journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 9 (2008), Issue 1, Article 25, 8 pp.

AN OPERATOR PRESERVING INEQUALITIES BETWEEN POLYNOMIALS

W. M. SHAH AND A. LIMAN

P.G. DEPARTMENT OF MATHEMATICS
BARAMULLA COLLEGE, KASHMIR
INDIA- 193101
wmshah@rediffmail.com

DEPARTMENT OF MATHEMATICS
NATIONAL INSTITUTE OF TECHNOLOGY
KASHMIR, INDIA-190006
abliman22@yahoo.com

Received 15 July, 2007; accepted 01 February, 2008
Communicated by I. Gavrea

ABSTRACT. Let $P(z)$ be a polynomial of degree at most n. We consider an operator B, which carries a polynomial $P(z)$ into

$$B\left[P(z)\right] := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right)\frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!},$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) = \lambda_0 + c(n,1)\lambda_1 z + c(n,2)\lambda_2 z^2$$

lie in the half plane

$$|z| \le \left| z - \frac{n}{2} \right|.$$

In this paper, we estimate the minimum and maximum modulii of $B[P(z)]$ on $|z| = 1$ with restrictions on the zeros of $P(z)$ and thereby obtain compact generalizations of some well known polynomial inequalities.

Key words and phrases: Polynomials, B operator, Inequalities in the complex domain.

2000 *Mathematics Subject Classification.* 30A06, 30A64.

1. INTRODUCTION

Let P_n be the class of polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n then

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)| \tag{1.1}$$

and

$$\max_{|z|=R>1} |P(z)| \le R^n \max_{|z|=1} |P(z)| . \tag{1.2}$$

235-07

Inequality (1.1) is an immediate consequence of a famous result due to Bernstein on the derivative of a trigonometric polynomial (for reference see [6, 9, 14]). Inequality (1.2) is a simple deduction from the maximum modulus principle (see [15, p.346], [11, p. 158, Problem 269]).

Aziz and Dawood [3] proved that if $P(z)$ has all its zeros in $|z| \leq 1,$ then

$$\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\min_{|z|=R>1} |P(z)| \geq R^n \min_{|z|=1} |P(z)|. \tag{1.4}$$

Inequalities (1.1), (1.2), (1.3) and (1.4) are sharp and equality holds for a polynomial having all its zeros at the origin.

For the class of polynomials having no zeros in $|z| < 1,$ inequalities (1.1) and (1.2) can be sharpened. In fact, if $P(z) \neq 0$ in $|z| < 1,$ then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.5}$$

and

$$\max_{|z|=R>1} |P(z)| \leq \left(\frac{R^n + 1}{2}\right) \max_{|z|=1} |P(z)|. \tag{1.6}$$

Inequality (1.5) was conjectured by Erdös and later verified by Lax [7], whereas Ankeny and Rivlin [1] used (1.5) to prove (1.6). Inequalities (1.5) and (1.6) were further improved in [3], where under the same hypothesis, it was shown that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\} \tag{1.7}$$

and

$$\max_{|z|=R>1} |P(z)| \leq \left(\frac{R^n + 1}{2}\right) \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{2}\right) \min_{|z|=1} |P(z)|. \tag{1.8}$$

Equality in (1.5), (1.6), (1.7) and (1.8) holds for polynomials of the form $P(z) = \alpha z^n + \beta,$ where $|\alpha| = |\beta|.$

Aziz [2], Aziz and Shah [5] and Shah [17] extended such well-known inequalities to the polar derivatives $D_\alpha\ P(z)$ of a polynomial $P(z)$ with respect to a point α and obtained several sharp inequalities. Like polar derivatives there are many other operators which are just as interesting (for reference see [13, 14]). It is an interesting problem, as pointed out by Professor Q. I. Rahman to characterize all such operators. As an attempt to this characterization, we consider an operator B which carries $P \in P_n$ into

$$B[P(z)] := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}, \tag{1.9}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) = \lambda_0 + c(n,1)\lambda_1 z + c(n,2)\lambda_2 z^2 \tag{1.10}$$

lie in the half plane

$$|z| \leq \left| z - \frac{n}{2} \right|, \tag{1.11}$$

and prove some results concerning the maximum and minimum modulii of $B[P(z)]$ and thereby obtain compact generalizations of some well-known theorems.

We first prove the following theorem and obtain a compact generalization of inequalities (1.3) and (1.4).

Theorem 1.1. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| > 1$, then*

$$|B[P(z)]| \geq |B[z^n]| \min_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1. \tag{1.12}$$

The result is sharp and equality holds for a polynomial having all its zeros at the origin.

Substituting for $B[P(z)]$, we have for $|z| \geq 1$,

$$\begin{aligned} &\left|\lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) P'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}\right| \\ &\qquad \geq \left|\lambda_0 z^n + \lambda_1 \left(\frac{nz}{2}\right) nz^{n-1} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{n(n-1)}{2} z^{n-2}\right| \min_{|z|=1} |P(z)|, \end{aligned} \tag{1.13}$$

where $\lambda_0,\ \lambda_1$ and λ_2 are such that all the zeros of (1.10) lie in the half plane represented by (1.11).

Remark 1.2. If we choose $\lambda_0 = 0 = \lambda_2$ in (1.13), and note that in this case all the zeros of $u(z)$ defined by (1.10) lie in (1.11), we get

$$|P'(z)| \geq n|z|^{n-1} \min_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1,$$

which in particular gives inequality (1.3).Next, choosing $\lambda_1 = 0 = \lambda_2$ in (1.13), which is possible in a similar way, we obtain

$$|P(z)| \geq |z|^n \min_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1.$$

Taking in particular $z = Re^{i\theta}$, $R \geq 1$, we get

$$\left|P(Re^{i\theta})\right| \geq R^n \min_{|z|=1} |P(z)|,$$

which is equivalent to (1.4).

As an extension of Bernstein's inequality, it was observed by Rahman [12], that if $P \in P_n$, then

$$|P(z)| \leq M, \quad |z| = 1$$

implies

$$|B[P(z)]| \leq M\,|B[z^n]|, \quad |z| \geq 1. \tag{1.14}$$

As an improvement to this result of Rahman, we prove the following theorem for the class of polynomials not vanishing in the unit disk and obtain a compact generalization of (1.5) and (1.6).

Theorem 1.3. *If $P \in P_n$, and $P(z) \neq 0$ in $|z| < 1$, then*

$$|B[P(z)]| \leq \frac{1}{2}\{|B[z^n]| + |\lambda_0|\} \max_{|z|=1} |P(z)|, \quad \text{for} \quad |z| \geq 1. \tag{1.15}$$

The result is sharp and equality holds for a polynomial whose zeros all lie on the unit disk.

Substituting for $B[P(z)]$ in inequality (1.15), we have for $|z| \geq 1$,

$$\begin{aligned} &\left|\lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) P'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2}\right| \\ &\qquad \leq \frac{1}{2}\left\{\left|\lambda_0 z^n + \lambda_1 \left(\frac{nz}{2}\right) nz^{n-1} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{n(n-1)}{2} z^{n-2}\right| + |\lambda_0|\right\} \max_{|z|=1} |P(z)|, \end{aligned} \tag{1.16}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of (1.10) lie in the half plane represented by (1.11).

Remark 1.4. Choosing $\lambda_0 = 0 = \lambda_2$ in (1.16) which is possible, we get

$$|P'(z)| \leq \frac{n}{2}|z|^{n-1} \max_{|z|=1} |P(z)| \,, \quad \text{for } |z| \geq 1$$

which in particular gives inequality (1.5).Next if we take $\lambda_1 = 0 = \lambda_2$ in (1.16) which is also possible, we obtain

$$|P(z)| \leq \frac{1}{2} \{|z|^n + 1\} \max_{|z|=1} |P(z)| \,,$$

for every z with $|z| \geq 1$. Taking $z = Re^{i\theta}$, so that $|z| = R \geq 1$, we get

$$\left|P(Re^{i\theta})\right| \leq \frac{1}{2}(R^n + 1) \max_{|z|=1} |P(z)|,$$

which in particular gives inequality (1.6).

As a refinement of Theorem 1.3, we next prove the following theorem, which provides a compact generalization of inequalities (1.7) and (1.8).

Theorem 1.5. *If $P \in P_n$, and $P(z) \neq 0$ in $|z| < 1$ then for $|z| \geq 1$,*

$$|B[P(z)]| \leq \frac{1}{2}\left[\{|B[z^n]| + |\lambda_0|\} \max_{|z|=1} |P(z)| - \{|B[z^n]| - |\lambda_0|\} \min_{|z|=1} |P(z)|\right]. \tag{1.17}$$

Equality holds for the polynomial having all zeros on the unit disk.

Substituting for $B[P(z)]$ in inequality (1.17),we get for $|z| \geq 1$,

$$\begin{aligned} &\left|\lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) P'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2}\right| \\ &\leq \frac{1}{2}\left[\left\{\left|\lambda_0 z^n + \lambda_1 \left(\frac{nz}{2}\right) nz^{n-1} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{n(n-1)}{2} z^{n-2}\right| + |\lambda_0|\right\} \max_{|z|=1} |P(z)|\right. \\ &\qquad \left. - \left\{\left|\lambda_0 z^n + \lambda_1 \left(\frac{nz}{2}\right) nz^{n-1} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{n(n-1)}{2} z^{n-2}\right| - |\lambda_0|\right\} \min_{|z|=1} |P(z)|\right], \end{aligned} \tag{1.18}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of $u(z)$ defined by (1.10) lie in (1.11).

Remark 1.6. Inequality (1.7) is a special case of inequality (1.18), if we choose $\lambda_0 = 0 = \lambda_2$, and inequality (1.8) immediately follows from it when $\lambda_1 = 0 = \lambda_2$.

If $P \in P_n$ is a self-inversive polynomial, that is, if $P(z) \equiv Q(z)$, where $Q(z) = z^n \overline{P(1/\bar{z})}$, then [10, 16],

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.19}$$

Lastly, we prove the following result which includes inequality (1.19) as a special case.

Theorem 1.7. *If $P \in P_n$ is a self-inversive polynomial, then for $|z| \geq 1$,*

$$|B[P(z)]| \leq \frac{1}{2} \{|B[z^n]| + |\lambda_0|\} \max_{|z|=1} |P(z)|. \tag{1.20}$$

The result is best possible and equality holds for $P(z) = z^n + 1$.

Substituting for $B[P(z)]$, we have for $|z| \geq 1$,

$$\left|\lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) P'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2}\right| \leq \frac{1}{2}\left\{\left|\lambda_0 z^n + \lambda_1 \left(\frac{nz}{2}\right) nz^{n-1} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{n(n-1)}{2} z^{n-2}\right| + |\lambda_0|\right\} \max_{|z|=1} |P(z)|, \tag{1.21}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of $u(z)$ defined by (1.10) lie in (1.11).

Remark 1.8. If we choose $\lambda_0 = 0 = \lambda_2$ in inequality (1.21), we get

$$|P'(z)| \leq \frac{n}{2}|z|^{n-1} \max_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1,$$

from which inequality (1.19) follows immediately.

Also if we take $\lambda_1 = 0 = \lambda_2$ in inequality (1.21), we obtain the following:

Corollary 1.9. *If $P \in P_n$ is a self-inversive polynomial, then*

$$|P(z)| \leq \frac{|z|^n + 1}{2} \max_{|z|=1} |P(z)|, \quad \textit{for } |z| \geq 1. \tag{1.22}$$

The result is best possible and equality holds for the polynomial $P(z) = z^n + 1$. Inequality (1.22) in particular gives

$$\max_{|z|=R>1} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|.$$

2. LEMMAS

For the proofs of these theorems we need the following lemmas. The first lemma follows from Corollary 18.3 of [8, p. 65].

Lemma 2.1. *If all the zeros of a polynomial $P(z)$ of degree n lie in a circle $|z| \leq 1$, then all the zeros of the polynomial $B[P(z)]$ also lie in the circle $|z| \leq 1$.*

The following two lemmas which we need are in fact implicit in [12, p. 305]; however, for the sake of completeness we give a brief outline of their proofs.

Lemma 2.2. *If $P \in P_n$, and $P(z) \neq 0$ in $|z| < 1$, then*

$$|B[P(z)]| \leq |B[Q(z)]| \quad \textit{for } |z| \geq 1, \tag{2.1}$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Proof of Lemma 2.2. Since $Q(z) = z^n \overline{P(1/\bar{z})}$, therefore $|Q(z)| = |P(z)|$ for $|z| = 1$ and hence $Q(z)/P(z)$ is analytic in $|z| \leq 1$. By the maximum modulus principle, $|Q(z)| \leq |P(z)|$ for $|z| \leq 1$, or equivalently, $|P(z)| \leq |Q(z)|$ for $|z| \geq 1$. Therefore, for every β with $|\beta| > 1$, the polynomial $P(z) - \beta Q(z)$ has all its zeros in $|z| \leq 1$. By Lemma 2.1, the polynomial $B[P(z) - \beta Q(z)] = B[P(z)] - \beta B[Q(z)]$ has all its zeros in $|z| \leq 1$, which in particular gives

$$|B[P(z)]| \leq |B[Q(z)]|, \quad \text{for } |z| \geq 1.$$

This proves Lemma 2.2. □

Lemma 2.3. *If $P \in P_n$, then for $|z| \geq 1$,*

$$|B[P(z)]| + |B[Q(z)]| \leq \{|B[z^n]| + |\lambda_0|\} \max_{|z|=1} |P(z)|, \tag{2.2}$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Proof of Lemma 2.3. Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M$ for $|z| \leq 1$. If λ is any real or complex number with $|\lambda| > 1$, then by Rouche's theorem, $P(z) - \lambda M$ does not vanish in $|z| \leq 1$. By Lemma 2.2, it follows that

$$|B[P(z) - M\lambda]| \leq |B[Q(z) - M\lambda z^n]|, \quad \text{for } |z| \geq 1. \tag{2.3}$$

Using the fact that B is linear and $B[1] = \lambda_0$, we have from (2.3)

$$|B[P(z) - M\lambda\lambda_0]| \leq |B[Q(z)] - M\lambda B[z^n]|, \quad \text{for } |z| \geq 1. \tag{2.4}$$

Choosing the argument of λ, which is possible by (1.14) such that

$$|B[Q(z)] - M\lambda B[z^n]| = M|\lambda|\,|B[z^n]| - |B[Q(z)]|,$$

we get from (2.4)

$$|B[P(z)]| - M|\lambda||\lambda_0| \leq M|\lambda||B[z^n]| - |B[Q(z)]| \quad \text{for } |z| \geq 1. \tag{2.5}$$

Making $|\lambda| \to 1$ in (2.5) we get

$$|B[P(z)]| + |B[Q(z)]| \leq \{|B[z^n]| + |\lambda_0|\}\, M$$

which is (2.2) and Lemma 2.3 is completely proved. □

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. If $P(z)$ has a zero on $|z| = 1$, then $m = \min_{|z|=1} |P(z)| = 0$ and there is nothing to prove. Suppose that all the zeros of $P(z)$ lie in $|z| < 1$, then $m > 0$, and we have $m \leq |P(z)|$ for $|z| = 1$. Therefore, for every real or complex number λ with $|\lambda| < 1$, we have $|m\lambda z^n| < |P(z)|$, for $|z| = 1$. By Rouche's theorem, it follows that all the zeros of $P(z) - m\lambda z^n$ lie in $|z| < 1$. Therefore, by Lemma 2.1, all the zeros of $B[P(z) - m\lambda z^n]$ lie in $|z| < 1$. Since B is linear, it follows that all the zeros of $B[P(z)] - m\lambda B[z^n]$ lie in $|z| < 1$, which gives

$$m|B[z^n]| \leq |B[P(z)]|, \quad \text{for } |z| \geq 1. \tag{3.1}$$

Because, if this is not true, then there is a point $z = z_0$, with $|z_0| \geq 1$, such that

$$(m|B[z^n]|)_{z=z_0} > (|B[P(z)]|)_{z=z_0}.$$

We take $\lambda = (B[P(z)])_{z=z_0} / (mB[z^n])_{z=z_0}$, so that $|\lambda| < 1$ and for this value of λ, $B[P(z)] - m\lambda B[z^n] = 0$ for $|z| \geq 1$, which contradicts the fact that all the zeros of $B[P(z)] - m\lambda B[z^n]$ lie in $|z| < 1$. Hence from (3.1) we conclude that

$$|B[P(z)]| \geq |B[z^n]| \min_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1,$$

which completes the proof of Theorem 1.1. □

Proof of Theorem 1.3. Combining Lemma 2.2 and Lemma 2.3 we have

$$\begin{aligned} 2|B[P(z)]| &\leq |B[P(z)]| + |B[Q(z)]| \\ &\leq \{|B[z^n]| + |\lambda_0|\} \max_{|z|=1} |P(z)|, \end{aligned}$$

which gives inequality (1.15) and Theorem 1.3 is completely proved. □

Proof of Theorem 1.5. If $P(z)$ has a zero on $|z| = 1$ then $m = \min_{|z|=1} |P(z)| = 0$ and the result follows from Theorem 1.3. We supppose that all the zeros of $P(z)$ lie in $|z| > 1$, so that $m > 0$ and

$$m \leq |P(z)|, \quad \text{for } |z| = 1. \tag{3.2}$$

Therefore, for every complex number β with $|\beta| < 1$, it follows by Rouche's theorem that all the zeros of $F(z) = P(z) - m\beta$ lie in $|z| > 1$. We note that $F(z)$ has no zeros on $|z| = 1$, because if for some $z = z_0$ with $|z_0| = 1$,

$$F(z_0) = P(z_0) - m\beta = 0,$$

then

$$|P(z_0)| = m|\beta| < m$$

which is a contradiction to (3.2). Now, if

$$G(z) = z^n\overline{F(1/\bar{z})} = z^n\overline{P(1/\bar{z})} - \bar{\beta}mz^n = Q(z) - \bar{\beta}mz^n,$$

then all the zeros of $G(z)$ lie in $|z| < 1$ and $|G(z)| = |F(z)|$ for $|z| = 1$. Therefore, for every γ with $|\gamma| > 1$, the polynomial $F(z) - \gamma G(z)$ has all its zeros in $|z| < 1$. By Lemma 2.1 all zeros of

$$B[F(z) - \gamma G(z)] = B[F(z)] - \gamma B[G(z)]$$

lie in $|z| < 1$, which implies

$$B[F(z)] \leq B[G(z)], \quad \text{for } |z| \geq 1. \tag{3.3}$$

Substituting for $F(z)$ and $G(z)$, making use of the facts that B is linear and $B[1] = \lambda_0$, we obtain from (3.3)

$$|B[P(z)] - \beta m\lambda_0| \leq |B[Q(z)] - \bar{\beta}mB[z^n]|, \quad \text{for } |z| \geq 1. \tag{3.4}$$

Choosing the argument of β on the right hand side of (3.4) suitably, which is possible by (3.1), and making $|\beta| \to 1$, we get

$$|B[P(z)]| - m|\lambda_0| \leq |B[Q(z)]| - m|B[z^n]|, \quad \text{for } |z| \geq 1.$$

This gives

$$|B[P(z)]| \leq |B[Q(z)]| - \{|B[z^n]| - \lambda_0\}\, m, \quad \text{for } |z| \geq 1. \tag{3.5}$$

Inequality (3.5) with the help of Lemma 2.3, yields

$$\begin{aligned} 2|B[P(z)]| &\leq |B[P(z)]| + |B[Q(z)]| - \{|B[z^n]| - \lambda_0\}\, m \\ &\leq \{|B[z^n]| + \lambda_0\} \max_{|z|=1} |P(z)| - \{|B[z^n]| - \lambda_0\} \min_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1, \end{aligned}$$

which is equivalent to (1.17) and this proves Theorem 1.5 completely. □

Proof of Theorem 1.7. Since $P(z)$ is a self-inversive polynomial, we have

$$P(z) \equiv Q(z) = z^n\overline{P(1/\bar{z})}.$$

Equivalently

$$B[P(z)] = B[Q(z)]. \tag{3.6}$$

Lemma 2.3 in conjunction with (3.6) gives

$$2|B[P(z)]| \leq \{|B[z^n]| + \lambda_0\} \max_{|z|=1} |P(z)|,$$

which is (1.20) and this completes the proof of Theorem 1.7. □

REFERENCES

[1] N.C. ANKENY AND T.J. RIVLIN, On a theorem of S. Bernstein, *Pacific J. Math.*, **5** (1955), 849–852.

[2] A. AZIZ, Inequalities for the polar derivatives of a polynomial, *J. Approx. Theory*, **55** (1988), 183–193.

[3] A. AZIZ AND Q. M. DAWOOD, Inequalities for a polynomial and its derivative, *J. Approx. Theory*, **53** (1988), 155–162.

[4] A. AZIZ AND Q.G. MOHAMMAND, Simple proof of a theorem of Erdös and Lax, *Proc. Amer. Math. Soc.*, **80** (1980), 119–122.

[5] A. AZIZ AND W.M. SHAH, Some inequalities for the polar derivative of a polynomial, *Indian Acad. Sci.* (*Math. Sci*.), **107** (1997), 263–170.

[6] S. BERNSTEIN, Sur la limitation des derivees des polnomes, *C. R. Acad. Sci. Paris*, **190** (1930), 338–341.

[7] P.D. LAX, Proof of a Conjecture of P. Erdös on the derivative of a polynomial, *Bull. Amer. Math. Soc.* (N. S.), **50** (1944), 509–513.

[8] M. MARDEN, *Geometry of Polynomials*, Math. Surveys, No.3, Amer. Math. Soc. Providence, RI, 1949.

[9] G.V. MILOVANOVIĆ, D.S. MITRINOVIĆ AND Th. M. RASSIAS, *Topics in Polynomials, Extremal Problems, Inequalities, Zeros* (Singapore: World Scientific), (1994).

[10] P.J. O'HARA AND R.S. RODRIGUEZ, Some properties of self-inversive polynomials, *Proc. Amer. Math. Soc.*, **44** (1974), 331–335.

[11] G. PÓLYA AND G. SZEGÖ, *Problems and Theorems in Analysis* (New York: Springer-Verlag) (1972), Vol. 1.

[12] Q.I. RAHMAN, Functions of Exponential type, *Trans. Amer. Math. Soc.*, **135** (1969), 295–309.

[13] Q.I. RAHMAN AND G. SCHMEISSER, Les inegalities de Markoff et de Bernstein, *Seminaire de mathematiques Superieures*, No. 86 (Ete 1981) (Les presses de Í Université de Montréal, Qué) (1983).

[14] Q.I. RAHMAN AND G. SCHMEISSER, *Analytic Theory of Polynomials*, Oxford University Press, Oxford, 2002.

[15] M. RIESZ, Über einen Satz des Herrn Serge Bernstein, *Acta Math.*, **40** (1916), 337–347.

[16] E.B. SAFF AND T. SHEIL-SMALL, Coefficient and integral mean estimates for algebraic and trigonometric polynomials with restricted zeros, *J. London Math. Soc.* (2), **9** (1975), 16–22.

[17] W.M. SHAH, A generalization of a theorem of Paul Turán, *J. Ramanujan Math. Soc.*, **11** (1996), 67–72.