Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 1, Article 18, 2005

INVARIANT MEANS, SET IDEALS AND SEPARATION THEOREMS

ROMAN BADORA

INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14, PL 40-007 KATOWICE
POLAND
robadora@ux2.math.us.edu.pl

Received 30 November, 2004; accepted 25 January, 2005
Communicated by K. Nikodem

ABSTRACT. We establish connections between invariant means and set ideals. As an application, we obtain a necessary and sufficient condition for the separation almost everywhere of two functions by an additive function. We also derive the stability results for Cauchy's functional equation.

Key words and phrases: Amenable group, Set ideal, Cauchy functional equation.

2000 *Mathematics Subject Classification.* Primary 43A07, 39B72.

1. INTRODUCTION

Let M be an invariant mean on the space $B(S,\mathbb{R})$ of all real bounded functions on a semigroup S. We say that the subset A of S is a zero set for M if $M(\chi_A) = 0$, where χ_A denotes the characteristic function of a set A. Zero sets for an invariant mean M are regarded as small sets. On the other hand, in literature we can find the axiomatic definition of a family, named set ideal, of a small subset of a semigroup S. In the first part we study connections between families of zero sets and set ideals. As a consequence, we obtain, for every set ideal $\mathcal{J}$ of subsets of S the existence of such an invariant mean M on $B(S,\mathbb{R})$ for which elements of $\mathcal{J}$ are zero sets for M.

In the second part of this paper we consider some functional inequalities. We give a necessary and sufficient condition for the existence of an additive function which separates almost everywhere two functions. As an application of our result, we derive a generalization of the Gajda-Kominek theorem on a separation of subadditive and superadditive functionals by an additive function. We also give stability properties of the Cauchy functional equation.

2. INVARIANT MEANS AND SET IDEALS

In this section we assume that $(S,+)$ is a semigroup.

ISSN (electronic): 1443-5756

231-04

Definition 2.1. A non-empty family $\mathcal{J}$ of subsets of S will be called a *proper set ideal* if

$$S \notin \mathcal{J}; \tag{2.1}$$

$$A, B \in \mathcal{J} \Longrightarrow A \cup B \in \mathcal{J}; \tag{2.2}$$

$$A \in \mathcal{J} \wedge B \subset A \Longrightarrow B \in \mathcal{J}. \tag{2.3}$$

Moreover, if the set ${}_aA = \{x \in S : a + x \in A\}$ belongs to the family $\mathcal{J}$ whenever $a \in S$ and $A \in \mathcal{J}$ then the set ideal $\mathcal{J}$ is called *proper left quasi-invariant* (in short p.l.q.i.). Analogously, the set ideal $\mathcal{J}$ is said to be *proper right quasi-invariant* (p.r.q.i.) if the set $A_a = \{x \in S : x + a \in A\}$ belongs to the family $\mathcal{J}$ whenever $a \in S$ and $A \in \mathcal{J}$. In the case where the set ideal satisfies both these conditions we shall call it *proper quasi-invariant* (p.q.i.).

The sets belonging to the set ideal are regarded as, in a sense, small sets (see Kuczma [13]). For example, if S is a second category subsemigroup of a topological group G then the family of all first category subsets of S is a p.q.i. ideal. If G is a locally compact topological group equipped with the left or right Haar measure μ and if S is a subsemigroup of G with positive measure μ then the family of all subsets of S which have zero measure μ is a p.q.i. ideal. Also, if S is a normed space ($\dim S \geq 1$) then the family of all bounded subsets of S is p.q.i. ideal.

Let $\mathcal{J}$ be a set ideal of subsets of S. For a real function f on S we define $\mathcal{J}_f$ to be the family of all sets $A \in \mathcal{J}$ such that f is bounded on the complement of A. A real function f on S is called *$\mathcal{J}$-essentially bounded* if and only if the family $\mathcal{J}_f$ is non-empty. The space of all $\mathcal{J}$-essentially bounded functions on S will be denoted by $B^{\mathcal{J}}(S, \mathbb{R})$.

For every element f of the space $B^{\mathcal{J}}(S, \mathbb{R})$ the real numbers

$$\mathcal{J} - \operatorname{ess}\inf_{x \in S} f(x) = \sup_{A \in \mathcal{J}_f} \inf_{x \in S \setminus A} f(x), \tag{2.4}$$

$$\mathcal{J} - \operatorname{ess}\sup_{x \in S} f(x) = \inf_{A \in \mathcal{J}_f} \sup_{x \in S \setminus A} f(x) \tag{2.5}$$

are correctly defined and are referred to as the *$\mathcal{J}$-essential infimum* and the *$\mathcal{J}$-essential supremum* of the function f, respectively.

Definition 2.2. A linear functional M on the space $B(S, \mathbb{R})$ is called a *left* (*right*) *invariant mean* if and only if

$$\inf_{x \in S} f(x) \leq M(f) \leq \sup_{x \in S} f(x); \tag{2.6}$$

$$M({}_af) = M(f) \quad (M(f_a) = M(f)) \tag{2.7}$$

for all $f \in B(S, \mathbb{R})$ and $a \in S$, where ${}_af$ and f_a are the left and right translates of f defined by

$${}_af(x) = f(a + x), \quad f_a(x) = f(x + a), \quad x \in S.$$

A semigroup S which admits a left (right) invariant mean on $B(S, \mathbb{R})$ will be termed *left* (*right*) *amenable*. If on the space $B(S, \mathbb{R})$ there exists a real linear functional which is simultaneously a left and right invariant mean then we say that S is *two-sided amenable* or just *amenable*.

One can prove that every Abelian semigroup is amenable. For the theory of amenability see, for example, Greenleaf [12].

Remark 2.1. In this paper, in the proofs of our theorems we restrict ourselves to the "left-hand side versions". The proofs of the "right-hand side versions" and "two-sided versions" are literally the same.

Let us start with the following observation.

Theorem 2.2. *If S is a semigroup and M is a left (right) invariant mean on $B(S,\mathbb{R})$ then $\mu_M : 2^S \to \mathbb{R}$ defined by the following formulae*

$$\mu_M(A) = M(\chi_A), \quad A \subset S, \tag{2.8}$$

where χ_A denotes the characteristic function of a set A, is an additive normed measure defined on the family of all subsets of S invariant with respect to the left (right) translations.

Proof. From (2.6) it follows immediately that $\mu_M(\emptyset) = 0$. The linearity of M shows that μ_M is additive:

$$\mu_M(A) + \mu_M(B) = M(\chi_A) + M(\chi_B) = M(\chi_{A\cup B}) = \mu_M(A \cup B),$$

for all $A, B \subset S$, $A \cap B = \emptyset$. The left invariance of M implies the left invariance of μ_M:

$$\mu_M({}_aA) = M(\chi_{{}_aA}) = M(\chi_A) = \mu_M(A),$$

for all $A \subset S$ and $a \in S$. Finally, from (2.6) we infer that $\mu_M(S) = M(\chi_S) = 1$. □

If M is an left (right) invariant mean on $B(S,\mathbb{R})$ then by $\mathcal{J}_M$ we denote the family of all subsets of S which have zero measure μ_M,

$$\mathcal{J}_M = \{A \subset S : \mu_M(A) = M(\chi_A) = 0\}. \tag{2.9}$$

Theorem 2.3. *If S is a semigroup and M is a left (right) invariant mean on $B(S,\mathbb{R})$ then the family $\mathcal{J}_M$ is a proper left (right) quasi-invariant ideal of subsets of S.*

Proof. By (2.6), $\mu_M(S) = 1$. Hence, $S \notin \mathcal{J}_M$.

Next we choose arbitrary $f, g \in B(S,\mathbb{R})$ such that $f \le g$. The additivity of M and (2.6) yields

$$0 \le M(g - f) = M(g) - M(f).$$

So, we get the monotonicity of M:

$$f, g \in B(S,\mathbb{R}) \wedge f \le g \Longrightarrow M(f) \le M(g). \tag{2.10}$$

Therefore, if $A \in \mathcal{J}_M$ and $B \subset A$ then

$$0 \le M(\chi_B) \le M(\chi_A) = 0,$$

which means that $B \in \mathcal{J}_M$ and for $A, B \in \mathcal{J}_M$ we have

$$0 \le M(\chi_{A\cup B}) \le M(\chi_A + \chi_B) = M(\chi_A) + M(\chi_B) = 0,$$

whence $A \cup B \in \mathcal{J}_M$. Moreover, for $A \in \mathcal{J}_M$ and $a \in S$, by the left invariance of M we obtain

$$0 \le M(\chi_{{}_aA}) = M(\chi_A) = 0,$$

which implies that ${}_aA \in \mathcal{J}_M$ and the proof is finished. □

Hence, the family $\mathcal{J}_M$ of all zero sets for every invariant mean M forms a proper set ideal of subsets of S. The following question arises: it is true that for every proper set ideal $\mathcal{J}$ of subsets of S there exists an invariant mean M on $B(S,\mathbb{R})$ for which elements of $\mathcal{J}$ are zero sets ($\mathcal{J} \subset \mathcal{J}_M$)? To answer to this question first we quote the following theorem which was proved using Silverman's extension theorem by Gajda in [9].

Theorem 2.4. *If $(S,+)$ is a left (right) amenable semigroup and $\mathcal{J}$ is a p.l.q.i. (p.r.q.i.) ideal of subsets of S, then there exists a real linear functional $M^{\mathcal{J}}$ on the space $B^{\mathcal{J}}(S,\mathbb{R})$ such that*

$$\mathcal{J} - \textit{ess}\inf_{x\in S} f(x) \leq M^{\mathcal{J}}(f) \leq \mathcal{J} - \textit{ess}\sup_{x\in S} f(x) \tag{2.11}$$

and

$$M^{\mathcal{J}}({}_af) = M^{\mathcal{J}}(f) \quad (M^{\mathcal{J}}(f_a) = M^{\mathcal{J}}(f)), \tag{2.12}$$

for all $f \in B^{\mathcal{J}}(S,\mathbb{R})$ and all $a \in S$.

We can find an elementary and short proof of this fact in [1] (see also [3]).

Remark 2.5. We already know that for every p.l.q.i. (p.r.q.i.) ideal $\mathcal{J}$ of subsets of the left (right) amenable semigroup S there exists a left (right) invariant mean $M^{\mathcal{J}}$ on the space $B^{\mathcal{J}}(S,\mathbb{R})$. Of course, the restriction of $M^{\mathcal{J}}$ to the space $B(S,\mathbb{R})$ is a left (right) invariant mean on this space. Moreover, by (2.11) we have

$$M^{\mathcal{J}}(\chi_A) = 0, \quad A \in \mathcal{J},$$

which means that for every p.l.q.i. (p.r.q.i.) ideal $\mathcal{J}$ of subsets of the left (right) amenable semigroup S there exists a left (right) invariant mean M ($M = M^{\mathcal{J}}|_{B(S,\mathbb{R})}$) on the space $B(S,\mathbb{R})$ such that

$$\mathcal{J} \subset \mathcal{J}_M. \tag{2.13}$$

As simple applications of our observation we obtain the following known facts.

Example 2.1. Let $(\mathbb{Z},+)$ be a group of integers and let $\mathbb{N}$ denote the set of positive integers. The family $\mathcal{J}$ of all subsets A of $\mathbb{Z}$ for which there exists $K \in \mathbb{Z}$ such that $A \subset \{k \in \mathbb{Z} : k \geq K\}$ forms a p.q.i. ideal of subsets of $\mathbb{Z}$. Hence, there exists an additive normed measure μ ($\mu = \mu_M$, for some invariant mean M) defined on the family of all subsets of $\mathbb{Z}$ invariant with respect to translations such that

$$\mu(\mathbb{N}) = 0.$$

Analogously, if $(S,+) = (\mathbb{R},+)$ and $A \in \mathcal{J}$ iff there exists $K \in \mathbb{R}$ such that $A \subset \{x \in \mathbb{R} : x \geq K\}$ then $\mathcal{J}$ is a p.q.i. ideal of subsets of $\mathbb{R}$ and there exists an additive normed measure μ defined on the family of all subsets of $\mathbb{R}$ invariant with respect to translations such that

$$\mu((a,+\infty) = 0,$$

for all $a \in \mathbb{R}$.

Now we formulate the theorem which generalized Cabello Sánchez's Lemma 6 from [6].

Theorem 2.6. *Let $\mathcal{J}$ be a p.l.q.i. (p.r.q.i.) ideal of subsets of a semigroup S. If the set ideal $\mathcal{J}$ satisfies the following condition*

$$\begin{gathered}\textit{for every element } A \textit{ of the set ideal} \mathcal{J} \textit{ there exists an element } a \textit{ of } S\\ \textit{such that } \ A \cap_a A = \emptyset (A \cap A_a = \emptyset),\end{gathered} \tag{2.14}$$

then

$$\mathcal{J} \subset \mathcal{J}_M,$$

for every left (right) invariant mean M on the space $B(S,\mathbb{R})$.

Proof. Let $A \in \mathcal{J}$ be fixed and let M be a left invariant mean on the space $B(S, \mathbb{R})$. Suppose to the contrary that

$$M(\chi_A) \neq 0.$$

Putting $A_0 = A$ and $f_0 = \chi_{A_0}$, by our hypothesis and condition (2.6) we have

$$0 = \inf_{x \in S} f_0(x) < M(f_0) \leq \sup_{x \in S} f_0(x) = 1.$$

Now, let f_1 be the real function on S defined by $f_1 = f_0 +_{a_0} f_0$, where the element $a_0 \in S$ is associated with the set A_0 by condition (2.14). Then the set $A_1 = A_0 \cup_{a_0} A_0$ is in $\mathcal{J}$. Moreover, applying the properties of the left invariant mean we have

$$M(f_1) = M(f_0 +_{a_0} f_0) = M(f_0) + M({}_{a_0}f_0) = M(f_0) + M(f_0) = 2M(f_0)$$

and

$$0 = \inf_{x \in S} f_1(x) < M(f_1) \leq \sup_{x \in S} f_1(x) = 1.$$

Next, let $f_2 = f_1 +_{a_1} f_1$, where the element $a_1 \in S$ is associated with the set A_1 by condition (2.14). Then $A_2 = A_1 \cup_{a_1} A_1 \in \mathcal{J}$ and

$$M(f_2) = M(f_1 +_{a_1} f_1) = 2M(f_1) = 2^2 M(f_0),$$

$$0 = \inf_{x \in S} f_2(x) < M(f_2) \leq \sup_{x \in S} f_2(x) = 1.$$

Inductively we construct the sequence of real functions f_n on S such that

$$0 = \inf_{x \in S} f_n(x) < M(f_n) = 2^n M(f_0) \leq \sup_{x \in S} f_2(x) = 1, \quad n \in \mathbb{N}$$

which is false. Hence, $M(f_0) = M(\chi_A) = 0$, which means that $A \in \mathcal{J}_M$ and thus ends the proof. □

Remark 2.7. Observe that the family $\mathcal{J}_b$ of all bounded sets of a normed space S ($\dim S \geq 1$) yields an example of a p.q.i. ideal of subsets of S fulfilling condition (2.14). Therefore,

$$\mathcal{J}_b \subset \mathcal{J}_M,$$

for every invariant mean M on $B(S, \mathbb{R})$. Moreover, the family $\mathcal{J}_f$ of all finite subsets of S also forms a p.q.i. ideal of subsets of S and $\mathcal{J}_f \subsetneq \mathcal{J}_b$. Hence

$$\mathcal{J}_f \subsetneq \mathcal{J}_b \subset \mathcal{J}_M$$

for every invariant mean M on $B(S, \mathbb{R})$ which shows that in (2.13) we have only inclusion. This answers the question posed by Zs. Páles on the equality in (2.13).

Finally, to summarize the results just obtained, we note the following.

Remark 2.8. Let S be a left amenable semigroup and let A be a subset of S.

If

$${}_{a_1}A \cup_{a_2} A \cup \ldots \cup_{a_n} A \neq S,$$

for all $a_1, a_2, \ldots, a_n \in S$ and $n \in \mathbb{N}$, then the set A generates a p.l.q.i. ideal of subsets of S. Hence, using Remark 2.5, the set A is a zero set for some invariant mean M on the space $B(S, \mathbb{R})$ ($A \in \mathcal{J}_M$).

If there exist $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in S$ such that

$${}_{a_1}A \cup_{a_2} A \cup \ldots \cup_{a_n} A = S,$$

then for every invariant mean M on the space $B(S,\mathbb{R})$ we have

$$\begin{aligned}1 &= M(\chi_S) = M(\chi_{a_1}A \cup_{a_2} A \cup \ldots \cup_{a_n} A)\\ &\leq M(\chi_{a_1}A + \chi_{a_2}A + \ldots + \chi_{a_1}A)\\ &= M(\chi_{a_1}A) + M(\chi_{a_2}A) + \ldots + M(\chi_{a_1}A)\\ &= nM(\chi A),\end{aligned}$$

which means that $A \notin \mathcal{J}_M$.

The "right-hand side version" of this observation is analogous to the one presented above.

3. Separation Theorems

Let S be a semigroup and let $\mathcal{J}$ be a proper ideal of subsets of S. Then we say that a given condition is satisfied *$\mathcal{J}$-almost everywhere on S* (written *$\mathcal{J}$-a.e. on S*) if there exists a set $A \in \mathcal{J}$ such that the condition in question is satisfied for every $x \in S \setminus A$.

Moreover, the symbol $\Omega(\mathcal{J})$ will stand for the family of all sets $N \subset S \times S$ with the property that

$$N[x] = \{y \in S : (x,y) \in N\} \in \mathcal{J} \ \ \mathcal{J} - \text{a.e. on } G.$$

The family $\Omega(\mathcal{J})$ forms a proper ideal of subsets of $S \times S$ (see Kuczma [13]).

We are now in a position to formulate and prove the main result of this section which is the "almost everywhere version" of the result proved by Páles in [14] (see also [4]).

Theorem 3.1. *Let $(S,+)$ be a left (right) amenable semigroup, let $\mathcal{J}$ be a p.l.q.i. (p.r.q.i.) ideal of subsets of S and let $p,\ q : S \to \mathbb{R}$. Then there exists a map $a : S \to \mathbb{R}$ such that*

$$a(x+y) = a(x) + a(y) \ \ \Omega(\mathcal{J}) - a.e. \text{ on } S \times S \tag{3.1}$$

and

$$p(x) \leq a(x) \leq q(x) \ \ \mathcal{J} - a.e. \text{ on } S \tag{3.2}$$

if and only if there exists a function $\varphi : S \to \mathbb{R}$ such that

$$p(x) \leq \varphi(x+y) - \varphi(y) \leq q(x) \ \ \Omega(\mathcal{J}) - a.e. \text{ on } S \times S. \tag{3.3}$$

Proof. Assume that a satisfies (3.1) and (3.2) and let $\varphi = a$. Condition (3.1) implies that there exists a set $M \in \Omega(\mathcal{J})$ such that

$$\varphi(x+y) - \varphi(y) = a(x+y) - a(y) = a(x), \ \ (x,y) \in S^2 \setminus M. \tag{3.4}$$

Now, choose $U \in \mathcal{J}$ such that $M[x] \in \mathcal{J}$, for all $x \in S \setminus U$. Next, by (3.2) we get the existence of a set $V \in \mathcal{J}$ such that

$$p(x) \leq a(x) \leq q(x), \ \ x \in S \setminus V. \tag{3.5}$$

By W we denote the set of all pairs $(x,y) \in S^2$ such that

$$p(x) \leq \varphi(x+y) - \varphi(y) \leq q(x)$$

do not hold. Putting (3.4) and (3.5) together, we infer that $W[x] \subseteq M[x] \in \mathcal{J}$, for all $x \in S \setminus (U \cup V)$, which implies $W \in \Omega(\mathcal{J})$. So, the function φ satisfies (3.3).

Assume that (3.3) is valid with a certain function $\varphi : S \to \mathbb{R}$. Then there exists a set $M \in \Omega(\mathcal{J})$ such that

$$p(x) \leq \varphi(x+y) - \varphi(y) \leq q(x), \ \ (x,y) \in S^2 \setminus M.$$

Since $M \in \Omega(\mathcal{J})$, one can find a set $U \in \mathcal{J}$ such that $M[x] \in \mathcal{J}$, for all $x \in S \setminus U$. Now, given an element $x \in S \setminus U$ we have

$$p(x) \leq \varphi(x+y) - \varphi(y) \leq q(x), \ \ y \in S \setminus M[x] \tag{3.6}$$

which means that for any fixed $x \in S \setminus U$ the function

$$S \ni y \longrightarrow \varphi(x+y) - \varphi(y) \in \mathbb{R}$$

belongs to the space $B^{\mathcal{J}}(S, \mathbb{R})$.

Let $M^{\mathcal{J}}$ represent a left invariant mean on the space $B^{\mathcal{J}}(S, \mathbb{R})$, whose existence results from Theorem 2.4. The function $a : S \to \mathbb{R}$ is defined by the formula

$$a(x) = \begin{cases} M_y^{\mathcal{J}}(\varphi(x+y) - \varphi(y)), & \text{for } x \in S \setminus U \\ 0, & \text{for } x \in U, \end{cases}$$

where the subscript y indicates that the mean $M^{\mathcal{J}}$ is applied to a function of the variable y.

If we choose $u, v \in S \setminus U$ in such a manner that $u + v \in S \setminus U$ too, then by the left invariance and linearity of $M^{\mathcal{J}}$, we get

$$\begin{aligned} a(u) + a(v) &= M_y^{\mathcal{J}}(\varphi(u+y) - \varphi(y)) + M_y^{\mathcal{J}}(\varphi(v+y) - \varphi(y)) \\ &= M_y^{\mathcal{J}}(\varphi(u+v+y) - \varphi(v+y)) + M_y^{\mathcal{J}}(\varphi(v+y) - \varphi(y)) \\ &= M_y^{\mathcal{J}}(\varphi(u+v+y) - \varphi(y)) = a(u+v). \end{aligned}$$

This means that $a(u+v) = a(u) + a(v)$, for all $(u, v) \in S^2 \setminus W$, where

$$W = (U \times S) \cup (S \times U) \cup \{(u, v) \in S^2 : u + v \in U\}.$$

It is clear that $W \in \Omega(\mathcal{J})$ and we get (3.1). Moreover, condition (2.11) jointly with the definition of a and (3.6) implies (3.2) and completes the proof. □

For groups we have the following.

Corollary 3.2. *Let $(S, +)$ be a left (right) amenable group, let $\mathcal{J}$ be a p.l.q.i. (p.r.q.i.) ideal of subsets of S and let $p, q : S \to \mathbb{R}$. Then there exists an additive function $A : S \to \mathbb{R}$ such that*

$$p(x) \le A(x) \le q(x) \;\; \mathcal{J} - a.e. \text{ on } S \tag{3.7}$$

if and only if there exists a function $\varphi : S \to \mathbb{R}$ such that

$$p(x) \le \varphi(x+y) - \varphi(y) \le q(x) \;\; \Omega(\mathcal{J}) - a.e. \text{ on } S \times S. \tag{3.8}$$

Proof. The proof of this theorem is a consequence of our previous result and the Cabello Sánchez theorem ([6, Theorem 8]) which is a version of the celebrated theorem of de Bruijn (see [5]) and its generalization given by Ger (see [10]) and which shows that for a map $a : S \to \mathbb{R}$ fulfilling (3.1) there exists an additive function $A : S \to \mathbb{R}$ such that

$$a(x) = A(x) \;\; \mathcal{J} - a.e. \text{ on } S.$$

□

As a consequence of this fact we obtain the following (see Gajda, Kominek [8] and Cabello Sánchez [6]).

Theorem 3.3. *Let $(S, +)$ be an Abelian group and let $\mathcal{J}$ be a p.l.q.i. (p.r.q.i.) ideal of subsets of S. If $f, g : S \to \mathbb{R}$ satisfy*

$$f(x+y) \le f(x) + f(y) \;\; \Omega(\mathcal{J}) - a.e. \text{ on } S \times S$$
$$g(x+y) \ge g(x) + g(y) \;\; \Omega(\mathcal{J}) - a.e. \text{ on } S \times S$$

and

$$g(x) \le f(x) \;\; \mathcal{J} - a.e. \text{ on } S$$

then there exists an additive function $A : S \to \mathbb{R}$ such that

$$g(x) \le a(x) \le f(x) \;\; \mathcal{J} - a.e. \text{ on } S.$$

Proof. Assume that $U_1, V_1 \in \mathcal{J}$ satisfy: for $x \in S \setminus U_1$

$$f(x+y) \le f(x) + f(y), \quad y \in S \setminus V_1$$

and let $U_2, V_2 \in \mathcal{J}$ satisfy: for $x \in S \setminus U_2$

$$g(x+y) \ge g(x) + g(y), \quad y \in S \setminus V_2.$$

Moreover, let U_0 be such that

$$g(x) \le f(x), \quad x \in S \setminus U_0.$$

Then, for $x \in S \setminus U$, where $U = U_0 \cup U_1 \cup U_2$ and for $y \in S \setminus (V_1 \cup V_2 \cup U_0 \cup_x U_0)$ we have

$$f(x+y) - g(y) \ge g(x+y) - g(y) \ge g(x).$$

Hence, one can define a function $\varphi : S \to \mathbb{R}$ by $\varphi(x) = 0$, if $x \in U$ and for $x \in S \setminus U$ by

$$\varphi(x) = \operatorname*{ess\,inf}_{t \in S} ({}_x f - g)(t).$$

Suppose that x and $x + y$ are in $S \setminus U$. Then, as in [6], we can show that

$$g(x) \le \varphi(x+y) - \varphi(y) \le f(x).$$

Now, taking $N = (U \times S) \cup \{(x, y) \in S^2 : x + y \in U\}$ we observe that $N \in \Omega(\mathcal{J})$ which means that φ satisfies condition (3.8) and an appeal to Corollary 3.2 completes the proof. □

The next application concerns the stability problem for Cauchy's functional equation. On account of similarity we restrict our considerations to the "Ger-additive" functions.

Theorem 3.4. *Let $(S, +)$ be a left (right) amenable semigroup, $\mathcal{J}$ be a p.l.q.i. (p.r.q.i.) ideal of subsets of S and let $\rho : S \to \mathbb{R}$. Moreover, let $f : S \to \mathbb{R}$ be a function such that for a certain set $N \in \Omega(\mathcal{J})$, the inequality*

$$|f(x+y) - f(x) - f(y)| \le \rho(x)$$

$$(|f(x+y) - f(x) - f(y)| \le \rho(y))$$

holds whenever $(x, y) \in S \times S \setminus N$. Then there exists a map $a : S \to \mathbb{R}$ such that

$$a(x+y) = a(x) + a(y) \;\; \Omega(\mathcal{J}) - a.e. \text{ on } S \times S \tag{3.9}$$

and

$$|f(x) - a(x)| \le \rho(x) \;\; \mathcal{J} - a.e. \text{ on } S. \tag{3.10}$$

Proof. The functions $p = f - \rho$, $q = f + \rho$ and $\varphi = f$ satisfy condition (3.3). Theorem 3.1 yields a map a fulfilling (3.9) and (3.10), and the proof is complete. □

For groups we have the following result.

Corollary 3.5. *Let $(S, +)$ be a left (right) amenable group, $\mathcal{J}$ be a p.l.q.i. (p.r.q.i.) ideal of subsets of S and let $\rho : S \to \mathbb{R}$. Moreover, let $f : S \to \mathbb{R}$ be a function such that for a certain set $N \in \Omega(\mathcal{J})$, the inequality*

$$|f(x+y) - f(x) - f(y)| \le \rho(x)$$

$$(|f(x+y) - f(x) - f(y)| \le \rho(y))$$

holds whenever $(x, y) \in S \times S \setminus N$. Then there exists an additive map $A : S \to \mathbb{R}$ such that

$$|f(x) - A(x)| \le \rho(x) \;\; \mathcal{J} - a.e. \text{ on } S.$$

Remark 3.6. The vector-valued versions of the above results can be obtained using the techniques presented in [4], [6] or [2].

REFERENCES

[1] R. BADORA, On an invariant mean for $\mathcal{J}$-essentially bounded functions, *Facta Universitatis (Niš), Ser. Math. Inform.*, **6** (1991), 95–106.

[2] R. BADORA, On some generalized invariant means and almost approximately additive functions, *Publ. Math. Debrecen*, **44**(1-2) (1994), 123–135.

[3] R. BADORA, On generalized invariant means, manuscript.

[4] R. BADORA, R. GER AND Zs. PÁLES, Additive selections and the stability of the Cauchy functional equation, *ANZIAM J.*, **44** (2003), 323–337.

[5] N.G.de BRUIJN, On almost additive functions, *Colloq.Math.*, **15**(1966), 59–63.

[6] F. CABELLO SÁNCHEZ, Stability of additive mappings on large subsets, *Proc. Amer. Math. Soc.*, **128** (1998), 1071–1077.

[7] M.M. DAY, *Normed Linear Spaces*, Springer Verlag, Berlin - Heidelberg - New York, 1973.

[8] Z. GAJDA AND Z. KOMINEK, On separation theorems for subadditive and superadditive functionals, *Studia Math.*, **100**(1) (1991), 25–38.

[9] Z. GAJDA, Invariant means and representations of semigroups in the theory of functional equations, Prace Naukowe Uniwersytetu Śląskiego 1273, Silesian University Press, Katowice, 1992.

[10] R. GER, Note on almost additive functions, *Aequationes Math.*, **17** (1978), 73–76.

[11] R. GER, On functional inequalities stemming from stability questions, *General Inequalities 6* (W. Walter, Ed.), ISNM **103** (1992), 227–240.

[12] F.P. GREENLEAF, *Invariant Mean on Topological Groups and their Applications*, Van Nostrand Mathematical Studies 16, New York - Toronto - London - Melbourne, 1969.

[13] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe and Uniwersytet Śląski, Warszawa - Kraków - Katowice, 1985.

[14] Zs. PÁLES, Generalized stability of the Cauchy functional equation, *Aequationes Math.*, **56** (1998), 222–232.