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DIFFERENTIAL ANALYSIS OF MATRIX CONVEX FUNCTIONS II

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ABSTRACT. We continue the analysis in [F. Hansen, and J. Tomiyama, Differential analysis of matrix convex functions. *Linear Algebra Appl.*, 420:102–116, 2007] of matrix convex functions of a fixed order defined in a real interval by differential methods as opposed to the characterization in terms of divided differences given by Kraus. We amend and improve some points in the previously given presentation, and we give a number of simple but important consequences of matrix convexity of low orders.

Key words and phrases: Matrix convex function, Polynomial.

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1. INTRODUCTION

Let f be a real function defined on an interval I. It is said to be n-convex if

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B) \qquad \lambda \in [0,1]$$

for arbitrary Hermitian $n \times n$ matrices A and B with spectra in I. It is said to be n-concave if $-f$ is n-convex, and it is said to be n-monotone if

$$A \le B \quad \Rightarrow \quad f(A) \le f(B)$$

for arbitrary Hermitian $n \times n$ matrices A and B with spectra in I. We denote by $P_n(I)$ the set of n-monotone functions defined on an interval I, and by $K_n(I)$ the set of n-convex functions defined in I.

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We analyzed in [3] the structure of the sets $K_n(I)$ by differential methods and proved, among other things, that $K_{n+1}(I)$ is strictly contained in $K_n(I)$ for every natural number n. We discovered that some improvements of the analysis and presentation is called for, and this is the topic of the next section. We also noticed that the theory has quite striking applications for monotone or convex functions of low order, and this is covered in the last section.

2. IMPROVEMENTS AND AMENDMENTS

Definition 2.1. Let $f\colon I \to \mathbb{R}$ be a function defined on an open interval. We say that f is strictly n-monotone, if f is n-monotone and $2n-1$ times continuously differentiable, and the determinant

$$\det\left(\frac{f^{(i+j-1)}(t)}{(i+j-1)!}\right)_{i,j=1}^{n} > 0$$

for every $t \in I$. Likewise, we say that f is strictly n-convex, if f is n-convex and $2n$ times continuously differentiable, and the determinant

$$\det\left(\frac{f^{(i+j)}(t)}{(i+j)!}\right)_{i,j=1}^{n} > 0$$

for every $t \in I$.

By inspecting the proof of [3, Proposition 1.3], we realize that we previously proved the following slightly stronger result.

Proposition 2.1. *Let I be a finite interval, and let m and n be natural numbers with $m \geq 2n$. There exists a strictly n-concave and strictly n-monotone polynomial $f_m\colon I \to \mathbb{R}$ of degree m. Likewise, there exists a strictly n-convex and strictly n-monotone polynomial $g_m\colon I \to \mathbb{R}$ of degree m.*

The above proposition is proved by introducing a polynomial $p_m(t)$ of degree m such that $M_n(p_m;t)$ is positive definite and $K_n(p_m;t)$ is negative definite for $t=0$. The last part of [3, Theorem 1.2] then directly ensures the existence of an $\alpha > 0$ such that p_m is n-monotone and n-concave in $(-\alpha,\alpha)$. It is somewhat misleading, as we did in the paper, to first consider the definiteness of $M_n(p_m;t)$ and $K_n(p_m;t)$ in a neighborhood of zero.

Remark 1. We would like to give some more detailed comments to the proof of the second part of [3, Theorem 1.2] (which is independent of the last assertion in the theorem). The statement is that if f is a real $2n$ times continuously differentiable function defined on an open interval I, then the matrix

$$K_n(f;t) = \left(\frac{f^{i+j}(t)}{(i+j)!}\right)_{i,j=1}^{n}$$

is positive semi-definite for each $t \in I$. We proved that the leading determinants of the matrix $K_n(f;t)$ are non-negative for each $t \in I$. It is well-known that this condition is not sufficient to insure that the matrix itself is positive semi-definite. In the proof we wave our hands and say that all principal submatrices of $K_n(f;t)$ may be obtained as a leading principal submatrix by first making a suitable joint permutation of the rows and columns in the Kraus matrix. But this common remedy is unfortunately not working in the present situation. We therefore owe it to readers to complete the proof correctly.

Proof. Let $D_m(K_n(f;t_0))$ for some $t_0 \in I$ denote the leading principal determinant of order m of the matrix $K_n(f;t_0)$. We may according to Proposition 2.1 choose a matrix convex function g such that

$$D_m(K_n(g;\, t_0)) > 0 \qquad m = 1,\ldots,n.$$

The polynomial p_m in ε defined by setting

$$p_m(\varepsilon) = D_m(K_n(f + \varepsilon g;\, t_0))$$

is of degree at most m, and $p_m(\varepsilon) \geq 0$ for $\varepsilon \geq 0$. However since the coefficient to ε^m in p_m is $D_m(K_n(g;\, t_0)) > 0$, we realize that p_m is not the zero polynomial. Let η_m be the smallest positive root of p_m, then

$$p_m(\varepsilon) > 0 \qquad 0 < \varepsilon < \eta_m.$$

Setting $\eta = \min\{\eta_1, \ldots, \eta_n)$, we obtain

$$K_n(f + \varepsilon g;\, t_0) > 0 \qquad 0 < \varepsilon < \eta.$$

By letting ε tend to zero, we finally conclude that $K_n(f;\, t_0)$ is positive semi-definite. □

We state in a remark after [3, Corollary 1.5] that the possible degrees of any polynomial in the gap between the matrix convex functions of order n and order $n+1$ defined on a finite interval are limited to $2n$ and $2n+1$. However, this is taken in the context of polynomials of degree less than or equal to $2n+1$ and may be misunderstood. There may well be polynomials of higher degrees in the gap.

3. SCATTERED OBSERVATIONS

It is well-known for which exponents the function $t \to t^p$ is either operator monotone or operator convex in the positive half-axis. It turns out that the same results apply if we ask for which exponents the function is 2-monotone or 2-convex on an open subinterval of the positive half-axis.

Proposition 3.1. *Consider the function*

$$f(t) = t^p \qquad t \in I$$

defined on any subinterval I of the positive half-axis. Then f is 2-monotone if and only if $0 \leq p \leq 1$, and it is 2-convex if and only if either $1 \leq p \leq 2$ or $-1 \leq p \leq 0$.

Proof. There is nothing to prove if f is constant or linear, so we may assume that $p \neq 0$ and $p \neq 1$. In the first case the derivative $f'(t) = pt^{p-1}$ should be non-negative so $p > 0$, and it may be written [2, Chapter VII Theorem IV] in the form

$$f'(t) = \frac{1}{c(t)^2} \qquad t \in I$$

for $c(t) = p^{-1/2}t^{(1-p)/2}$ and this function is concave only for $0 < p \leq 1$. One may alternatively consider the determinant

$$\det\begin{pmatrix} f'(t) & \frac{f''(t)}{2!} \\ \frac{f''(t)}{2!} & \frac{f^{(3)}(t)}{3!} \end{pmatrix} = \det\begin{pmatrix} pt^{p-1} & \frac{p(p-1)t^{p-2}}{2} \\ \frac{p(p-1)t^{p-2}}{2} & \frac{p(p-1)(p-2)t^{p-3}}{6} \end{pmatrix}$$
$$= -\frac{1}{12}p^2(p-1)(p+1)t^{2p-4}$$

and note that the matrix is positive semi-definite only for $0 \leq p \leq 1$.

The second derivative may be written [3, Theorem 2.3] in the form

$$f''(t) = p(p-1)t^{p-2} = \frac{1}{d(t)^3} \qquad t \in I$$

for $d(t) = (p(p-1))^{-1/3} t^{(2-p)/3}$, and this function is concave only for $-1 \le p < 0$ or $1 < p \le 2$. One may alternatively consider the determinant

$$\det \begin{pmatrix} \frac{f''(t)}{2} & \frac{f^{(3)}(t)}{6} \\ \frac{f^{(3)}(t)}{6} & \frac{f^{(4)}(t)}{24} \end{pmatrix} = \det \begin{pmatrix} \frac{p(p-1)t^{p-2}}{2} & \frac{p(p-1)(p-2)t^{p-3}}{6} \\ \frac{p(p-1)(p-2)t^{p-3}}{6} & \frac{p(p-1)(p-2)(p-3)t^{p-4}}{24} \end{pmatrix}$$
$$= -\frac{1}{144} p^2 (p-1)^2 (p-2)(p+1) t^{2p-6}$$

and note that the matrix is positive semi-definite only for $-1 \le p \le 0$ or $1 \le p \le 2$. □

The observation that the function $t \to t^p$ is 2-monotone only for $0 \le p \le 1$ has appeared in the literature in different forms, cf. [6, 1.3.9 Proposition] or [4].

It is known that the derivative of an operator monotone function defined on an infinite interval (α, ∞) is completely monotone [2, Page 86]. We give a parallel result for matrix monotone functions which implies this observation, and extend the analysis to matrix convex functions.

Theorem 3.2. *Consider a function f defined on an interval of the form (α, ∞) for some real α.*

(1) *If f is n-monotone and $2n-1$ times continuously differentiable, then*

$$(-1)^k f^{(k+1)}(t) \ge 0 \qquad k = 0, 1, \dots, 2n-2.$$

Therefore, the function f and its even derivatives up to order $2n-4$ are concave functions, and the odd derivatives up to order $2n-3$ are convex functions.

(2) *If f is n-convex and $2n$ times continuously differentiable, then*

$$(-1)^k f^{(k+2)}(t) \ge 0 \qquad k = 0, 1, \dots, 2n-2.$$

Therefore, the function f and its even derivatives up to order $2n-2$ are convex functions, and the odd derivatives up to order $2n-3$ are concave functions.

Proof. We may assume that $n \ge 2$. To prove the first assertion we may write [2, Chapter VII Theorem IV] the derivative f' in the form

$$f'(t) = \frac{1}{c(t)^2},$$

where c is a positive concave function. Since c is defined on an infinite interval it has to be increasing, therefore f' is decreasing and thus $f'' \le 0$. Since f is n-monotone, it follows from Dobsch's condition [1] that the odd derivatives satisfy

$$f^{(2k+1)} \ge 0 \qquad k = 0, 1, \dots, n-1.$$

The odd derivatives $f^{(2k+1)}$ are thus convex for $k = 0, 1, \dots, n-2$. If the third derivative $f^{(3)}$, which is a convex function, were strictly increasing at any point, then it would go towards infinity and the second derivative would eventually be positive for large t. However, this contradicts $f'' \le 0$, so $f^{(3)}$ is decreasing and thus the fourth derivative $f^{(4)} \le 0$. This argument may now be continued to prove the first assertion.

To prove the second assertion we may write [3, Theorem 2.3] the second derivative f'' in the form

$$f''(t) = \frac{1}{d(t)^3},$$

where d is a positive concave function. Since d is defined on an infinite interval it has to be increasing, therefore f'' is decreasing and thus $f^{(3)} \le 0$. Since f is n-convex, it follows [3, Theorem 1.2] that the even derivatives satisfy

$$f^{(2k)} \ge 0 \qquad k = 1, \dots, n.$$

The statement now follows in a similar way as for the first assertion. □

Corollary 3.3. *The second derivative of an operator convex function defined on an infinite interval (α, ∞) is completely monotone.*

Remark 2. The indefinite integral $g(t) = \int f(t)\,dt$ of a 2-monotone function f is 2-convex.

Proof. The second derivative may be written in the form

$$g''(t) = f'(t) = \frac{1}{c(t)^2} = \frac{1}{(c(t)^{2/3})^3}$$

for some positive concave function c. Since the function $t \to t^{2/3}$ is increasing and concave, we conclude that $t \to c(t)^{2/3}$ is concave. The statement then follows from the characterization of 2-convexity. □

It is known in the literature that operator monotone or operator convex functions defined on the whole real line are either affine or quadratic, and this fact is established by appealing to the representation theorem of Pick functions. However, the situation is far more general, and the results only depend on the monotonicity or convexity of two by two matrices.

Theorem 3.4. *Let f be a function defined on the whole real line. If f is 2-monotone then it is necessarily affine. If f is 2-convex then it is necessarily quadratic.*

Proof. Let $(\rho_n)_{n=1,2,\dots}$ be an approximate unit of positive and even C^∞-functions defined on the real axis, vanishing outside the closed interval $[-1, 1]$. The convolutions $\rho_n * f$ are infinitely many times differentiable, and they are 2-monotone if f is 2-monotone and 2-convex if f is 2-convex. Since f is continuous $\rho_n * f$ converge uniformly on any bounded interval to f. We may therefore assume that f is four times differentiable.

In the first case, the derivative f' may be written [2, Chapter VII Theorem IV] in the form $f'(t) = c(t)^{-2}$ for some positive concave function c defined on the real line, while in the second case the second derivative f'' may be written [3, Theorem 2.3] in the form $f''(t) = d(t)^{-3}$ for some positive concave function d defined on the real line. The assertions now follow since a positive concave function defined on the whole real line is necessarily constant. □

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