THE ARITHMETIC-ALGEBRAIC MEAN INEQUALITY VIA SYMMETRIC MEAN

OSCAR G. VILLAREAL
Department of Mathematics
University of California, Irvine
340 Rowland Hall, Irvine, CA 92697-3875, USA
EMail: ovillare@math.uci.edu
URL: http://math.uci.edu/∼oscar

Received: 02 August, 2008

Accepted: 06 August, 2008

Communicated by: P.S. Bullen

2000 AMS Sub. Class.: Primary 26D15

Key words: Arithmetic mean, Geometric mean, Symmetric mean, Inequality

Abstract: We give two proofs of the arithmetic-algebraic mean inequality by giving a characterization of symmetric means.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$ be an n-tuple of positive real numbers. The inequality of arithmetic-algebraic means states that

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

The left-hand side of the inequality is called the geometric mean and the right-hand side the arithmetic mean. We will refer to this inequality as AG_n to specify the size of the n-tuple. This inequality has been known in one form or another since antiquity and numerous proofs have been given over the centuries. Bullen's book [1], for example, gives over seventy proofs. We give two proofs based on a characterization of symmetric means as the smallest among the means constructed by homogeneous symmetric polynomials. The main result is

Theorem 1.1. *Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$ be an n-tuple of positive real numbers, $f(x_1, \ldots, x_n)$ be a homogenous symmetric polynomial of degree k, $1 \leq k \leq n$, having positive coefficients, and let $s_k(x_1, \ldots, x_n)$ be the k-th elementary symmetric polynomial. Then*

$$\frac{s_k(a_1, \ldots, a_n)}{\binom{n}{k}} \leq \frac{f(a_1, \ldots, a_n)}{f(1, \ldots, 1)}.$$

There is equality if and only if the a_i's are all equal.

Note that $\binom{n}{k} = s_k(1, \ldots, 1)$. Similarly we note that if the coefficents of f are all equal to one, then $f(1, \ldots, 1)$ is the number of monomials comprising f. Thus it is reasonable to think of $\frac{f(a_1,\ldots,a_n)}{f(1,\ldots,1)}$ as a mean for f as in the theorem. The theorem implies the arithmetic-algebraic mean inequality by taking $k = n$, $f(x_1, \ldots, x_n) = (x_1 + \cdots + x_n)^n$ so that $f(1, \ldots, 1) = n^n$, and then taking n-th roots.

We shall give two proofs of Theorem 1.1. The first depends on Muirhead's Theorem. The second proves AG_n and Theorem 1.1 in one induction step.

AG mean
Oscar G. Villareal
vol. 9, iss. 3, art. 78, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. First Proof of Theorem 1.1

For any function $f(x_1, \dots, x_n)$, the symmetric group S_n acts on the x_k's, and so we set

$$\sum! f(x_1, \dots, x_n) = \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

In particular, for an n-tuple of nonnegative real numbers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, when

$$f(x_1, \dots, x_n) = x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

we set

$$[\alpha] = \frac{1}{n!} \sum! x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Note that $[1, 0, \dots, 0]$ is the arithmetic mean while $[\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}]$ is the geometric mean.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ be two n-tuples of nonnegative real numbers. Muirhead's theorem gives conditions under which an inequality exists of the form

$$[\alpha] = \frac{1}{n!} \sum! x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \le [\beta] = \frac{1}{n!} \sum! x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$$

valid for all positve x_i's. To do this we first note that $[\alpha]$ is invariant under permutations of the α_i's and so we introduce an eqivalence relation as follows. We write $\alpha \preceq \beta$ if some permutation of the coordinates of α and β satisfies

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \ldots + \beta_n,$$
$$\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n \text{ and } \beta_1 \ge \beta_2 \ge \ldots \ge \beta_n,$$
$$\alpha_1 + \alpha_2 + \cdots + \alpha_k \le \beta_1 + \beta_2 + \ldots + \beta_k \text{ for } k = 1, 2, \dots, n.$$

Muirhead's Theorem states

AG mean
Oscar G. Villareal
vol. 9, iss. 3, art. 78, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 2.1. *The inequality*

$$[\alpha] = \frac{1}{n!}\sum! x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \leq [\beta] = \frac{1}{n!}\sum! x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$$

is valid for all positve x_i's if and only if $\alpha \leq \beta$. There is equality only when $\alpha = \beta$ or the x_i's are all equal.

We refer to [2] for the proof of this theorem and further discussion. Before giving the first proof of Theorem 1.1 we need a lemma.

Lemma 2.2. *Let $(a_{1j}, \dots, a_{n_j j}) \in \mathbb{R}^{n_j}$ for $j = 1, \dots, m$, and let $c_1, \dots, c_m$ be positive real numbers. Suppose $a \leq \frac{a_{1j}+\cdots+a_{n_j j}}{n_j}$ for each j. Then*

$$a \leq \frac{c_1(a_{11} + \cdots + a_{n_1 1}) + c_2(a_{12} + \cdots + a_{n_2 2}) + \cdots + c_m(a_{1m} + \cdots + a_{n_m m})}{c_1 n_1 + c_2 n_2 + \cdots + c_m n_m}.$$

There is equality if and only if the original inequalities are all equalities.

Proof. For each j we rewrite $a \leq \frac{a_{1j}+\cdots+a_{n_j j}}{n_j}$ as $n_j a \leq a_{1j} + \cdots + a_{n_j j}$. We then multiply by c_j to obtain $c_j n_j a \leq c_j(a_{1j} + \cdots + a_{n_j j})$. We now add over all j to obtain

$$\begin{aligned}&(c_1 n_1 + c_2 n_2 + \cdots + c_m n_m)a \\ &\quad \leq c_1(a_{11} + \cdots + a_{n_1 1}) + c_2(a_{12} + \cdots + a_{n_2 2}) + \cdots + c_m(a_{1m} + \cdots + a_{n_m m}).\end{aligned}$$

By dividing by the coefficient of a we get the lemma. Note that if at least one of the original inequalities is strict, then the argument shows the final inequality is also strict. □

Proof of Theorem 1.1. Let $f(x_1, \dots, x_n)$ be a homogenous symmetric polynomial of degree k with positive coefficients. The monomials of f break up into orbits under

journal of inequalities in pure and applied mathematics
issn: 1443-5756

AG mean
Oscar G. Villareal
vol. 9, iss. 3, art. 78, 2008

the action of the symmetric group S_n and so we may write $f = c_1 f_1 + \cdots + c_m f_m$, $c_j > 0$ where each f_j is a homogenous polynomial with all non-zero coefficients equal to one and for which S_n acts transitively. In view of Lemma 2.2, for the proof of Theorem 1.1 we may assume $f(x_1, \ldots, x_n)$ itself is a homogenous polynomial of degree k with all non-zero coefficients equal to one and for which S_n acts transitively.

For such an f, it follows that there exists an α such that $f(x_1, \ldots, x_n) = t[\alpha]$, where $t = f(1, 1, \ldots, 1)$ is the number of monomials comprising f. We note that $s_k(x_1, \ldots, x_n) = \binom{n}{k}[1, 1, \ldots, 1, 0, \ldots, 0]$ with k 1's and $n - k$ 0's. Since $[1, 1, \ldots, 1, 0, \ldots, 0] \leq \alpha$, Theorem 2.1 gives the result. $\square$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. Second Proof of Theorem 1.1

The inequality of arithmetic-geometric means can be stated in polynomial form in two ways. By taking n-th powers we get

$$a_1 \cdots a_n \leq \left(\frac{a_1 + \cdots + a_n}{n} \right)^n .$$

Alternately, if we let $a_i = A_i^n$ we get

$$A_1 \cdots A_n \leq \frac{A_1^n + \cdots + A_n^n}{n}.$$

We will refer to these equivalent inequalities also as AG_n.

Let $f(x_1, \dots, x_n)$ be a homogenous symmetric polynomial. The monomials of f break up into orbits under the action of the symmetric group S_n and so we may write $f = c_1 f_1 + \cdots + c_m f_m$, $c_j \in \mathbb{R}$ where each f_j is a homogenous polynomial with all non-zero coefficients equal to one and for which S_n acts transitively. In view of Lemma 2.2, for the proof of Theorem 1.1 we may assume $f(x_1, \dots, x_n)$ itself is a homogenous polynomial with all non-zero coefficients equal to one and for which S_n acts transitively.

Proposition 3.1. *Assume AG_2,…,AG_{n-1}. Let $f(x_1, \dots, x_n)$ be a homogenous symmetric polynomial of degree k, $1 \leq k \leq n$, with all non-zero coefficients equal to one and for which S_n acts transitively. Assume $f(x_1, \dots, x_n) \neq x_1^n + \cdots + x_n^n$. Then the conclusion of Theorem 1.1 holds.*

Proof. The polynomial $f(x_1, \dots, x_n)$ has a monomial of the form $x_1^{\ell_1} x_2^{\ell_2} \cdots x_s^{\ell_s}$ where $k = \deg f = \ell_1 + \cdots + \ell_s$ and $0 < \ell_j < n$. By AG_{ℓ_j} for each j we have

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$
\begin{aligned}
\ell_1 x_1 x_2 \cdots x_{\ell_1} &\le x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}, \\
\ell_2 x_{\ell_1+1} x_{\ell_1+2} \cdots x_{\ell_1+\ell_2} &\le x_{\ell_1+1}^{\ell_2} + \cdots + x_{\ell_1+\ell_2}^{\ell_2}, \\
&\vdots \\
\ell_s x_{\ell_1+\cdots+\ell_{s-1}+1} x_{\ell_1+\cdots+\ell_{s-1}+2} \cdots x_{\ell_1+\cdots+\ell_s} &\le x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s}.
\end{aligned}
$$

Since $k = \deg f = \ell_1 + \cdots + \ell_s$, we multiply the inequalities to obtain

$$
\ell_1 \cdots \ell_s x_1 \cdots x_k \le (x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s}). \tag{3.1}
$$

Inequality (3.1) now yields

$$
\begin{aligned}
\textstyle\sum! \, \ell_1 \cdots \ell_s x_1 \cdots x_k \\
\le \textstyle\sum! \, (x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s}).
\end{aligned} \tag{3.2}
$$

Since $\sum! \, x_1 \cdots x_k$ consists of $n!$ monomials with coefficient one, we get

$$
\textstyle\sum! \, x_1 \cdots x_k = \dfrac{n!}{\binom{n}{k}} s_k(x_1, \dots, x_n).
$$

Similarly since $(x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s})$ consists of $\ell_1 \cdots \ell_s$ monomials with coefficient one, it follows that $\sum! \, ((x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s})$ consists of $\ell_1 \cdots \ell_s n!$ monomials with coefficient one. Thus we have

$$
\textstyle\sum! \, ((x_1^{\ell_1} + \cdots + x_{\ell_1}^{\ell_1}) \cdots (x_{\ell_1+\cdots+\ell_{s-1}+1}^{\ell_s} + \cdots + x_{\ell_1+\cdots+\ell_s}^{\ell_s}) = \dfrac{\ell_1 \cdots \ell_s n!}{t} f(x_1, \dots, x_n),
$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

where $t = f(1, \dots, 1)$ is the number of monomials of f. Plugging this into (3.2), then we see that if the x_i's are not all equal then at least one permutation of (3.1) is a strict inequality and hence inequality (3.2) is also strict. □

By the previous proposition and the discussion preceding it, in order to prove Theorem 1.1, it suffices to prove AG_n for all $n \geq 2$.

Theorem 3.2. *AG_n is true for all $n \geq 2$.*

Proof. The proof is by induction on n. The case $n = 2$ is standard. For $x, y \in \mathbb{R}$, $x, y > 0$ we have $(\sqrt{x} - \sqrt{y})^2 \geq 0$ with equality if and only if $x = y$. Expanding we get.

$$\begin{aligned} x - 2\sqrt{xy} + y &\geq 0, \\ x + y &\geq 2\sqrt{xy}, \\ \frac{x+y}{2} &\geq \sqrt{xy}. \end{aligned}$$

We now assume $AG_2, \dots, AG_n$ and we prove AG_{n+1}. To this end, it suffices to show that

$$x_1 \cdots x_{n+1} \leq \left(\frac{x_1 + \cdots + x_{n+1}}{n+1} \right)^{n+1}.$$

Now, by AG_2 and AG_n we have for each k,

$$\begin{aligned} \sqrt{x_k \sqrt[n]{x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n+1}}} &\leq \frac{x_k + \sqrt[n]{x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n+1}}}{2} \\ &\leq \frac{x_1 + \cdots + n x_k + \cdots + x_{n+1}}{2n} \\ &= \frac{s + (n-1) x_k}{2n}. \end{aligned}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Where we have set $s = x_1 + \cdots + x_{n+1}$. Multiplying these inequalities over k, we get

$$x_1 \cdots x_{n+1} \le \prod_{k=1}^{n+1} \frac{s + (n-1)x_k}{2n} = \frac{1}{(2n)^{n+1}} \prod_{k=1}^{n+1} (s + (n-1)x_k).$$

Multiplying through by $(2n)^{n+1}$ and expanding we get,

$$(2n)^{n+1} x_1 \cdots x_{n+1} \le \sum_{k=0}^{n+1} (n-1)^k s_k(x_1, \dots, x_{n+1}) s^{n+1-k}. \tag{3.3}$$

We now use Proposition 3.1 and the discussion preceding it to conclude

$$s_k(x_1, \dots, x_{n+1}) \le \binom{n+1}{k} \frac{s^k}{(n+1)^k}$$

for $0 < k < n+1$. Plugging this into (3.3), we get,

$$(2n)^{n+1} x_1 \cdots x_{n+1} \le \sum_{k=0}^{n} \binom{n+1}{k} \left(\frac{n-1}{n+1}\right)^k s^{n+1} + (n-1)^{n+1} s_{n+1}(x_1, \dots, x_{n+1}). \tag{3.4}$$

Moving

$$(n-1)^{n+1} s_{n+1}(x_1, \dots, x_{n+1}) = (n-1)^{n+1} x_1 \cdots x_{n+1}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756
AG mean
Oscar G. Villareal
vol. 9, iss. 3, art. 78, 2008

to the other side, we get

$$\begin{aligned}\left((2n)^{n+1}-(n-1)^{n+1}\right)x_1\cdots x_{n+1} &\le \sum_{k=0}^{n}\binom{n+1}{k}\left(\frac{n-1}{n+1}\right)^k s^{n+1}\\ &= \left[\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{n-1}{n+1}\right)^k-\left(\frac{n-1}{n+1}\right)^{n+1}\right]s^{n+1}\\ &= \left[\left(\frac{n-1}{n+1}+1\right)^{n+1}-\left(\frac{n-1}{n+1}\right)^{n+1}\right]s^{n+1}\\ &= \left[\left(\frac{2n}{n+1}\right)^{n+1}-\left(\frac{n-1}{n+1}\right)^{n+1}\right]s^{n+1}\\ &= \left((2n)^{n+1}-(n-1)^{n+1}\right)\frac{s^{n+1}}{(n+1)^{n+1}}.\end{aligned}$$

Cancelling $((2n)^{n+1}-(n-1))^{n+1}$, we get

$$x_1\cdots x_{n+1}\le\left(\frac{x_1+\cdots+x_{n+1}}{n+1}\right)^{n+1}, \tag{3.5}$$

as desired. We note that if the x_k's are distinct, then by Proposition 3.1, the inequalites used in equation (3.4) are strict. It follows that in this case inequality (3.5) is also strict. □

To recap our argument, Lemma 2.2 reduces the proof of Theorem 1.1 to the case where $f(x_1,\ldots,x_n)$ is a homogenous polynomial with all non-zero coefficients equal to one, for which S_n acts transitively. Proposition 3.1 further reduces the proof to the AG_n. Finally, the proof of AG_n is achieved in Theorem 3.2.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] P.S. BULLEN, *Handbook of Means and their Inequalities*, Kluwer Acad. Press, Dordrecht, 2003.

[2] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, 1964.

AG mean

Oscar G. Villareal

vol. 9, iss. 3, art. 78, 2008

journal of inequalities in pure and applied mathematics

issn: 1443-5756