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INEQUALITIES FOR WEIGHTED POWER PSEUDO MEANS

VASILE MIHEŞAN

DEPARTMENT OF MATHEMATICS,
TECHNICAL UNIVERSITY OF CLUJ-NAPOCA
STR. C.DAICOVICIU NR.15
400020 CLUJ-NAPOCA
ROMANIA
Vasile.Mihesan@math.utcluj.ro

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ABSTRACT. In this paper we denote by $m_n^{[r]}$ the following expression, which is closely connected to the weighted power means of order r, $M_n^{[r]}$.

Let $n \geq 2$ be a fixed integer and

$$m_n^{[r]}(\mathbf{x};\mathbf{p}) = \begin{cases} \left(\frac{P_n}{p_1}x_1^r - \frac{1}{p_1}\sum\limits_{i=2}^{n} p_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0 \\ x_1^{P_n/p_1} \Big/ \prod\limits_{i=2}^{n} x_i^{p_i/p_1}, & r = 0 \end{cases} \qquad (\mathbf{x} \in R_r),$$

where $P_n = \sum_{i=1}^n p_i$ and R_r denotes the set of the vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ for which $x_i > 0$ $(i = 1, 2, \dots, n)$, $\mathbf{p} = (p_1, p_2, \dots, p_n)$, $p_1 > 0$, $p_i \geq 0$ $(i = 1, 2, \dots, n)$ and $P_n x_1^r > \sum_{i=2}^n p_i x_i^r$.

Three inequalities are presented for $m_n^{[r]}$. The first is a comparison theorem. The second and the third is Rado type inequalities. The proofs show that the above inequalities are consequences of some well-known inequalities for weighted power means.

Key words and phrases: Weighted power pseudo means, Inequalities.

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1. INTRODUCTION

Let $\mathbf{y} = (y_1, y_2, \dots, y_n)$ and $\mathbf{q} = (q_1, q_2, \dots, q_n)$ be positive n-tuples, then the arithmetic and geometric means of $\mathbf{y}$ with weights $\mathbf{q}$ are defined by

$$A_n(\mathbf{y};\mathbf{q}) = \frac{1}{Q_n}\sum_{i=1}^{n} q_i y_i \quad \text{and} \quad G_n(\mathbf{y};\mathbf{q}) = \left(\prod_{i=1}^{n} y_i^{q_i}\right)^{\frac{1}{Q_n}}, \quad \text{where} \quad Q_n = \sum_{i=1}^{n} q_i.$$

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If r is a real number, then the r-th power means of $\mathbf{y}$ with weights $\mathbf{q}$, $M_n^{[r]}(\mathbf{y};\mathbf{q})$ is defined by

$$M_n^{[r]}(\mathbf{y};\mathbf{q}) = \begin{cases} \left(\frac{1}{Q_n}\sum\limits_{i=1}^n q_i y_i^r\right)^{\frac{1}{r}}, & r \neq 0; \\ \left(\prod\limits_{i=1}^n y_i^{q_i}\right)^{\frac{1}{Q_n}}, & r = 0. \end{cases} \tag{1.1}$$

If $r, s \in \mathbb{R}$, $r \leq s$ then [11]

$$M_n^{[r]}(\mathbf{y};\mathbf{q}) \leq M_n^{[s]}(\mathbf{y},\mathbf{q}) \tag{1.2}$$

is valid for all positive real numbers y_i and q_i $(i = 1, 2, \ldots, n)$. For $r = 0$ and $s = 1$ we obtain the clasical inequality between the weighted arithmetic and geometric means

$$G_n = G_n(\mathbf{y};\mathbf{q}) = \prod_{i=1}^n y_i^{q_i/Q_n} \leq \frac{1}{Q_n}\sum_{i=1}^n q_i y_i = A_n(\mathbf{y},\mathbf{q}) = A_n. \tag{1.3}$$

In this paper we denote by $m_n^{[r]}(\mathbf{y};\mathbf{q})$ the following expression which is closely connected to $M_n^{[r]}(\mathbf{y};\mathbf{q})$.

Let $n \geq 2$ be an integer (considered fixed throughout the paper) and define

$$m_n^{[r]}(\mathbf{x};\mathbf{p}) = \begin{cases} \left(\frac{P_n}{p_1}x_1^r - \frac{1}{p_1}\sum\limits_{i=2}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0 \\ x_1^{P_n/p_1} \Big/ \prod\limits_{i=2}^n x_i^{p_i/p_1}, & r = 0 \end{cases} \qquad (\mathbf{x} \in R_r) \tag{1.4}$$

where $P_n = \sum_{i=1}^n p_i$ and R_r denotes the set of the vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for which $x_i > 0$ $(i = 1, 2, \ldots, n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $p_1 > 0$, $p_i \geq 0$ $(i = 2, 3, \ldots, n)$ and $P_n x_1^r > \sum_{i=2}^n p_i x_i^r$.

Although there is no general agreement in literature about what constitutes a mean value most authors consider the intermediate property as the main feature. Since $m_n^{[r]}(\mathbf{x};\mathbf{p})$ do not satisfy this condition, this means that the double inequalities

$$\min_{1\leq i\leq n} x_i \leq m_n^{[r]}(\mathbf{x};\mathbf{p}) \leq \max_{1\leq i\leq n} x_i$$

are not true for all positive x_i, we call $m_n^{[r]}$ the weighted power pseudo means of order r.

For $r = 1$ we obtain by (1.4) the pseudo arithmetic means $a_n(\mathbf{x},\mathbf{p})$ for $r = 0$ the pseudo geometric means, $g_n(\mathbf{x},\mathbf{p})$, see [2]. In 1990, H. Alzer [2] published the following companion of inequality (1.3):

$$a_n(\mathbf{x};\mathbf{p}) \leq g_n(\mathbf{x};\mathbf{p}). \tag{1.5}$$

For the special case $p_1 = p_2 = \cdots = p_n$ the inequality (1.5) was proved by S. Iwamoto, R.J. Tomkins and C.L. Wang [6].

Rado and Popoviciu type inequalities for pseudo arithmetic and geometric means were given in [2], [9], [10].

We note that inequality (1.5) is an example of a so called reverse inequality. One of the first reverse inequalities was published by J. Aczél [1] who proved the following intriguing variant of the Cauchy-Schwarz inequality:

If x_i and y_i $(i = 1, 2, \ldots, n)$ are real numbers with $x_1^2 > \sum_{i=2}^n x_i^2$ and $y_1^2 > \sum_{i=2}^n y_i^2$, then

$$\left(x_1 y_1 - \sum_{i=2}^n x_i y_i\right)^2 \geq \left(x_1^2 - \sum_{i=2}^n x_i^2\right)\left(y_1^2 - \sum_{i=2}^n y_i^2\right). \tag{1.6}$$

Further interesting reverse inequalities were given in [3], [5], [6], [7], [8], [11], [12].

The aim of this paper is to prove a comparison theorem and Rado type inequalities for the weighted power pseudo means.

2. COMPARISON THEOREM

Our first result is a comparison theorem for the weighted power pseudo means.

Theorem 2.1. *If* $0 \leq r \leq s,\ \text{x} \in R_s$ *then* $\text{x} \in R_r$ *and*

$$m_n^{[s]}(\text{x,p}) \leq m_n^{[r]}(\text{x,p}). \tag{2.1}$$

If $r \leq s \leq 0,\ \text{x} \in R_r$ *then* $\text{x} \in R_s$ *and*

$$m_n^{[s]}(\text{x;p}) \leq m_n^{[r]}(\text{x;p}). \tag{2.2}$$

If $r < 0 < s$ *then* $R_r \cap R_s = \emptyset$*, hence* $m_n^{[r]}(\text{x,p}),\ m_n^{[s]}(\text{x,p})$ *cannot both be defined, they are not comparable.*

Proof. To prove (2.1) let $a = m_n^{[s]}(\mathbf{x}, \mathbf{p}) > 0$, then we obtain by (1.4) and (1.2)

$$x_1 = \left(\frac{p_1 a^s + \sum_{i=2}^n p_i x_i^s}{P_n} \right)^{\frac{1}{s}} \geq \left(\frac{p_1 a^r + \sum_{i=2}^n p_i x_i^r}{P_n} \right)^{\frac{1}{r}},$$

hence

$$\frac{P_n}{p_1} x_1^r - \frac{\sum_{i=2}^n p_i x_i^r}{p_1} \geq a^r > 0$$

which shows that $\mathbf{x} \in R_r$. Taking the r th root, we obtain (2.1).

To prove (2.2) let $b = m_n^{[r]}(\mathbf{x}, \mathbf{p}) > 0$, then we obtain by (1.4) and (1.2),

$$x_1 = \left(\frac{p_1 b^r + \sum_{i=2}^n p_i x_i^r}{P_n} \right)^{\frac{1}{r}} \leq \left(\frac{p_1 b^s + \sum_{i=2}^n p_i x_i^b}{P_n} \right)^{\frac{1}{s}}.$$

Hence

$$x_1^s \geq \frac{p_1 b^s + \sum_{i=2}^n p_i x_i^s}{P_n}$$

and

$$\frac{P_n}{p_1} x_1^s - \frac{\sum_{i=2}^n p_i x_i^s}{p_1} \geq b^s > 0,$$

which shows that $\mathbf{x} \in R_s$. Taking the $(-s)$ th root, we obtain (2.2).

If $r < 0 < s$ we infer for $n = 2$, $p_1 = p_2$ that $x_1, x_2 > 0$, $x_1^r > x_2^r$, $x_1^s > x_2^s$ hence $x_1 < x_2$ and $x_1 > x_2$, which is impossible. □

3. RADO TYPE INEQUALITIES FOR WEIGHTED POWER PSEUDO MEANS

The well-known extension of the arithmetic mean-geometric mean inequality (1.3) is the following inequality of Rado [11]:

$$Q_n(A_n(\mathbf{y};\mathbf{q}) - G_n(\mathbf{y};\mathbf{q})) \geq Q_{n-1}(A_{n-1}(\mathbf{y};\mathbf{q}) - G_{n-1}(\mathbf{y};\mathbf{q})). \tag{3.1}$$

The next proposition provides an analog of the Rado inequality (3.1) for pseudo arithmetic and geometric means [2].

Proposition 3.1. *For all positive real numbers* x_i $(i = 1, 2, \ldots, n;\ n \geq 2)$ *we have*

$$g_n(\text{x,p}) - a_n(\text{x;p}) \geq g_{n-1}(\text{x;p}) - a_{n-1}(\text{x;p}). \tag{3.2}$$

The most obvious extension is to allow the means in the Rado inequality to have different weights [4]

$$Q_nA_n(\mathbf{y};\mathbf{q}) - \frac{q_n}{p_n}P_nG_n(\mathbf{y};\mathbf{p}) \geq Q_{n-1}A_{n-1}(\mathbf{y};\mathbf{q}) - \frac{q_n}{p_n}P_{n-1}G_{n-1}(\mathbf{y};\mathbf{p}).$$

Using this inequality we obtain the following generalization of the inequality (3.2) [10].

Proposition 3.2. *For all positive real numbers* x_i $(i = 1, 2, \dots, n;\ n \geq 2)$ *we have*

$$g_n(\text{x;p}) - a_n(\text{x;q}) \geq g_{n-1}(\text{x;p}) - a_{n-1}(\text{x;q}). \tag{3.3}$$

An extension of the Rado inequality for weighted power means is the following inequality [4]: If $r, s, t \in \mathbb{R}$ such that $r/t \leq 1$ and $s/t \geq 1$ then

$$P_n\left(\left(M_n^{[s]}(\mathbf{y};\mathbf{p})\right)^t - \left(M_n^{[r]}(\mathbf{y};\mathbf{p})\right)^t\right) \geq P_{n-1}\left(\left(M_{n-1}^{[s]}(\mathbf{y};\mathbf{p})\right)^t - \left(M_{n-1}^{[r]}(\mathbf{y},\mathbf{p})\right)^t\right). \tag{3.4}$$

Using inequality (3.4) we obtain generalizations of the inequality of Rado type (3.2) for the weighted power pseudo means.

Theorem 3.3. *If* $r \leq 1, \text{x} \in R_r$ *and* $x_1^r \leq x_n^r$ *then*

$$m_n^{[r]}(\text{x,p}) - a_n(\text{x,p}) \geq m_{n-1}^{[r]}(\text{x,p}) - a_{n-1}(\text{x,p}). \tag{3.5}$$

If $s \geq 1, \text{x} \in R_s$ *and* $x_1 \leq x_n$ *then*

$$a_n(\text{x,p}) - m_n^{[s]}(\text{x,p}) \geq a_{n-1}(\text{x,p}) - m_{n-1}^{[s]}(\text{x,p}). \tag{3.6}$$

Proof. To prove (3.5) we put in (3.4) $s = t = 1$ and we obtain for $r \leq 1$ the inequality:

$$P_n\left(A_n(\mathbf{y};\mathbf{p}) - M_n^{[r]}(\mathbf{y};\mathbf{p})\right) \geq P_{n-1}\left(A_{n-1}(\mathbf{y};\mathbf{p}) - M_{n-1}^{[r]}(\mathbf{y};\mathbf{p})\right). \tag{3.7}$$

If we set in (3.7) $y_1 = m_n^{[r]}(\mathbf{x},\mathbf{p})$, $y_i = x_i$ $(i = 2, 3, \dots, n)$ then we have:

$$P_n\left(A_n(\mathbf{y};\mathbf{p}) - M_n^{[r]}(\mathbf{y};\mathbf{p})\right) = p_1\left(m_n^{[r]}(\mathbf{x};\mathbf{p}) - a_n(\mathbf{x};\mathbf{p})\right),$$

which leads to inequality (3.5). We observe that for $r \leq 1$, $\mathbf{x} \in R_r$ and $x_1^r \leq x_n^r$ we have

$$0 < P_nx_1^r - \sum_{i=2}^{n} p_ix_i^r \leq P_{n-1}x_1^r - \sum_{i=2}^{n-1} p_ix_i^r$$

and $m_{n-1}^{[r]}(\mathbf{x},\mathbf{p})$ exist.

To prove (3.6) we set in (3.4) $r = t = 1$ and we obtain for $s \geq 1$ the inequality

$$P_n\left(M_n^{(s)}(\mathbf{y};\mathbf{p}) - A_n(\mathbf{y};\mathbf{p})\right) \geq P_{n-1}\left(M_{n-1}^{[s]}(\mathbf{y};\mathbf{p}) - A_{n-1}(\mathbf{y};\mathbf{p})\right). \tag{3.8}$$

If we put in (3.8) $y_1 = m_n^{[s]}(\mathbf{x},\mathbf{p})$, $y_i = x_i$ $(i = 2, 3, \dots, n)$ then we have

$$P_n\left(M_n^{[s]}(\mathbf{y};\mathbf{p}) - A_n(\mathbf{y};\mathbf{p})\right) = p_1\left(a_n(\mathbf{x};\mathbf{p}) - m_n^{[s]}(\mathbf{x};\mathbf{p})\right),$$

which leads to inequality (3.6). For $\mathbf{s} \geq 1$, $x \in R_s$ and $x_1 \leq x_n,\ m_{n-1}^{[s]}(\mathbf{x};\mathbf{p})$ exist. □

Theorem 3.4. *If* $0 < r \leq s, \text{x} \in R_s$ *and* $x_1 \leq x_n$ *then*

$$\left(m_n^{[r]}(\text{x;p})\right)^s - \left(m_n^{[s]}(\text{x;p})\right)^s \geq \left(m_{n-1}^{[r]}(\text{x;p})\right)^s - \left(m_{n-1}^{[s]}(\text{x;p})\right)^s \tag{3.9}$$

and

$$\left(m_n^{[r]}(\text{x;p})\right)^r - \left(m_n^{[s]}(\text{x;p})\right)^r \geq \left(m_{n-1}^{[s]}(\text{x;p})\right)^r - \left(m_{n-1}^{[s]}(\text{x;p})\right)^r. \tag{3.10}$$

Proof. To prove (3.9) we put in (3.4) $t = s$ and we obtain for $0 < r \leq s$ the inequality

$$P_n \left(\left(M_n^{[s]}(\mathbf{y};\mathbf{p})\right)^s - \left(M_n^{[r]}(\mathbf{y};\mathbf{p})\right)^s \right) \geq P_{n-1} \left(\left(M_{n-1}^{[s]}(\mathbf{y},\mathbf{p})\right)^s - \left(M_{n-1}^{[r]}(\mathbf{y},\mathbf{p})\right)^s \right). \tag{3.11}$$

If we set in (3.11) $y_1 = m_n^{[r]}(\mathbf{x};\mathbf{p})$, $y_i = x_i$ $(i = 2, 3, \ldots, n)$ then we have

$$P_n \left(\left(M_n^{[s]}(\mathbf{y};\mathbf{p})\right)^s - \left(M_n^{[r]}(\mathbf{y};\mathbf{p})\right)^s \right) = p_1 \left(\left(m_n^{[r]}(\mathbf{x};\mathbf{p})\right)^s - \left(m_n^{[s]}(\mathbf{x};\mathbf{p})\right)^s \right),$$

which leads to inequality (3.9) If $0 < r \leq s$, $\mathbf{x} \in R_s$ then $\mathbf{x} \in R_r$ and if $x_1 \leq x_n$ then $m_{n-1}^{[r]}(\mathbf{x};\mathbf{p})$ exists.

To prove (3.10) we set in (3.4) $t = r$ and we obtain for $0 < r \leq s$ the inequality

$$P_n \left(\left(M_n^{[s]}(\mathbf{y};\mathbf{p})\right)^r - \left(M_n^{[r]}(\mathbf{y};\mathbf{p})\right)^r \right) \geq P_{n-1} \left(\left(M_{n-1}^{[s]}(\mathbf{y};\mathbf{p})\right)^r - \left(M_{n-1}^{[r]}(\mathbf{y};\mathbf{p})\right)^r \right). \tag{3.12}$$

If we put in (3.12) $y_1 = m_n^{[s]}(\mathbf{x},\mathbf{p}) y_i = x_i$ $(i = 2, 3, \ldots, n)$ then we have

$$P_n \left(\left(M_n^{[s]}(\mathbf{y};\mathbf{p})\right)^r - \left(M_n^{[r]}(\mathbf{y};\mathbf{p})^r\right) \right) = p_1 \left(\left(m_n^{[r]}(\mathbf{x};\mathbf{p})\right)^r - \left(m_n^{[s]}(\mathbf{x};\mathbf{p})\right)^r \right),$$

which leads to inequality (3.10). □

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