Journal of Inequalities in Pure and Applied Mathematics

INEQUALITIES FOR WEIGHTED POWER PSEUDO MEANS

VASILE MIHESAN

Department of Mathematics
Technical University of Cluj-Napoca
Str. C.Daicoviciu nr.15
400020 Cluj-Napoca
Romania.

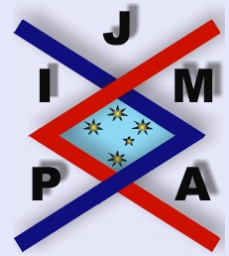*EMail*: Vasile.Mihesan@math.utcluj.ro

Abstract

Contents

◀◀   ▶▶

◀   ▶

Home Page

Go Back

Close

Quit

Abstract

In this paper we denote by $m_n^{[r]}$ the following expression, which is closely connected to the weighted power means of order r, $M_n^{[r]}$.

Let $n \geq 2$ be a fixed integer and

$$m_n^{[r]}(\mathbf{x}; \mathbf{p}) = \begin{cases} \left(\frac{P_n}{p_1} x_1^r - \frac{1}{p_1} \sum_{i=2}^{n} p_i x_i^r \right)^{\frac{1}{r}}, & r \neq 0 \\ x_1^{P_n/p_1} \Big/ \prod_{i=2}^{n} x_i^{p_i/p_1}, & r = 0 \end{cases} \quad (\mathbf{x} \in R_r),$$

where $P_n = \sum_{i=1}^{n} p_i$ and R_r denotes the set of the vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for which $x_i > 0$ $(i = 1, 2, \ldots, n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $p_1 > 0$, $p_i \geq 0$ $(i = 1, 2, \ldots, n)$ and $P_n x_1^r > \sum_{i=2}^{n} p_i x_i^r$.

Three inequalities are presented for $m_n^{[r]}$. The first is a comparison theorem. The second and the third is Rado type inequalities. The proofs show that the above inequalities are consequences of some well-known inequalities for weighted power means.

Contents

Inequalities for Weighted Power Pseudo Means

Vasile Mihesan

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 2 of 13

1. Introduction

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be positive n-tuples, then the arithmetic and geometric means of $\mathbf{y}$ with weights $\mathbf{q}$ are defined by

$$A_n(\mathbf{y}; \mathbf{q}) = \frac{1}{Q_n} \sum_{i=1}^{n} q_i y_i \quad \text{and} \quad G_n(\mathbf{y}; \mathbf{q}) = \left(\prod_{i=1}^{n} y_i^{q_i} \right)^{\frac{1}{Q_n}},$$

$$\text{where} \quad Q_n = \sum_{i=1}^{n} q_i.$$

If r is a real number, then the r-th power means of $\mathbf{y}$ with weights $\mathbf{q}$, $M_n^{[r]}(\mathbf{y}; \mathbf{q})$ is defined by
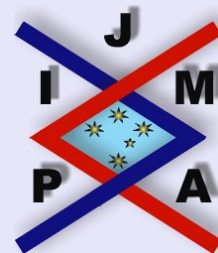
$$(1.1) \qquad M_n^{[r]}(\mathbf{y}; \mathbf{q}) = \begin{cases} \left(\frac{1}{Q_n} \sum_{i=1}^{n} q_i y_i^r \right)^{\frac{1}{r}}, & r \neq 0; \\ \left(\prod_{i=1}^{n} y_i^{q_i} \right)^{\frac{1}{Q_n}}, & r = 0. \end{cases}$$

If $r, s \in \mathbb{R}$, $r \leq s$ then [11]

$$(1.2) \qquad M_n^{[r]}(\mathbf{y}; \mathbf{q}) \leq M_n^{[s]}(\mathbf{y}, \mathbf{q})$$

is valid for all positive real numbers y_i and q_i $(i = 1, 2, \ldots, n)$. For $r = 0$ and $s = 1$ we obtain the clasical inequality between the weighted arithmetic and geometric means

$$(1.3) \qquad G_n = G_n(\mathbf{y}; \mathbf{q}) = \prod_{i=1}^{n} y_i^{q_i/Q_n} \leq \frac{1}{Q_n} \sum_{i=1}^{n} q_i y_i = A_n(\mathbf{y}, \mathbf{q}) = A_n.$$

J. Ineq. Pure and Appl. Math. **6**(3) Art. 75, 2005

http://jipam.vu.edu.au

In this paper we denote by $m_n^{[r]}(\mathbf{y}; \mathbf{q})$ the following expression which is closely connected to $M_n^{[r]}(\mathbf{y}; \mathbf{q})$.

Let $n \geq 2$ be an integer (considered fixed throughout the paper) and define

$$(1.4) \qquad m_n^{[r]}(\mathbf{x}; \mathbf{p}) = \begin{cases} \left(\frac{P_n}{p_1} x_1^r - \frac{1}{p_1} \sum_{i=2}^{n} p_i x_i^r \right)^{\frac{1}{r}}, & r \neq 0 \\ x_1^{P_n/p_1} \Big/ \prod_{i=2}^{n} x_i^{p_i/p_1}, & r = 0 \end{cases} \qquad (\mathbf{x} \in R_r)$$

where $P_n = \sum_{i=1}^{n} p_i$ and R_r denotes the set of the vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for which $x_i > 0$ $(i = 1, 2, \ldots, n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $p_1 > 0$, $p_i \geq 0$ $(i = 2, 3, \ldots, n)$ and $P_n x_1^r > \sum_{i=2}^{n} p_i x_i^r$.
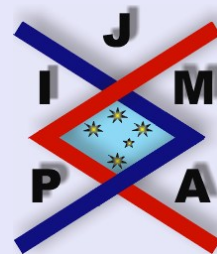
Although there is no general agreement in literature about what constitutes a mean value most authors consider the intermediate property as the main feature. Since $m_n^{[r]}(\mathbf{x}; \mathbf{p})$ do not satisfy this condition, this means that the double inequalities

$$\min_{1 \leq i \leq n} x_i \leq m_n^{[r]}(\mathbf{x}; \mathbf{p}) \leq \max_{1 \leq i \leq n} x_i$$

are not true for all positive x_i, we call $m_n^{[r]}$ the weighted power pseudo means of order r.

For $r = 1$ we obtain by (1.4) the pseudo arithmetic means $a_n(\mathbf{x}, \mathbf{p})$ for $r = 0$ the pseudo geometric means, $g_n(\mathbf{x}, \mathbf{p})$, see [2]. In 1990, H. Alzer [2] published the following companion of inequality (1.3):

$$(1.5) \qquad a_n(\mathbf{x}; \mathbf{p}) \leq g_n(\mathbf{x}; \mathbf{p}).$$

J. Ineq. Pure and Appl. Math. 6(3) Art. 75, 2005

http://jipam.vu.edu.au

For the special case $p_1 = p_2 = \cdots = p_n$ the inequality (1.5) was proved by S. Iwamoto, R.J. Tomkins and C.L. Wang [6].

Rado and Popoviciu type inequalities for pseudo arithmetic and geometric means were given in [2], [9], [10].

We note that inequality (1.5) is an example of a so called reverse inequality. One of the first reverse inequalities was published by J. Aczél [1] who proved the following intriguing variant of the Cauchy-Schwarz inequality:

If x_i and y_i $(i = 1, 2, \ldots, n)$ are real numbers with $x_1^2 > \sum_{i=2}^n x_i^2$ and $y_1^2 > \sum_{i=2}^n y_i^2$, then

$$(1.6) \qquad \left(x_1 y_1 - \sum_{i=2}^n x_i y_i \right)^2 \geq \left(x_1^2 - \sum_{i=2}^n x_i^2 \right) \left(y_1^2 - \sum_{i=2}^n y_i^2 \right).$$

Further interesting reverse inequalities were given in [3], [5], [6], [7], [8], [11], [12].

The aim of this paper is to prove a comparison theorem and Rado type inequalities for the weighted power pseudo means.
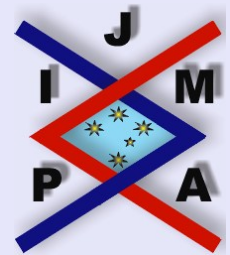
J. Ineq. Pure and Appl. Math. **6**(3) Art. 75, 2005

http://jipam.vu.edu.au

2.   Comparison Theorem

Our first result is a comparison theorem for the weighted power pseudo means.

Theorem 2.1. *If* $0 \leq r \leq s$, $\mathbf{x} \in R_s$ *then* $\mathbf{x} \in R_r$ *and*

$$(2.1) \qquad m_n^{[s]}(\mathbf{x},\mathbf{p}) \leq m_n^{[r]}(\mathbf{x},\mathbf{p}).$$

If $r \leq s \leq 0$, $\mathbf{x} \in R_r$ *then* $\mathbf{x} \in R_s$ *and*

$$(2.2) \qquad m_n^{[s]}(\mathbf{x};\mathbf{p}) \leq m_n^{[r]}(\mathbf{x};\mathbf{p}).$$

If $r < 0 < s$ *then* $R_r \cap R_s = \emptyset$, *hence* $m_n^{[r]}(\mathbf{x},\mathbf{p})$, $m_n^{[s]}(\mathbf{x},\mathbf{p})$ *cannot both be defined, they are not comparable.*

Proof. To prove (2.1) let $a = m_n^{[s]}(\mathbf{x}, \mathbf{p}) > 0$, then we obtain by (1.4) and (1.2)

$$x_1 = \left(\frac{p_1 a^s + \sum_{i=2}^n p_i x_i^s}{P_n} \right)^{\frac{1}{s}} \geq \left(\frac{p_1 a^r + \sum_{i=2}^n p_i x_i^r}{P_n} \right)^{\frac{1}{r}},$$
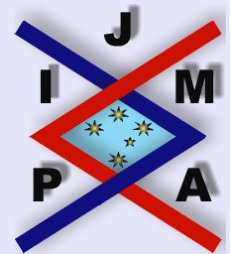
hence

$$\frac{P_n}{p_1} x_1^r - \frac{\sum_{i=2}^n p_i x_i^r}{p_1} \geq a^r > 0$$

which shows that $\mathbf{x} \in R_r$. Taking the r th root, we obtain (2.1).

To prove (2.2) let $b = m_n^{[r]}(\mathbf{x}, \mathbf{p}) > 0$, then we obtain by (1.4) and (1.2),

$$x_1 = \left(\frac{p_1 b^r + \sum_{i=2}^n p_i x_i^r}{P_n} \right)^{\frac{1}{r}} \leq \left(\frac{p_1 b^s + \sum_{i=2}^n p_i x_i^b}{P_n} \right)^{\frac{1}{s}}.$$

Title Page

Contents

◀◀   ▶▶
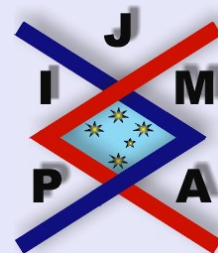
◀   ▶

Go Back

Close

Quit

Hence

$$x_1^s \geq \frac{p_1 b^s + \sum_{i=2}^n p_i x_i^s}{P_n}$$

and

$$\frac{P_n}{p_1} x_1^s - \frac{\sum_{i=2}^n p_i x_i^s}{p_1} \geq b^s > 0,$$

which shows that $\mathbf{x} \in R_s$. Taking the $(-s)$ th root, we obtain (2.2).

If $r < 0 < s$ we infer for $n = 2$, $p_1 = p_2$ that $x_1, x_2 > 0$, $x_1^r > x_2^r$, $x_1^s > x_2^s$ hence $x_1 < x_2$ and $x_1 > x_2$, which is impossible. $\qquad\square$

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **6**(3) Art. 75, 2005

http://jipam.vu.edu.au

3. Rado Type Inequalities for Weighted Power Pseudo Means

The well-known extension of the arithmetic mean-geometric mean inequality (1.3) is the following inequality of Rado [11]:

$$(3.1) \qquad Q_n(A_n(\mathbf{y};\mathbf{q}) - G_n(\mathbf{y};\mathbf{q})) \geq Q_{n-1}(A_{n-1}(\mathbf{y};\mathbf{q}) - G_{n-1}(\mathbf{y};\mathbf{q})).$$

The next proposition provides an analog of the Rado inequality (3.1) for pseudo arithmetic and geometric means [2].

Proposition 3.1. *For all positive real numbers x_i ($i = 1, 2, \ldots, n$; $n \geq 2$) we have*

$$(3.2) \qquad g_n(\mathbf{x},\mathbf{p}) - a_n(\mathbf{x};\mathbf{p}) \geq g_{n-1}(\mathbf{x};\mathbf{p}) - a_{n-1}(\mathbf{x};\mathbf{p}).$$
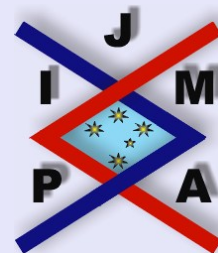
The most obvious extension is to allow the means in the Rado inequality to have different weights [4]

$$Q_n A_n(\mathbf{y};\mathbf{q}) - \frac{q_n}{p_n} P_n G_n(\mathbf{y};\mathbf{p}) \geq Q_{n-1} A_{n-1}(\mathbf{y};\mathbf{q}) - \frac{q_n}{p_n} P_{n-1} G_{n-1}(\mathbf{y};\mathbf{p}).$$

Using this inequality we obtain the following generalization of the inequality (3.2) [10].

Proposition 3.2. *For all positive real numbers x_i ($i = 1, 2, \ldots, n$; $n \geq 2$) we have*

$$(3.3) \qquad g_n(\mathbf{x};\mathbf{p}) - a_n(\mathbf{x};\mathbf{q}) \geq g_{n-1}(\mathbf{x};\mathbf{p}) - a_{n-1}(\mathbf{x};\mathbf{q}).$$

J. Ineq. Pure and Appl. Math. **6**(3) Art. 75, 2005

http://jipam.vu.edu.au

An extension of the Rado inequality for weighted power means is the following inequality [4]: If $r, s, t \in \mathbb{R}$ such that $r/t \leq 1$ and $s/t \geq 1$ then

$$(3.4) \quad P_n \left(\left(M_n^{[s]}(\mathbf{y}; \mathbf{p}) \right)^t - \left(M_n^{[r]}(\mathbf{y}; \mathbf{p}) \right)^t \right)$$
$$\geq P_{n-1} \left(\left(M_{n-1}^{[s]}(\mathbf{y}; \mathbf{p}) \right)^t - \left(M_{n-1}^{[r]}(\mathbf{y}, \mathbf{p}) \right)^t \right).$$

Using inequality (3.4) we obtain generalizations of the inequality of Rado type (3.2) for the weighted power pseudo means.

Theorem 3.3. *If* $r \leq 1$, $\mathrm{x} \in R_r$ *and* $x_1^r \leq x_n^r$ *then*

$$(3.5) \qquad m_n^{[r]}(\mathrm{x}, \mathrm{p}) - a_n(\mathrm{x}, \mathrm{p}) \geq m_{n-1}^{[r]}(\mathrm{x}, \mathrm{p}) - a_{n-1}(\mathrm{x}, \mathrm{p}).$$

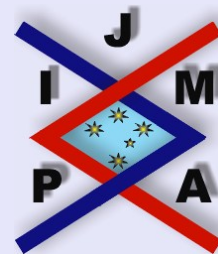*If* $s \geq 1$, $\mathrm{x} \in R_s$ *and* $x_1 \leq x_n$ *then*

$$(3.6) \qquad a_n(\mathrm{x}, \mathrm{p}) - m_n^{[s]}(\mathrm{x}, \mathrm{p}) \geq a_{n-1}(\mathrm{x}, \mathrm{p}) - m_{n-1}^{[s]}(\mathrm{x}, \mathrm{p}).$$

Proof. To prove (3.5) we put in (3.4) $s = t = 1$ and we obtain for $r \leq 1$ the inequality:

$$(3.7) \qquad P_n \left(A_n(\mathbf{y}; \mathbf{p}) - M_n^{[r]}(\mathbf{y}; \mathbf{p}) \right) \geq P_{n-1} \left(A_{n-1}(\mathbf{y}; \mathbf{p}) - M_{n-1}^{[r]}(\mathbf{y}; \mathbf{p}) \right).$$

If we set in (3.7) $y_1 = m_n^{[r]}(\mathbf{x}, \mathbf{p})$, $y_i = x_i$ $(i = 2, 3, \ldots, n)$ then we have:

$$P_n \left(A_n(\mathbf{y}; \mathbf{p}) - M_n^{[r]}(\mathbf{y}; \mathbf{p}) \right) = p_1 \left(m_n^{[r]}(\mathbf{x}; \mathbf{p}) - a_n(\mathbf{x}; \mathbf{p}) \right),$$

J. Ineq. Pure and Appl. Math. **6**(3) Art. 75, 2005

http://jipam.vu.edu.au

which leads to inequality (3.5). We observe that for $r \leq 1$, $\mathbf{x} \in R_r$ and $x_1^r \leq x_n^r$ we have

$$0 < P_n x_1^r - \sum_{i=2}^{n} p_i x_i^r \leq P_{n-1} x_1^r - \sum_{i=2}^{n-1} p_i x_i^r$$

and $m_{n-1}^{[r]}(\mathbf{x}, \mathbf{p})$ exist.

To prove (3.6) we set in (3.4) $r = t = 1$ and we obtain for $s \geq 1$ the inequality

$$(3.8) \quad P_n \left(M_n^{(s)}(\mathbf{y}; \mathbf{p}) - A_n(\mathbf{y}; \mathbf{p}) \right) \geq P_{n-1} \left(M_{n-1}^{[s]}(\mathbf{y}; \mathbf{p}) - A_{n-1}(\mathbf{y}; \mathbf{p}) \right).$$

If we put in (3.8) $y_1 = m_n^{[s]}(\mathbf{x}, \mathbf{p})$, $y_i = x_i$ $(i = 2, 3, \ldots, n)$ then we have

$$P_n \left(M_n^{[s]}(\mathbf{y}; \mathbf{p}) - A_n(\mathbf{y}; \mathbf{p}) \right) = p_1 \left(a_n(\mathbf{x}; \mathbf{p}) - m_n^{[s]}(\mathbf{x}; \mathbf{p}) \right),$$
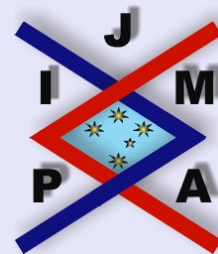
which leads to inequality (3.6). For $\mathbf{s} \geq 1$, $x \in R_s$ and $x_1 \leq x_n$, $m_{n-1}^{[s]}(\mathbf{x}; \mathbf{p})$ exist. $\qquad\square$

Theorem 3.4. *If* $0 < r \leq s$, $\mathbf{x} \in R_s$ *and* $x_1 \leq x_n$ *then*

$$(3.9) \quad \left(m_n^{[r]}(\mathbf{x};\mathbf{p}) \right)^s - \left(m_n^{[s]}(\mathbf{x};\mathbf{p}) \right)^s \geq \left(m_{n-1}^{[r]}(\mathbf{x};\mathbf{p}) \right)^s - \left(m_{n-1}^{[s]}(\mathbf{x};\mathbf{p}) \right)^s$$

and

$$(3.10) \quad \left(m_n^{[r]}(\mathbf{x};\mathbf{p}) \right)^r - \left(m_n^{[s]}(\mathbf{x};\mathbf{p}) \right)^r \geq \left(m_{n-1}^{[s]}(\mathbf{x};\mathbf{p}) \right)^r - \left(m_{n-1}^{[s]}(\mathbf{x};\mathbf{p}) \right)^r.$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **6**(3) Art. 75, 2005

http://jipam.vu.edu.au

Proof. To prove (3.9) we put in (3.4) $t = s$ and we obtain for $0 < r \leq s$ the inequality

$$(3.11) \quad P_n \left(\left(M_n^{[s]}(\mathbf{y}; \mathbf{p}) \right)^s - \left(M_n^{[r]}(\mathbf{y}; \mathbf{p}) \right)^s \right)$$
$$\geq P_{n-1} \left(\left(M_{n-1}^{[s]}(\mathbf{y}, \mathbf{p}) \right)^s - \left(M_{n-1}^{[r]}(\mathbf{y}, \mathbf{p}) \right)^s \right).$$

If we set in (3.11) $y_1 = m_n^{[r]}(\mathbf{x}; \mathbf{p})$, $y_i = x_i$ $(i = 2, 3, \ldots, n)$ then we have

$$P_n \left(\left(M_n^{[s]}(\mathbf{y}; \mathbf{p}) \right)^s - \left(M_n^{[r]}(\mathbf{y}; \mathbf{p}) \right)^s \right) = p_1 \left(\left(m_n^{[r]}(\mathbf{x}; \mathbf{p}) \right)^s - \left(m_n^{[s]}(\mathbf{x}; \mathbf{p}) \right)^s \right),$$

which leads to inequality (3.9) If $0 < r \leq s$, $\mathbf{x} \in R_s$ then $\mathbf{x} \in R_r$ and if $x_1 \leq x_n$ then $m_{n-1}^{[r]}(\mathbf{x}; \mathbf{p})$ exists.
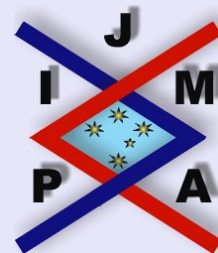
To prove (3.10) we set in (3.4) $t = r$ and we obtain for $0 < r \leq s$ the inequality

$$(3.12) \quad P_n \left(\left(M_n^{[s]}(\mathbf{y}; \mathbf{p}) \right)^r - \left(M_n^{[r]}(\mathbf{y}; \mathbf{p}) \right)^r \right)$$
$$\geq P_{n-1} \left(\left(M_{n-1}^{[s]}(\mathbf{y}; \mathbf{p}) \right)^r - \left(M_{n-1}^{[r]}(\mathbf{y}; \mathbf{p}) \right)^r \right).$$

If we put in (3.12) $y_1 = m_n^{[s]}(\mathbf{x}, \mathbf{p})$ $y_i = x_i$ $(i = 2, 3, \ldots, n)$ then we have

$$P_n \left(\left(M_n^{[s]}(\mathbf{y}; \mathbf{p}) \right)^r - \left(M_n^{[r]}(\mathbf{y}; \mathbf{p})^r \right) \right) = p_1 \left(\left(m_n^{[r]}(\mathbf{x}; \mathbf{p}) \right)^r - \left(m_n^{[s]}(\mathbf{x}; \mathbf{p}) \right)^r \right),$$

which leads to inequality (3.10). □

References

[1] J. ACZÉL, Some general methods in the theory of functional equations in one variable.New applications of functional equations (Russian), *Uspehi Mat. Nauk* (N.S.), **11**(3) (69) (1956), 3–58.

[2] H. ALZER, Inequalities for pseudo arithmetic and geometric means, *International Series of Numerical Mathematics*, Vol. **103**, Birkhauser-Verlag Basel, 1992, 5–16.

[3] R. BELLMAN, On an inequality concerning an indefinite form, *Amer. Math. Monthly*, **63** (1956), 108–109.

[4] P.S. BULLEN, D.S. MITRINOVIĆ AND P.M. VASIĆ, *Means and Their Inequalities,* Reidel Publ. Co., Dordrecht, 1988.

[5] Y.J. CHO, M. MATIĆ AND J. PEČARIĆ, Improvements of some inequalities of Aczél's type, *J. Math. Anal. Appl.*, **256** (2001), 226–240.

[6] S. IWAMOTO, R.J. TOMKINS AND C.L. WANG, Some theorems on reverse inequalities, *J. Math. Anal. Appl.*, **119** (1986), 282–299.

[7] L. LOSONCZI, Inequalities for indefinite forms, *J. Math. Anal. Appl.*, **285** (1997),148–156.

[8] V. MIHEŞAN, Applications of continuous dynamic programing to inverse inequalities, *General Mathematics*, **2**(1994), 53–60.

[9] V. MIHEŞAN, Popoviciu type inequalities for pseudo arithmetic and geometric means, (in press)
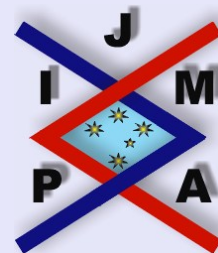
Title Page

Contents

◀◀   ▶▶

◀   ▶
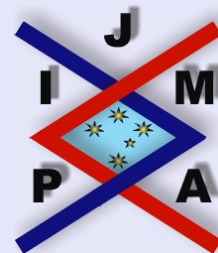
Go Back

Close

Quit

[10] V. MIHEŞAN, Rado and Popoviciu type inequalities for pseudo arithmetic and geometric means, (in press)

[11] D.S. MITRINOVIĆ, *Analytic Inequalities,* Springer Verlag, New York, 1970.

[12] X.H. SUN, Aczél-Chebyshev type inequality for positive linear functional, *J. Math. Anal. Appl.*, **245** (2000), 393–403.

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit