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THE COUNTERPART OF FAN'S INEQUALITY AND ITS RELATED RESULTS

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ABSTRACT. In the present paper we give the new proofs for the counterpart of Fan's inequality, and establish several refinements and converses of it. The method is based on an idea of J. Sándor and V.E.S. Szabò in [19]. It must be noted that the technique has been replaced by a more effective one.

Key words and phrases: Inequality, counterpart, Sándor-Szabò's idea, refinement, converse.

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1. NOTATION AND INTRODUCTION

We need the following notation and symbols used in the papers [18], [22], [4], [21], [15], [1], [2], [9], [23]:

$$a_i \in (0, 1/2],\ p_i > 0,\ i = 1, \dots, n, \quad P := p_1 + \cdots + p_n, \quad \mathbb{N} := \text{the natural numbers set},$$

$$A := A(a) := P^{-1} \cdot \sum p_i a_i, \quad G := G(a) := \prod a_i^{p_i/P}, \quad H := H(a) := P\left(\sum p_i a_i^{-1}\right)^{-1},$$

$$A' := A(1-a) := P^{-1} \cdot \sum p_i(1-a_i), \quad G' := G(1-a), \quad H' := H(1-a),$$

$$m := \min\{a_1, \dots, a_n\}, \quad M := \max\{a_1, \dots, a_n\}, \quad \exp\{x\} := e^x.$$

Here and in what follows $\sum$ and $\prod$ are used to indicate $\sum_{i=1}^n$ and $\prod_{i=1}^n$, respectively.

In 1996, J.Sándor and V.E.S. Szabò [19] discovered an interesting method of establishing inequalities, that is, they established inequalities by means of the following:

$$\sum \inf_{x \in E} F_i(x) \le \inf_{x \in E} \sum F_i(x). \tag{1.1}$$

Since 1999 [21], the present authors have been studying the following inequalities:

$$\frac{H}{H'} \le \frac{G}{G'} \le \frac{A}{A'}. \tag{1.2}$$

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The second inequality in (1.2) was published in 1961 and is due to Ky Fan [7, p. 5]; the first inequality with equal weights was established by W.-l.Wang and P.-F.Wang [22] in 1984. Clearly, the first is a counterpart of Fan's inequality. It seems that the counterpart is called Wang-Wang's inequality in the current literature [15], [1], [8], [11], [12]. The inequalities in (1.2) have evoked the interest of several mathematicians, and many new proofs as well as some generalizations and refinements have been published (see [8], [11], [12], [13], [3], [10], [14], [16], [17], [5], [20], [6], etc.). We refer to H. Alzer's brilliant exposition [4] for the inequalities (1.2) and some related subjects. In this paper we improve the Sándor-Szabò technique. We shall apply the inequality (1.1) and the following facts:

$$\inf_{x\in E} F_i(x) \le F_i(y), \tag{1.3}$$

$$\sum \inf_{x\in E} F_i(x) \le \inf_{x\in E} \sum F_i(x) \le \sum F_i(y) \quad \text{for all } y \in E \tag{1.4}$$

to two proofs of the counterpart (i.e., (2.1) below), and establish several refinements and converses. Indeed, the following process will reveal the simplicity, adaptability and reliability of using (1.3) and (1.4). In Section 2, we give theorems and their proofs. As an application of the new results, in Section 3, we discuss a connection between the results of [23] and our result (2.3). In Section 4, we give some concluding remarks.

2. Proofs of the Counterpart and Related Results

First we reprove the first inequality in (1.2).

Theorem 2.1. *If* $a_i \in (0, 1/2]$, $(i = 1, \dots, n)$, *then the first inequality in (1.2) holds, that is, the following result holds:*

$$\frac{H}{H'} \le \frac{G}{G'}. \tag{2.1}$$

First Proof. We first choose the function in the argument of Theorem 3 of [21], namely, ϕ_i : $(0, 1/2] \to \mathbb{R}$ $(i = 1, \dots, n)$ defined by

$$\phi_i(x) := p_i \left(\frac{x}{a_i} - \frac{1-x}{1-a_i} + \log \frac{1-x}{x} \right).$$

Since ϕ_i is strictly convex and $x_{i,0} = a_i$ is the unique critical point in $(0, 1/2]$, then for every ϕ_i and any $y \in (0, 1/2]$, using the inequality (1.3) we have

$$\phi_i(a_i) = \log \left(\frac{1-a_i}{a_i} \right)^{p_i} \le \phi_i(y) = p_i \left(\frac{y}{a_i} - \frac{1-y}{1-a_i} + \log \frac{1-y}{y} \right).$$

Summing up over i from 1 to n we get

$$\log \prod \left(\frac{1-a_i}{a_i} \right)^{p_i} \le \left(\sum \frac{p_i}{a_i} \right) y - \left(\sum \frac{p_i}{1-a_i} \right) (1-y) + P \log \frac{1-y}{y}.$$

Dividing both sides by P, we have

$$\log \prod \left(\frac{1-a_i}{a_i} \right)^{p_i/P} \le \frac{y}{H} - \frac{1-y}{H'} + \log \frac{1-y}{y}. \tag{2.2}$$

Taking $y = H/(H+H')$, clearly $y \in (0, 1/2]$, a simple calculation yields that

$$\log \frac{G'}{G} \le \log \frac{H'}{H},$$

which is equivalent to (2.1). This completes the first proof of Theorem 2.1. □

Second Proof. Along the same lines of the first proof, we obtain (2.2). If we take $y = G/(G + G')$ in (2.2), clearly $y \in (0, 1/2]$, then

$$\log \frac{G'}{G} \le \frac{G}{(G+G')H} - \frac{G'}{(G+G')H'} + \log \frac{G'}{G}$$

or,

$$0 \le \frac{G}{H} - \frac{G'}{H'},$$

which is equivalent to (2.1). This completes the second proof of Theorem 2.1. $\square$

Remark 1. We can also give an equality condition from the argument in the first proof. In fact, we have known that all these functions are strictly convex in $(0, 1/2]$, so the equality condition of (2.1) should be "if and only if $a_1 = \cdots = a_n$".

Remark 2. There are already at least eight proofs of (2.1) (see [22], [4], [15], [1], [2], [9], [23], [12]). The author believes that the proofs of this paper are extremely simple, interesting and elementary.

Remark 3. By a procedure analogous to [22], [4], [21], we can deduce the well-known inequality $H \le G$. In fact, if we choose $t/2 \ge M = \max\{a_1, \dots, a_n\}$, then $a_i/t \in (0, 1/2]$ $(i = 1, \dots, n)$. Replacing successively a_i by a_i/t in (2.1), and then simplifying the resulting inequality, we have

$$\frac{\left(\sum p_i a_i^{-1}\right)^{-1}}{\left[\sum p_i (1 - a_i/t)^{-1}\right]^{-1}} \le \frac{\prod a_i^{p_i/P}}{\prod (1 - a_i/t)^{p_i/P}}.$$

Now passing to the limit as $t \to +\infty$, the desired $H \le G$ can be deduced.

Theorem 2.2. *If* $a_i \in (0, 1/2],\ (i = 1, \dots, n)$*, then we have the following refinement of (2.1):*

$$\frac{H}{H'} \le \frac{x_0}{1 - x_0} \exp\left[\frac{1}{H'} - \left(\frac{1}{H} + \frac{1}{H'}\right) x_0\right] \le \frac{G}{G'}, \tag{2.3}$$

where

$$x_0 = \frac{1}{2} - \frac{\sqrt{(H + H')(H + H' - 4HH')}}{2(H + H')} \tag{2.4}$$

and $x_0 \in [m, M]$.

Proof. Choose the above functions ϕ_i, $(i = 1, \dots, n)$ in the argument of Theorem 2.1. We observe that

$$\sum \inf_{x \in (0,1/2]} \phi_i(x) = \log \prod \left(\frac{1 - a_i}{a_i}\right)^{p_i}.$$

Let $\Phi := \sum \phi_i$. Then

$$\Phi(x) = \sum \phi_i(x) = P\left(\frac{x}{H} - \frac{1 - x}{H'} + \log \frac{1 - x}{x}\right).$$

By Theorem 3 in [21], Φ has minimum at

$$x_0 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4P\left[\sum \frac{p_i}{a_i(1 - a_i)}\right]^{-1}}. \tag{2.5}$$

Combining (2.5) with the following relationship

$$\left[\sum \frac{p_i}{a_i(1 - a_i)}\right]^{-1} = P^{-1}\left[\frac{1}{H} + \frac{1}{H'}\right]^{-1},$$

we can obtain the expression (2.4).

As to $x_0 \in [m, M]$, this is also a conclusion of Theorem 3 in [21].

Using the inequality (1.4), for any $y \in (0, 1/2]$ we get

$$\log \prod \left(\frac{1 - a_i}{a_i} \right)^{p_i} \leq \Phi(x_0) \leq \Phi(y). \tag{2.6}$$

Taking $y = H/(H + H')$ and dividing both sides by P, (2.6) gives

$$\log \frac{G'}{G} \leq \frac{x_0}{H} - \frac{1 - x_0}{H'} + \log \frac{1 - x_0}{x_0} \leq \log \frac{H'}{H}$$

which is equivalent to (2.3). The proof of Theorem 2.2 is therefore complete. □

Remark 4. Clearly, the inequality (2.1) is a natural consequence of (2.3). We may give a numerical example of (2.3): In $n = 5$, we take

$$a_1 = 0.1, \quad a_2 = 0.15, \quad a_3 = 0.2, \quad a_4 = 0.35, \quad a_5 = 0.4,$$
$$p_1 = 0.1, \quad p_2 = 0.3, \quad p_3 = 0.2, \quad p_4 = 0.25, \quad p_5 = 0.15,$$

arbitrarily. The results are generated via use of Mathematica, and as expected:

$$\begin{aligned} \frac{H}{H'} &= 0.265001 \\ &< \frac{x_0}{1 - x_0} \exp \left[\frac{1}{H'} - \left(\frac{1}{H} + \frac{1}{H'} \right) x_0 \right] \\ &= 0.265939 < \frac{G}{G'} = 0.291434. \end{aligned}$$

Proposition 2.3. *If* $a_i \in (0, 1/2], \ (i = 1, \dots, n)$*, we have* $\frac{A}{1-A} = \frac{A}{A'}, \ \frac{H}{1-H} \leq \frac{H}{H'}$ *and* $\frac{G}{1-G} \leq \frac{G}{G'}$*.*

In fact, we can obtain the desired result from the inequalities

$$H + H' \leq G + G' \leq A + A' = 1.$$

Theorem 1 of [21] uses (1.1) and some functions to prove Fan's inequality and its generalization. The functions chosen in the argument are $f_i : (0, 1/2] \to \mathbb{R}, \ (i = 1, \dots, n)$ defined by

$$f_i(x) := p_i \left(\frac{a_i}{x} - \frac{1 - a_i}{1 - x} - \log \frac{1 - x}{x} \right). \tag{2.7}$$

By using (1.4) and (2.7), we shall renew our efforts to further establish the converses of $H/H' \leq G/G'$ and $H/H' \leq A/A'$ as follows:

Theorem 2.4. *If* $a_i \in (0, 1/2], \ (i = 1, \dots, n)$*, we have*

$$\frac{G}{G'} \leq \frac{A}{A'} \leq \frac{H}{H'} \exp \left[\left(1 + \frac{H'}{H} \right) A - \left(1 + \frac{H}{H'} \right) A' \right]; \tag{2.8}$$

$$\frac{G}{G'} \leq \frac{A}{A'} \leq \frac{G}{G'} \exp \left[\left(1 + \frac{G'}{G} \right) A - \left(1 + \frac{G}{G'} \right) A' \right]; \tag{2.9}$$

$$\frac{G}{G'} \leq \frac{A}{A'} \leq \frac{H}{H'} \exp \left[\frac{A}{H} - \frac{A'}{1 - H} \right]; \tag{2.10}$$

$$\frac{G}{G'} \leq \frac{A}{A'} \leq \frac{G}{G'} \exp \left[\frac{A}{G} - \frac{A'}{1 - G} \right]. \tag{2.11}$$

Proof. Choose the above functions in (2.7). Since f_i has a minimum at $x_{i,0} = a_i$ and its value is $f_i(a_i) = -\log[(1-a_i)/a_i]^{p_i}$, then

$$\sum \inf_{x\in(0,1/2]} f_i(x) = \sum f_i(a_i) = \log \prod \left[\frac{a_i}{1-a_i}\right]^{p_i}.$$

Similarly, the function

$$f(x) := \sum f_i(x) = P\left(\frac{A}{x} - \frac{1-A}{1-x} - \log\frac{1-x}{x}\right)$$

has a minimum at $x_0 = A$ and its value is $f(x_0) = f(A) = P\log\frac{A}{A'}$.

Using (1.4) we get

$$\log \prod \left(\frac{a_i}{1-a_i}\right)^{p_i} \le P\log\frac{A}{A'} \le P\left(\frac{A}{y} - \frac{1-A}{1-y} - \log\frac{1-y}{y}\right), \tag{2.12}$$

where $y \in (0, 1/2]$. Taking $y = H/(H+H')$ in (2.12), we have

$$\begin{aligned}\log \prod \left(\frac{a_i}{1-a_i}\right)^{p_i} &\le P\log\frac{A}{A'} \\ &\le P\log\frac{H}{H'} + P\left[\left(1+\frac{H'}{H}\right)A - \left(1+\frac{H}{H'}\right)A'\right].\end{aligned}$$

Dividing both sides by P, we get

$$\log\frac{G}{G'} \le \log\frac{A}{A'} \le \log\frac{H}{H'} + \left(1+\frac{H'}{H}\right)A - \left(1+\frac{H}{H'}\right)A',$$

which is equivalent to (2.8).

By a similar argument to the above, taking $y = G/(G+G')$ in (2.12), we can obtain (2.9); taking $y = H$ and $y = G$ in (2.12) respectively, and then combining the resulting inequalities with Proposition 2.3, we respectively obtain (2.10) and (2.11). The proof of Theorem 2.4 is therefore complete. □

Remark 5. The data in Remark 4 is used below so as to save space. Using those values, we have

$$\begin{aligned}\frac{G}{G'} &= 0.291434 \le \frac{A}{A'} = 0.320132 \\ &\le \frac{H}{H'}\exp\left[\left(1+\frac{H'}{H}\right)A - \left(1+\frac{H}{H'}\right)A'\right] = 0.323423;\end{aligned}$$

$$\begin{aligned}\frac{G}{G'} &= 0.291434 \le \frac{A}{A'} = 0.320132 \\ &\le \frac{G}{G'}\exp\left[\left(1+\frac{G'}{G}\right)A - \left(1+\frac{G}{G'}\right)A'\right] = 0.320905;\end{aligned}$$

$$\frac{G}{G'} = 0.291434 \le \frac{A}{A'} = 0.320132 \le \frac{H}{H'}\exp\left[\frac{A}{H} - \frac{A'}{1-H}\right] = 0.332691;$$

$$\frac{G}{G'} = 0.291434 \le \frac{A}{A'} = 0.320132 \le \frac{G}{G'}\exp\left[\frac{A}{G} - \frac{A'}{1-G}\right] = 0.438724.$$

Remark 6. Notice that the given inequalities $0 < m \le a_i \le M \le 1/2$ imply the following:

$$m \le H \le M, m \le A \le M, 1-M \le H' \le 1-m, 1-M \le A' \le 1-m,$$

$$\frac{1-M}{M} \le \frac{H'}{H} \le \frac{1-m}{m}, \qquad \frac{m}{1-m} \le \frac{H}{H'} \le \frac{M}{1-M}.$$

It follows from the above that

$$\left(1+\frac{H'}{H}\right)A - \left(1+\frac{H}{H'}\right)A' \le \left(1+\frac{1-m}{m}\right)M - \left(1+\frac{m}{1-m}\right)(1-M) = \frac{M-m}{m(1-m)}. \tag{2.13}$$

Combining (2.13) with (2.8), (2.8) can also be rewritten as

$$\frac{G}{G'} \le \frac{A}{A'} \le \frac{H}{H'} \exp\left[\left(1+\frac{H'}{H}\right)A - \left(1+\frac{H}{H'}\right)A'\right] \tag{2.14}$$

$$\le \frac{M}{1-M} \exp\left[\frac{M-m}{m(1-m)}\right]. \tag{2.15}$$

Similarly, we can obtain several estimations for (2.9), (2.10) and (2.11) that are similar to (2.14).

3. AN APPLICATION

We shall consider a connection between the above inequalities (2.3) and a useful result which is due to G.-S.Yang and C.-S.Wang.

The first part in Theorem 2 of [23] is the following

Proposition 3.1. *Given a sequence $\{a_1, a_2, \dots, a_n\}$ with $a_i \in (0, 1/2]$, $i = 1, \dots, n$, which do not all coincide. Let*

$$p(t) = \prod_{i=1}^{n} \left[\frac{1}{a_i} + t\sum_{j=1}^{n}\left(\frac{1}{a_j} - \frac{1}{a_i}\right) - 1\right]^{-\frac{1}{n}}, \quad t \in \left[0, \frac{1}{n}\right]. \tag{3.1}$$

Then $p(t)$ is continuous, strictly decreasing, and

$$\frac{H}{1-H} = p\left(\frac{1}{n}\right) \le p(t) \le p(0) = \frac{G}{G'}$$

on $[0, 1/n]$.

Theorem 3.2. *Under the hypotheses of Proposition 3.1 and $p_1 = \cdots = p_n = 1$ in (2.3), there exist three points $0, \xi, t_0 \in [0, 1/n],\ 0 \le \xi \le t_0$ such that*

$$\begin{aligned} p(t_0) &= \frac{H}{H'} \\ &\le p(\xi) = \frac{x_0}{1-x_0} \exp\left[\frac{1}{H'}\left(\frac{1}{H} + \frac{1}{H'}\right)x_0\right] \\ &\le p(0) = \frac{G}{G'}, \end{aligned} \tag{3.2}$$

where

$$H = \frac{n}{\sum \frac{1}{a_i}}, \quad H' = \frac{n}{\sum \frac{1}{1-a_i}}, \quad G = \prod a_i^{1/n}, \quad G' = \prod (1-a_i)^{1/n},$$

and $p(t)$ is defined by (2.15).

Proof. On the one hand, by Proposition 2.3 and Theorem 2.1 we get $H/(1-H) \le H/H' \le G/G'$. On the other hand, by Proposition 3.1, we know that $p(t)$ is a strictly decreasing and continuous function on $[0, 1/n]$ and $p(0) = G/G'$, $p(1/n) = H/(1-H)$. Based on these facts and the intermediate value theorem of continuous functions, there exists a unique $t_0 \in [0, 1/n]$ such that $p(t_0) = H/H'$.

Combining the above facts and Proposition 3.1 with (2.3) in Theorem 2.2, the intermediate value theorem implies the existence of a ξ on the interval $[0, t_0]$ with the property that

$$p(\xi) = \frac{x_0}{1-x_0} \exp\left[\frac{1}{H'}\left(\frac{1}{H} + \frac{1}{H'}\right) x_0\right].$$

In conclusion, there exist three points $0, \xi, t_0 \in [0, 1/n],\ 0 \le \xi \le t_0$ such that (3.1) holds. Thus the proof of Theorem 3.2 is completed. □

4. CONCLUDING REMARKS

The result given above as well as those in [21] have revealed that inequalities (1.1), (1.3) and (1.4) are based on the same idea. However, their roles are different in applying these inequalities. Inequalities that can be established by (1.1) cannot necessarily be established by (1.3) and/or (1.4). We have noticed that using (1.3) and/or (1.4) is more convenient for proving or discovering the refinements of some inequalities. For these reasons, they can be applied in a wider scope. Several advantages that the technique has are its simplicity, adaptability and reliability. In other words, the method of using (1.3) and/or (1.4) provided in this paper is superior to the original approach that only uses (1.1).

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