^{+1} f(x)dx + (-1)^n \frac{2^n n!}{(2n)!} \sum_{p=0}^{n-1} (-1)^p \big\{ [f^{(n-1-p)}(1) + (-1)^{n+p+1} f^{(n-1-p)}(-1)] P_n^{(p)}(1) \big\} = \frac{1}{(2n)!} I_{2n} = (-1)^n \frac{2^n n!}{(2n)!} \int_{-1}^{1} f^{(n)}(x) P_n(x) dx.$$

*Here we have chosen to write the right hand side of (2.5) in terms of* $I_{2n}$ *but, because of (2.1), it could be replaced by any of the following*:

$$\frac{1}{(2n)!}(-1)^k \int_{-1}^{1} f^{(2n-k)}(x) D^k[(x^2-1)^n] dx \quad \text{with} \quad k = 0, 1, 2, \dots, n.$$

So our result to this stage - after changing the order of summation - reads:

$$\text{(2.6)}\quad \int_{-1}^{+1} f(x)dx + \frac{2^n n!}{(2n)!} \sum_{q=0}^{n-1} (-1)^{q+1} \left\{ [f^{(q)}(1) + (-1)^q f^{(q)}(-1)] P_n^{(n-1-q)}(1) \right\} = \frac{1}{(2n)!}(-1)^k \int_{-1}^{1} f^{(2n-k)}(x) D^k[(x^2-1)^n] dx \quad \text{with} \quad k = 0, 1, 2, \dots, n$$

These are, of course, quadrature formulae - with error terms - which involve only the end points of the interval.

## 3. The Perturbed Trapezoid Inequalities

From (2.6) we get the following family of inequalities:

$$\text{(3.1)}\quad \left| \int_{-1}^{+1} f(x)dx + \frac{2^n n!}{(2n)!} \sum_{q=0}^{n-1} (-1)^{q+1} \left\{ [f^{(q)}(1) + (-1)^q f^{(q)}(-1)] P_n^{(n-1-q)}(1) \right\} \right|$$
$$\le \frac{1}{2}(\Gamma_{2n-k} - \gamma_{2n-k}) \frac{1}{(2n)!} \int_{-1}^{1} \left| D^k[(x^2-1)^n] \right| dx \quad \text{with} \quad k = 1, 2, \dots, n$$
$$\text{or} \quad \le M_{2n} \frac{1}{(2n)!} \int_{-1}^{1} \left| (x^2-1)^n \right| dx \quad \text{when} \quad k = 0,$$

wherein it remains just to evaluate the last term in particular cases.

*In this we have invoked the lemma above, since* $D^k[(x^2-1)^n]$ *satisfies the requirements put on* $g(x)$ *for* $k = 1, 2, \dots, n$ *(though not for* $k = 0$*)*

We now give some examples.

(1) Take $n = 2, k = 2$ in (3.1). and, since $P_2(1) = 1$ and $P_2^{(1)}(1) = 3$

$$\left| \int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] + \frac{1}{3}[f^{(1)}(1) - f^{(1)}(-1)] \right| \le \frac{1}{3} \cdot \frac{1}{2}(\Gamma_2 - \gamma_2) \int_{-1}^{1} |P_2(x)| dx.$$

That is

$$\text{(3.2)}\quad \left| \int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] + \frac{1}{3}[f^{(1)}(1) - f^{(1)}(-1)] \right| \le \frac{2}{9\sqrt{3}}(\Gamma_2 - \gamma_2)$$

which is the $[-1, +1]$ form of (1.3).

(2) Take $n = 2, k = 1$ in (3.1). and, in a similar fashion, we get :

$$\left|\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] \ + \frac{1}{3}[f^{(1)}(1) - f^{(1)}(-1)]\right| \leq \frac{1}{24}(\Gamma_3 - \gamma_3)$$

which is (1.4) for the interval $[-1, +1]$.
(3) Take $n = 2, k = 0$ in (3.1). and we get:

$$\left|\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] \ + \frac{1}{3}[f^{(1)}(1) - f^{(1)}(-1)]\right| \leq \frac{2}{45}M_4$$

which is (1.5) for the interval $[-1, +1]$.

We now give four more examples of these inequalities, which we believe to be new.

These will be the cases of (3.1) in which $n = 3$ and $k = 0, 1, 2, 3$. Since $n = 3$ is fixed for each case, the left hand side for each will be:

$$L_2 = \left|\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] \ + \frac{2}{5}[f^{(1)}(1) - f^{(1)}(-1)] \ - \frac{1}{15}[f^{(2)}(1) + f^{(2)}(-1)]\right|$$

and the inequalities are :
(4) $n = 3, k = 3$.

$$L_2 \leq \frac{13}{600}(\Gamma_3 - \gamma_3).$$

(5) $n = 3, k = 2$.

$$L_2 \leq \frac{4\sqrt{5}}{1875}(\Gamma_4 - \gamma_4).$$

(6) $n = 3, k = 1$.

$$L_2 \leq \frac{1}{720}(\Gamma_5 - \gamma_5).$$

(7) $n = 3, k = 0$.

$$L_2 \leq \frac{2}{1575}M_6.$$

## 4. FINAL REMARKS

As mentioned above, most of the inequalities in the literature are limited to those involving $f$ and $f^{(1)}$. There are exceptions however, one being the following (which we give in its $[-1, 1]$ form):

$$\left|\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] + \frac{1}{12}[f^{(1)}(1) - f^{(1)}(-1)] \right.$$
$$\left. - \frac{1}{45}[f^{(3)}(1) - f^{(3)}(-1)]\right| \leq \frac{4}{945}M_6.$$

This inequality, in which $f^{(2)}$ does not appear, is obviously not a member of the family derived above. It is to be found in [4].

We conclude this note by returning to those inequalities of the type of (1.3) in which the second derivative appears on the right and only $f$ and $f^{(1)}$ appear on the left.

Again we integrate by parts but starting 'from the other end' so to speak. We integrate

$$\int_{-1}^{+1} f(x)1dx$$

by parts twice, allowing the $1$ to integrate to $x$ and then the $x$ to $\frac{1}{2}(x^2 - a)$ ($a$ constant) getting the quadrature formula:

$$\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] + \frac{1-a}{2}[f^{(1)}(1) - f^{(1)}(-1)] = \int_{-1}^{+1} f^{(2)}(x)\frac{x^2-a}{2}dx, \tag{4.1}$$

which leads to

$$\left|\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] + \frac{1-a}{2}[f^{(1)}(1) - f^{(1)}(-1)]\right| \le M_2 \int_{-1}^{+1} \left|\frac{x^2-a}{2}\right| dx. \tag{4.2}$$

Taking $a = \frac{1}{3}, 0$ and $-1$ in this gives

$$\left|\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] + \frac{1}{3}[f^{(1)}(1) - f^{(1)}(-1)]\right| \le \frac{4}{9\sqrt{3}}M_2,$$

$$\left|\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] + \frac{1}{2}[f^{(1)}(1) - f^{(1)}(-1)]\right| \le \frac{1}{3}M_2$$

and

$$\left|\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] + [f^{(1)}(1) - f^{(1)}(-1)]\right| \le \frac{4}{3}M_2$$

respectively. The first of these is the $M$ form of (3.2).

(4.2) illustrates that there is an infinity of perturbed trapezoid inequalities, even when the derivatives appearing on each side are the same. Also, (4.2) leads one to seek the value of $a$ which minimises

$$\int_{-1}^{+1} \left|\frac{x^2-a}{2}\right| dx.$$

This value is easily found. It is given by

$$a = \frac{1}{4} \quad \text{when} \quad \int_{-1}^{+1} \left|\frac{x^2-a}{2}\right| dx = \frac{1}{4}$$

So the "tightest" inequality of this type ($f$, $f^{(1)}$, $M_2$) is

$$\left|\int_{-1}^{+1} f(x)dx - [f(1) + f(-1)] + \frac{3}{8}[f^{(1)}(1) - f^{(1)}(-1)]\right| \le \frac{1}{4}M_2.$$

**Note:** The constants on the right hand sides of all the examples in this note are *best possible* in the sense that equality can be achieved, at least in the limit, by a chosen sequence of functions. We illustrate this for the case of (4.2).

In view of (4.1) and (4.2), equality in (4.2) would be obtained by a function $f$ for which

$$f^{(2)}(x) = \operatorname{sgn}\left(\frac{x^2-a}{2}\right) \qquad \text{(with} \quad M_2 = 1\text{)}.$$

This last function, however, is not continuous in $[-1, 1]$. But we can, instead, find a function $f$ and an infinite sequence of functions $f_n \in C^{(2)}[-1, 1]$ such that

$$f_n \to f \quad \text{uniformly} \quad \text{in } [-1, 1]$$

and

$$f_n^{(2)}(x) \to \operatorname{sgn}\left(\frac{x^2 - a}{2}\right)$$

boundedly and almost everywhere (i.e. except possibly at $\pm\sqrt{a}$) in $[-1, 1]$. And so equality in (4.2) is obtained in the limit.

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