

# THE ARITHMETIC-ALGEBRAIC MEAN INEQUALITY VIA SYMMETRIC MEAN

**OSCAR G. VILLAREAL**

Department of Mathematics  
University of California, Irvine  
340 Rowland Hall, Irvine, CA 92697-3875, USA  
EMail: [ovillare@math.uci.edu](mailto:ovillare@math.uci.edu)  
URL: <http://math.uci.edu/~oscar>

*Received:* 02 August, 2008

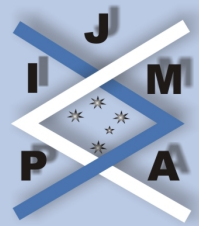
*Accepted:* 06 August, 2008

*Communicated by:* **P.S. Bullen**

*2000 AMS Sub. Class.:* Primary 26D15

*Key words:* Arithmetic mean, Geometric mean, Symmetric mean, Inequality

*Abstract:* We give two proofs of the arithmetic-algebraic mean inequality by giving a characterization of symmetric means.



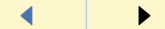
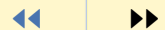
**AG mean**

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

[Title Page](#)

[Contents](#)



Page 1 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

# Contents

1	Introduction	3
2	First Proof of Theorem 1.1	4
3	Second Proof of Theorem 1.1	7



---

AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

---

Title Page

Contents



Page 2 of 12

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

Title Page

Contents



Page 3 of 12

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

## 1. Introduction

Let  $(a_1, \dots, a_n) \in \mathbb{R}^n$  be an  $n$ -tuple of positive real numbers. The inequality of arithmetic-algebraic means states that

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

The left-hand side of the inequality is called the geometric mean and the right-hand side the arithmetic mean. We will refer to this inequality as  $AG_n$  to specify the size of the  $n$ -tuple. This inequality has been known in one form or another since antiquity and numerous proofs have been given over the centuries. Bullen's book [1], for example, gives over seventy proofs. We give two proofs based on a characterization of symmetric means as the smallest among the means constructed by homogeneous symmetric polynomials. The main result is

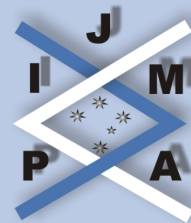
**Theorem 1.1.** *Let  $(a_1, \dots, a_n) \in \mathbb{R}^n$  be an  $n$ -tuple of positive real numbers,  $f(x_1, \dots, x_n)$  be a homogenous symmetric polynomial of degree  $k$ ,  $1 \leq k \leq n$ , having positive coefficients, and let  $s_k(x_1, \dots, x_n)$  be the  $k$ -th elementary symmetric polynomial. Then*

$$\frac{s_k(a_1, \dots, a_n)}{\binom{n}{k}} \leq \frac{f(a_1, \dots, a_n)}{f(1, \dots, 1)}.$$

*There is equality if and only if the  $a_i$ 's are all equal.*

Note that  $\binom{n}{k} = s_k(1, \dots, 1)$ . Similarly we note that if the coefficients of  $f$  are all equal to one, then  $f(1, \dots, 1)$  is the number of monomials comprising  $f$ . Thus it is reasonable to think of  $\frac{f(a_1, \dots, a_n)}{f(1, \dots, 1)}$  as a mean for  $f$  as in the theorem. The theorem implies the arithmetic-algebraic mean inequality by taking  $k = n$ ,  $f(x_1, \dots, x_n) = (x_1 + \cdots + x_n)^n$  so that  $f(1, \dots, 1) = n^n$ , and then taking  $n$ -th roots.

We shall give two proofs of Theorem 1.1. The first depends on Muirhead's Theorem. The second proves  $AG_n$  and Theorem 1.1 in one induction step.



AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

[Title Page](#)

[Contents](#)



Page 4 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

## 2. First Proof of Theorem 1.1

For any function  $f(x_1, \dots, x_n)$ , the symmetric group  $S_n$  acts on the  $x_k$ 's, and so we set

$$\sum! f(x_1, \dots, x_n) = \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

In particular, for an  $n$ -tuple of nonnegative real numbers  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , when

$$f(x_1, \dots, x_n) = x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

we set

$$[\alpha] = \frac{1}{n!} \sum! x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Note that  $[1, 0, \dots, 0]$  is the arithmetic mean while  $[\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}]$  is the geometric mean.

Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $\beta = (\beta_1, \beta_2, \dots, \beta_n)$  be two  $n$ -tuples of nonnegative real numbers. Muirhead's theorem gives conditions under which an inequality exists of the form

$$[\alpha] = \frac{1}{n!} \sum! x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \leq [\beta] = \frac{1}{n!} \sum! x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$$

valid for all positive  $x_i$ 's. To do this we first note that  $[\alpha]$  is invariant under permutations of the  $\alpha_i$ 's and so we introduce an equivalence relation as follows. We write  $\alpha \leq \beta$  if some permutation of the coordinates of  $\alpha$  and  $\beta$  satisfies

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_n,$$

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \text{ and } \beta_1 \geq \beta_2 \geq \cdots \geq \beta_n,$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k \leq \beta_1 + \beta_2 + \cdots + \beta_k \text{ for } k = 1, 2, \dots, n.$$

Muirhead's Theorem states

[Title Page](#)[Contents](#)

Page 5 of 12

[Go Back](#)[Full Screen](#)[Close](#)**Theorem 2.1.** *The inequality*

$$[\alpha] = \frac{1}{n!} \sum! x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \leq [\beta] = \frac{1}{n!} \sum! x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$$

is valid for all positive  $x_i$ 's if and only if  $\alpha \leq \beta$ . There is equality only when  $\alpha = \beta$  or the  $x_i$ 's are all equal.

We refer to [2] for the proof of this theorem and further discussion. Before giving the first proof of Theorem 1.1 we need a lemma.

**Lemma 2.2.** *Let  $(a_{1j}, \dots, a_{n_jj}) \in \mathbb{R}^{n_j}$  for  $j = 1, \dots, m$ , and let  $c_1, \dots, c_m$  be positive real numbers. Suppose  $a \leq \frac{a_{1j} + \dots + a_{n_jj}}{n_j}$  for each  $j$ . Then*

$$a \leq \frac{c_1(a_{11} + \dots + a_{n_11}) + c_2(a_{12} + \dots + a_{n_22}) + \dots + c_m(a_{1m} + \dots + a_{n_mm})}{c_1n_1 + c_2n_2 + \dots + c_mn_m}.$$

*There is equality if and only if the original inequalities are all equalities.*

*Proof.* For each  $j$  we rewrite  $a \leq \frac{a_{1j} + \dots + a_{n_jj}}{n_j}$  as  $n_j a \leq a_{1j} + \dots + a_{n_jj}$ . We then multiply by  $c_j$  to obtain  $c_j n_j a \leq c_j(a_{1j} + \dots + a_{n_jj})$ . We now add over all  $j$  to obtain

$$\begin{aligned} & (c_1n_1 + c_2n_2 + \dots + c_mn_m)a \\ & \leq c_1(a_{11} + \dots + a_{n_11}) + c_2(a_{12} + \dots + a_{n_22}) + \dots + c_m(a_{1m} + \dots + a_{n_mm}). \end{aligned}$$

By dividing by the coefficient of  $a$  we get the lemma. Note that if at least one of the original inequalities is strict, then the argument shows the final inequality is also strict.  $\square$

*Proof of Theorem 1.1.* Let  $f(x_1, \dots, x_n)$  be a homogenous symmetric polynomial of degree  $k$  with positive coefficients. The monomials of  $f$  break up into orbits under



AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

Title Page

Contents



Page 6 of 12

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

the action of the symmetric group  $S_n$  and so we may write  $f = c_1 f_1 + \cdots + c_m f_m$ ,  $c_j > 0$  where each  $f_j$  is a homogenous polynomial with all non-zero coefficients equal to one and for which  $S_n$  acts transitively. In view of Lemma 2.2, for the proof of Theorem 1.1 we may assume  $f(x_1, \dots, x_n)$  itself is a homogenous polynomial of degree  $k$  with all non-zero coefficients equal to one and for which  $S_n$  acts transitively.

For such an  $f$ , it follows that there exists an  $\alpha$  such that  $f(x_1, \dots, x_n) = t[\alpha]$ , where  $t = f(1, 1, \dots, 1)$  is the number of monomials comprising  $f$ . We note that  $s_k(x_1, \dots, x_n) = \binom{n}{k}[1, 1, \dots, 1, 0, \dots, 0]$  with  $k$  1's and  $n - k$  0's. Since  $[1, 1, \dots, 1, 0, \dots, 0] \leq \alpha$ , Theorem 2.1 gives the result.  $\square$



AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

Title Page

Contents



Page 7 of 12

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

### 3. Second Proof of Theorem 1.1

The inequality of arithmetic-geometric means can be stated in polynomial form in two ways. By taking  $n$ -th powers we get

$$a_1 \cdots a_n \leq \left( \frac{a_1 + \cdots + a_n}{n} \right)^n.$$

Alternately, if we let  $a_i = A_i^n$  we get

$$A_1 \cdots A_n \leq \frac{A_1^n + \cdots + A_n^n}{n}.$$

We will refer to these equivalent inequalities also as  $AG_n$ .

Let  $f(x_1, \dots, x_n)$  be a homogenous symmetric polynomial. The monomials of  $f$  break up into orbits under the action of the symmetric group  $S_n$  and so we may write  $f = c_1 f_1 + \cdots + c_m f_m$ ,  $c_j \in \mathbb{R}$  where each  $f_j$  is a homogenous polynomial with all non-zero coefficients equal to one and for which  $S_n$  acts transitively. In view of Lemma 2.2, for the proof of Theorem 1.1 we may assume  $f(x_1, \dots, x_n)$  itself is a homogenous polynomial with all non-zero coefficients equal to one and for which  $S_n$  acts transitively.

**Proposition 3.1.** *Assume  $AG_2, \dots, AG_{n-1}$ . Let  $f(x_1, \dots, x_n)$  be a homogenous symmetric polynomial of degree  $k$ ,  $1 \leq k \leq n$ , with all non-zero coefficients equal to one and for which  $S_n$  acts transitively. Assume  $f(x_1, \dots, x_n) \neq x_1^n + \cdots + x_n^n$ . Then the conclusion of Theorem 1.1 holds.*

*Proof.* The polynomial  $f(x_1, \dots, x_n)$  has a monomial of the form  $x_1^{\ell_1} x_2^{\ell_2} \cdots x_s^{\ell_s}$  where  $k = \deg f = \ell_1 + \cdots + \ell_s$  and  $0 < \ell_j < n$ . By  $AG_{\ell_j}$  for each  $j$  we have



AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

Title Page

Contents



Page 8 of 12

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

$$\begin{aligned} \ell_1 x_1 x_2 \cdots x_{\ell_1} &\leq x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}, \\ \ell_2 x_{\ell_1+1} x_{\ell_1+2} \cdots x_{\ell_1+\ell_2} &\leq x_{\ell_1+1}^{\ell_2} + \cdots + x_{\ell_1+\ell_2}^{\ell_2}, \\ &\vdots \\ \ell_s x_{\ell_1+\cdots+\ell_{s-1}+1} x_{\ell_1+\cdots+\ell_{s-1}+2} \cdots x_{\ell_1+\cdots+\ell_s} &\leq x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s}. \end{aligned}$$

Since  $k = \deg f = \ell_1 + \cdots + \ell_s$ , we multiply the inequalities to obtain

$$(3.1) \quad \ell_1 \cdots \ell_s x_1 \cdots x_k \leq (x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s}).$$

Inequality (3.1) now yields

$$(3.2) \quad \sum! \ell_1 \cdots \ell_s x_1 \cdots x_k \leq \sum! (x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s}).$$

Since  $\sum! x_1 \cdots x_k$  consists of  $n!$  monomials with coefficient one, we get

$$\sum! x_1 \cdots x_k = \frac{n!}{\binom{n}{k}} s_k(x_1, \dots, x_n).$$

Similarly since  $(x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s})$  consists of  $\ell_1 \cdots \ell_s$  monomials with coefficient one, it follows that  $\sum! ((x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s}))$  consists of  $\ell_1 \cdots \ell_s n!$  monomials with coefficient one. Thus we have

$$\sum! ((x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s})) = \frac{\ell_1 \cdots \ell_s n!}{t} f(x_1, \dots, x_n),$$



[Title Page](#)[Contents](#)

Page 9 of 12

[Go Back](#)[Full Screen](#)[Close](#)

where  $t = f(1, \dots, 1)$  is the number of monomials of  $f$ . Plugging this into (3.2), then we see that if the  $x_i$ 's are not all equal then at least one permutation of (3.1) is a strict inequality and hence inequality (3.2) is also strict.  $\square$

By the previous proposition and the discussion preceding it, in order to prove Theorem 1.1, it suffices to prove  $AG_n$  for all  $n \geq 2$ .

**Theorem 3.2.**  $AG_n$  is true for all  $n \geq 2$ .

*Proof.* The proof is by induction on  $n$ . The case  $n = 2$  is standard. For  $x, y \in \mathbb{R}$ ,  $x, y > 0$  we have  $(\sqrt{x} - \sqrt{y})^2 \geq 0$  with equality if and only if  $x = y$ . Expanding we get.

$$\begin{aligned}x - 2\sqrt{xy} + y &\geq 0, \\x + y &\geq 2\sqrt{xy}, \\ \frac{x + y}{2} &\geq \sqrt{xy}.\end{aligned}$$

We now assume  $AG_2, \dots, AG_n$  and we prove  $AG_{n+1}$ . To this end, it suffices to show that

$$x_1 \cdots x_{n+1} \leq \left( \frac{x_1 + \cdots + x_{n+1}}{n+1} \right)^{n+1}.$$

Now, by  $AG_2$  and  $AG_n$  we have for each  $k$ ,

$$\begin{aligned}\sqrt{x_k \sqrt[n]{x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n+1}}} &\leq \frac{x_k + \sqrt[n]{x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n+1}}}{2} \\ &\leq \frac{x_1 + \cdots + nx_k + \cdots + x_{n+1}}{2n} \\ &= \frac{s + (n-1)x_k}{2n}.\end{aligned}$$



AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

Title Page

Contents



Page 10 of 12

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-575b

© 2007 Victoria University. All rights reserved.

Where we have set  $s = x_1 + \cdots + x_{n+1}$ . Multiplying these inequalities over  $k$ , we get

$$\begin{aligned}x_1 \cdots x_{n+1} &\leq \prod_{k=1}^{n+1} \frac{s + (n-1)x_k}{2n} \\ &= \frac{1}{(2n)^{n+1}} \prod_{k=1}^{n+1} (s + (n-1)x_k).\end{aligned}$$

Multiplying through by  $(2n)^{n+1}$  and expanding we get,

$$(3.3) \quad (2n)^{n+1} x_1 \cdots x_{n+1} \leq \sum_{k=0}^{n+1} (n-1)^k s_k(x_1, \dots, x_{n+1}) s^{n+1-k}.$$

We now use Proposition 3.1 and the discussion preceding it to conclude

$$s_k(x_1, \dots, x_{n+1}) \leq \binom{n+1}{k} \frac{s^k}{(n+1)^k}$$

for  $0 < k < n+1$ . Plugging this into (3.3), we get,

$$(3.4) \quad \begin{aligned}(2n)^{n+1} x_1 \cdots x_{n+1} &\leq \sum_{k=0}^n \binom{n+1}{k} \left(\frac{n-1}{n+1}\right)^k s^{n+1} + (n-1)^{n+1} s_{n+1}(x_1, \dots, x_{n+1}).\end{aligned}$$

Moving

$$(n-1)^{n+1} s_{n+1}(x_1, \dots, x_{n+1}) = (n-1)^{n+1} x_1 \cdots x_{n+1}$$



AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

Title Page

Contents



Page 11 of 12

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

to the other side, we get

$$\begin{aligned}((2n)^{n+1} - (n-1)^{n+1}) x_1 \cdots x_{n+1} &\leq \sum_{k=0}^n \binom{n+1}{k} \left(\frac{n-1}{n+1}\right)^k s^{n+1} \\&= \left[ \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{n-1}{n+1}\right)^k - \left(\frac{n-1}{n+1}\right)^{n+1} \right] s^{n+1} \\&= \left[ \left(\frac{n-1}{n+1} + 1\right)^{n+1} - \left(\frac{n-1}{n+1}\right)^{n+1} \right] s^{n+1} \\&= \left[ \left(\frac{2n}{n+1}\right)^{n+1} - \left(\frac{n-1}{n+1}\right)^{n+1} \right] s^{n+1} \\&= ((2n)^{n+1} - (n-1)^{n+1}) \frac{s^{n+1}}{(n+1)^{n+1}}.\end{aligned}$$

Cancelling  $((2n)^{n+1} - (n-1)^{n+1})$ , we get

$$(3.5) \quad x_1 \cdots x_{n+1} \leq \left( \frac{x_1 + \cdots + x_{n+1}}{n+1} \right)^{n+1},$$

as desired. We note that if the  $x_k$ 's are distinct, then by Proposition 3.1, the inequalities used in equation (3.4) are strict. It follows that in this case inequality (3.5) is also strict.  $\square$

To recap our argument, Lemma 2.2 reduces the proof of Theorem 1.1 to the case where  $f(x_1, \dots, x_n)$  is a homogenous polynomial with all non-zero coefficients equal to one, for which  $S_n$  acts transitively. Proposition 3.1 further reduces the proof to the  $AG_n$ . Finally, the proof of  $AG_n$  is achieved in Theorem 3.2.

## References

- [1] P.S. BULLEN, *Handbook of Means and their Inequalities*, Kluwer Acad. Press, Dordrecht, 2003.
- [2] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, 1964.



---

AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

---

Title Page

Contents



Page 12 of 12

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756