THE BEST CONSTANTS FOR A DOUBLE INEQUALITY IN A TRIANGLE

YU-DONG WU
Department of Mathematics
Zhejiang Xinchang High School
Shaoxing 312500, Zhejiang
People's Republic of China.
EMail: yudong.wu@yahoo.com.cn

NU-CHUN HU
Department of Mathematics
Zhejiang Normal University
Jinhua 321004, Zhejiang
People's Republic of China.
EMail: nuchun@zjnu.cn

WEI-PING KUANG
Department of Mathematics
Huaihua University
Huaihua 418008, Hunan
People's Republic of China.
EMail: sy785153@126.com

Received: 08 August, 2008

Accepted: 02 February, 2009

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: 51M16

Key words: Inequality; Best Constant; Triangle.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Abstract: In this short note, by using some of Chen's theorems and classic analysis, we obtain a double inequality for triangle and give a positive answer to a problem posed by Yang and Yin [6].

Acknowledgment: The authors would like to thank Prof. Zhi-Hua Zhang and Dr. Zhi-Gang Wang for their careful reading and making some valuable suggestions in the preparation of this paper.

Dedicatory: Dedicated to Professor Bi-Cheng Yang on the occasion of his 63rd birthday.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

Double Inequality in a Triangle
Yu-Dong Wu, Nu-Chun Hu and Wei-Ping Kuang
vol. 10, iss. 1, art. 29, 2009

journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Introduction and Main Results

For $\triangle ABC$, let a, b, c denote the side-lengths, A, B, C the angles, s the semi-perimeter, R the circumradius and r the inradius, respectively.

In 1957, Kooistra (see [1]) built the following double inequality for any triangle:

$$2 < \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2}. \tag{1.1}$$

In 2000, Yang and Yin [6] considered a new bound of inequality (1.1) and posed a problem as follows:

Problem 1. Determine the best constant μ such that

$$\left(\frac{3\sqrt{3}}{2}\right)^{\mu} \cdot \left(\frac{s}{R}\right)^{1-\mu} \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \tag{1.2}$$

holds for any $\triangle ABC$.

In this short note, we solve the above problem and obtain the following result.

Theorem 1.1. *Let*

$$\lambda \geq \lambda_0 = 1$$

and

$$\mu \leq \mu_0 = \frac{2\ln(2-\sqrt{2}) + \ln 2}{4\ln 2 - 3\ln 3} \approx 0.7194536993.$$

Then the double inequality

$$\left(\frac{3\sqrt{3}}{2}\right)^{\mu} \cdot \left(\frac{s}{R}\right)^{1-\mu} \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \left(\frac{3\sqrt{3}}{2}\right)^{\lambda} \cdot \left(\frac{s}{R}\right)^{1-\lambda} \tag{1.3}$$

Double Inequality in a Triangle
Yu-Dong Wu, Nu-Chun Hu and Wei-Ping Kuang
vol. 10, iss. 1, art. 29, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

holds for any $\triangle ABC$, while the constants λ_0 and μ_0 are both the best constant for inequality (1.3).

Remark 1. When $\lambda_0 = 1$, the right hand of inequality (1.3) is just the right hand of inequality (1.1).

Remark 2. It is not difficult to demonstrate that:

$$\begin{cases} \left(\frac{3\sqrt{3}}{2}\right)^{\mu_0} \cdot \left(\frac{s}{R}\right)^{1-\mu_0} < 2 \left(0 < \frac{s}{R} < 2^{\frac{1}{1-\mu_0}} \left(\frac{2}{3\sqrt{3}}\right)^{\frac{\mu_0}{1-\mu_0}}\right), \\ \left(\frac{3\sqrt{3}}{2}\right)^{\mu_0} \cdot \left(\frac{s}{R}\right)^{1-\mu_0} \geq 2 \left(2^{\frac{1}{1-\mu_0}} \left(\frac{2}{3\sqrt{3}}\right)^{\frac{\mu_0}{1-\mu_0}} \leq \frac{s}{R} \leq \frac{3\sqrt{3}}{2}\right). \end{cases}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Preliminary Results

In order to establish our main theorem, we shall require the following lemmas.

Lemma 2.1 (see [3, 4, 5]). *If the inequality $s \geq (>) f(R,r)$ holds for any isosceles triangle whose top-angle is greater than or equal to $\frac{\pi}{3}$, then the inequality $s \geq (>) f(R,r)$ holds for any triangle.*

Lemma 2.2 (see [2, 3]). *The homogeneous inequality*

$$s \geq (>) f(R,r) \tag{2.1}$$

holds for any acute-angled triangle if and only if it holds for any acute isosceles triangle whose top-angle $A \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$ with $2r \leq R < (\sqrt{2}+1)r$ and any right-angled triangle with $R \geq (\sqrt{2}+1)r$.

For the convenience of our readers, we give below the proof by Chen in [2, 3].

Proof. Let $\odot O$ denote the circumcircle of $\triangle ABC$. Necessity is obvious from Lemma 2.1. Thus we only need to prove the sufficiency. It is well known that $R \geq 2r$ for any acute-angled triangle. So we consider the following two cases:

(i) When $2r \leq R < (\sqrt{2}+1)r$: In this case , we can construct an isosceles triangle $A_1B_1C_1$ whose circumcircle is also $\odot O$ and the top-angle of $\triangle A_1B_1C_1$(see Figure 1) is

$$A_1 = 2\arcsin\frac{1}{2}\left(1+\sqrt{1-\frac{2r}{R}}\right).$$

It is easy to see that (see [4, 5]):

$$R_1 = R, \quad r_1 = r, \quad s_1 \leq s \quad \text{and} \quad \frac{\pi}{3} \leq A_1 < \frac{\pi}{2}.$$

vol. 10, iss. 1, art. 29, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

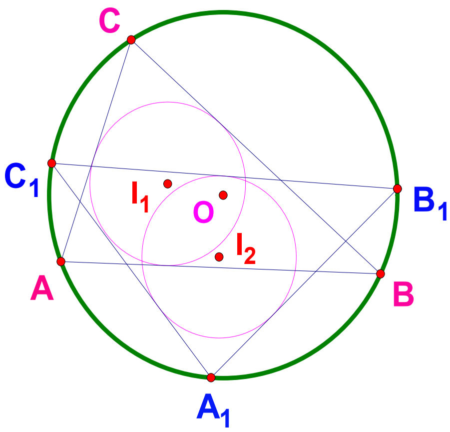

Figure 1:

Thus we have

$$s \geq s_1 \geq f(R_1, r_1) = f(R, r). \tag{2.2}$$

because the inequality (2.1) holds for any acute isosceles triangle whose top-angle $A \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$.

(ii) When $R \geq (\sqrt{2}+1)r$: In this case we can construct a right-angled triangle $A_2B_2C_2$ whose inscribed circle is also $\odot I$ and the length of its hypotenuse is $c_2 = 2R$ (see Figure 2). This implies that

Double Inequality in a Triangle
Yu-Dong Wu, Nu-Chun Hu and Wei-Ping Kuang
vol. 10, iss. 1, art. 29, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

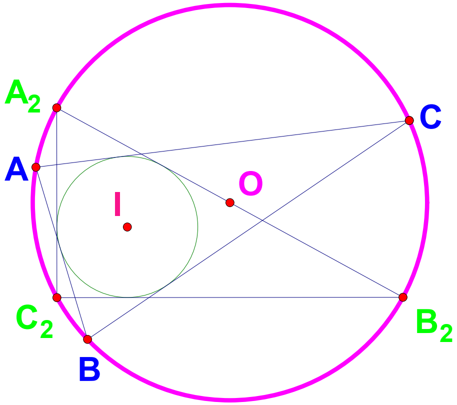

Figure 2:

$$r_2 = \frac{1}{2}(a_2 + b_2 - c_2) = r, \quad R_2 = \frac{1}{2}c_2 = R,$$
$$s_2 = \frac{1}{2}(a_2 + b_2 + c_2) = 2R_2 + r_2 = 2R + r < s.$$

Thus we have the inequality (2.2) since the inequality (2.1) holds for any right-angled triangle.

□

Lemma 2.3 (see [2, 3]). *The homogeneous inequality* (2.1) *holds for any acute-angled triangle if and only if*

$$\sqrt{(1-x)(3+x)^3} \geq (>) f(2, 1-x^2) \qquad \left(0 \leq x < \sqrt{2} - 1\right), \tag{2.3}$$

Double Inequality in a Triangle
Yu-Dong Wu, Nu-Chun Hu and Wei-Ping Kuang
vol. 10, iss. 1, art. 29, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

and

$$5 - x^2 \geq (>) f(2, 1 - x^2) \qquad \left(\sqrt{2} - 1 \leq x < 1\right). \tag{2.4}$$

Proof. Since the inequality (2.1) is homogeneous, we may assume $R = 2$ without losing generality.

(i) When $2r \leq R < (\sqrt{2}+1)r$: By Lemma 2.2, we only need to consider the isosceles triangle whose top-angle $A \in [\frac{\pi}{3}, \frac{\pi}{2})$. Let

$$t = \sin\frac{A}{2} \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right).$$

Then we have (see [4, 5])

$$r = 4t(1-t) \quad \text{and} \quad s = 4(1+t)\sqrt{1-t^2}. \tag{2.5}$$

Let $x = 2t - 1$. Then the inequality (2.1) is just the inequality (2.3).

(ii) When $R \geq (\sqrt{2}+1)r$: We only need to consider a right-angled triangle. Let

$$r = \frac{2r}{R} = 4t(1-t) \in \left(0, \sqrt{2}-1\right) \qquad \left(\frac{\sqrt{2}}{2} \leq t < 1\right).$$

Thus we have

$$s = 2R + r = 4 + 4t(1-t). \tag{2.6}$$

Let $x = 2t - 1$. Then the inequality (2.1) is just the inequality (2.4).

This completes the proof Lemma 2.3. $\square$

vol. 10, iss. 1, art. 29, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Lemma 2.4 ([3, 4, 5]). *The homogeneous inequality*

$$s \leq (<) f(R, r) \tag{2.7}$$

holds for any triangle if and only if it holds for any isosceles triangle whose top-angle $A \in (0, \frac{\pi}{3}]$, or the following inequality holds

$$\sqrt{(1-x)(3+x)^3} \leq (<) f\left(2, 1-x^2\right) \qquad (-1 < x \leq 0)\,. \tag{2.8}$$

Lemma 2.5 (see [2, 3]). *The homogeneous inequality (2.7) holds for any acute-angled triangle if and only if it holds for any isosceles triangle whose top-angle $A \in (0, \frac{\pi}{3}]$, or the inequality (2.8) holds.*

Proof. As acute-angled triangles include all isosceles triangles whose top-angle is less than or equal to $\frac{\pi}{3}$, Lemma 2.5 straightforwardly follows from Lemma 2.4 and Lemma 2.1. □

Lemma 2.6. *Define*

$$G_1(x) := \frac{2\ln(1-x) + 2\ln(1+x)}{3\ln(1-x) + 3\ln(3+x) + 2\ln(1+x) - 3\ln 3}. \tag{2.9}$$

Then G_1 is decreasing on $(-1, \sqrt{2}-1)$, and

$$\lim_{x\to(\sqrt{2}-1)^-} G_1(x) = \frac{2\ln(2-\sqrt{2}) + \ln 2}{4\ln 2 - 3\ln 3} < G_1(x) < 1 = \lim_{x\to-1^+} G_1(x). \tag{2.10}$$

Proof. Let G_1' be the derivative of G_1. It is easy to see that

$$G_1'(x) = \frac{4x g_1(x)}{(3\ln(1-x)+3\ln(3+x)+2\ln(1+x)-3\ln 3)^2(1-x^2)(3+x)} \tag{2.11}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

with

$$g_1(x) := (x-1)\ln(1-x) - 3(x+3)[\ln(3+x) - \ln 3] + 2(x+1)\ln(1+x). \tag{2.12}$$

Moreover, we know that

$$g_1'(x) = \ln(1-x) - 3\ln(3+x) + 2\ln(1+x) + 3\ln 3 \tag{2.13}$$

and

$$g_1''(x) = \frac{-8x}{(1-x^2)(3+x)}. \tag{2.14}$$

Now we show that G_1 is decreasing on $\left(-1, \sqrt{2}-1\right)$.

(i) It is easy to see that $g_1''(x) \geq 0$ when $-1 < x \leq 0$, and g_1' is increasing on $(-1, 0]$. Thus, $g_1'(x) \leq g_1'(0) = 0$, and g_1 is decreasing on $(-1, 0]$. Therefore, $g_1(x) \geq g_1(0) = 0$, and $G_1'(x) \leq 0$. This means that G_1 is decreasing on $(-1, 0]$.

(ii) It is easy to see that $g_1''(x) \leq 0$ when $0 \leq x < \sqrt{2}-1$, and g_1' is decreasing on $\left[0, \sqrt{2}-1\right)$. Thus, $g_1'(x) \leq g_1'(0) = 0$, and g_1 is decreasing on $\left[0, \sqrt{2}-1\right)$. Therefore, $g_1(x) \leq g_1(0) = 0$, and $G_1'(x) \leq 0$. This means that G_1 is decreasing on $\left[0, \sqrt{2}-1\right)$.

Combining (i) and (ii), it follows that G_1 is decreasing on $(-1, \sqrt{2}-1)$ and (2.10) holds, and hence the proof is complete. □

Lemma 2.7. *Define*

$$G_2(x) := \frac{2\ln(1-x^2)}{2\ln(5-x^2) + 2\ln(1-x^2) - 3\ln 3}. \tag{2.15}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Then G_2 is increasing on $\left[\sqrt{2}-1,1\right)$, and

$$(2.16) \qquad G_2(x) \ge G_2(\sqrt{2}-1) = \frac{2\ln\left(2-\sqrt{2}\right)+\ln 2}{4\ln 2 - 3\ln 3}.$$

Proof. Let G_2' be the derivative of G_2. It is easy to see that

$$(2.17) \qquad G_2'(x) = \frac{4xg_2(x)}{(2\ln\left(5-x^2\right)+2\ln\left(1-x^2\right)-3\ln 3)^2(1-x^2)(5-x^2)},$$

$$g_2'(x) = -2xh_2(x),$$

and

$$(2.18) \qquad h_2'(x) = \frac{-16x}{\left(1-x^2\right)\left(5-x^2\right)};$$

where

$$g_2(x) := 2\left(1-x^2\right)\left[\ln\left(1-x\right)+\ln\left(1+x\right)\right]+2\left(x^2-5\right)\ln\left(5-x^2\right)+3\left(5-x^2\right)\ln 3,$$

and

$$(2.19) \qquad h_2(x) = 2\ln\left(1-x^2\right)-2\ln\left(5-x^2\right)+3\ln 3.$$

Thus it follows that $h_2'(x) < 0$ when $\sqrt{2}-1 \le x < 1$, and h_2 is decreasing on $\left[\sqrt{2}-1,1\right)$, and

$$h_2(x) \le h_2(\sqrt{2}-1) = \ln\frac{27}{34+24\sqrt{2}} < 0.$$

Therefore $g_2'(x) > 0$, and g_2 is increasing on $\left[\sqrt{2}-1,1\right)$, and

$$g_2(x) \ge g_2(\sqrt{2}-1) = 6(\sqrt{2}+1)\ln 3 - 8\ln 2 - 8\sqrt{2}\ln\left(\sqrt{2}+1\right) > 0.$$

This means that $G_2'(x) > 0$, and G_2 is increasing on $\left[\sqrt{2}-1,1\right)$, and (2.16) holds. The proof of Lemma 2.7 is thus completed. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. The Proof of Theorem 1.1

Proof. (i) The first inequality of (1.3) is equivalent to

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\geq\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\cdot(\sin A+\sin B+\sin C)^{1-\mu} \tag{3.1}$$

with application to the well known identity

$$\sin A+\sin B+\sin C=\frac{s}{R}.$$

Taking

$$A\to\pi-2A,\quad B\to\pi-2B\quad\text{and}\quad C\to\pi-2C,$$

then inequality (3.1) is equivalent to

$$\sin A+\sin B+\sin C\geq\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\cdot(\sin 2A+\sin 2B+\sin 2C)^{1-\mu} \tag{3.2}$$

for an acute-angled triangle ABC.

By the well known identities

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C,$$

and

$$\sin A\sin B\sin C=\frac{rs}{2R^2},$$

the inequality (3.2) can be written as follows:

$$\frac{s}{R}\geq\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\cdot\left(\frac{2rs}{R^2}\right)^{1-\mu}\Longleftrightarrow\left(\frac{s}{R}\right)^{\mu}\geq\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\cdot\left(\frac{2r}{R}\right)^{1-\mu}. \tag{3.3}$$

vol. 10, iss. 1, art. 29, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Furthermore, by Lemma 2.3, the inequality (3.3) holds if and only if the following two inequalities

$$\left(\frac{\sqrt{(1-x)(3+x)^3}}{2}\right)^{\mu} \geq \left(\frac{3\sqrt{3}}{2}\right)^{\mu} \left(1-x^2\right)^{1-\mu} \qquad \left(0 \leq x < \sqrt{2}-1\right) \tag{3.4}$$

and

$$\left(\frac{5-x^2}{2}\right)^{\mu} \geq \left(\frac{3\sqrt{3}}{2}\right)^{\mu} \left(1-x^2\right)^{1-\mu} \qquad \left(\sqrt{2}-1 \leq x < 1\right) \tag{3.5}$$

hold. In other words,

$$\mu \leq \min_{0 \leq x < 1} G(x) \tag{3.6}$$

where

$$G(x) = \begin{cases} G_1(x) & \left(0 \leq x < \sqrt{2}-1\right), \\ G_2(x) & \left(\sqrt{2}-1 \leq x < 1\right), \end{cases} \tag{3.7}$$

while $G_1(x)$ and $G_2(x)$ are defined by (2.9) and (2.15) respectively.

By Lemma 2.6 and Lemma 2.7, it follows that

$$\min_{0 \leq x < 1} G(x) = G\left(\sqrt{2}-1\right).$$

Thus the first inequality of (1.3) holds, and the best constant μ for inequality (1.3) is

$$\mu_0 = \frac{2\ln\left(2-\sqrt{2}\right)+\ln 2}{4\ln 2 - 3\ln 3}.$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

(ii) By applying a similar method to (i), it follows that the second inequality of (1.3) is equivalent to

$$\left(\frac{s}{R}\right)^{\lambda} \leq \left(\frac{3\sqrt{3}}{2}\right)^{\lambda} \cdot \left(\frac{2r}{R}\right)^{1-\lambda}. \tag{3.8}$$

By Lemma 2.5, the inequality (3.8) holds if and only if the following inequality holds:

$$\left(\frac{\sqrt{(1-x)(3+x)^3}}{2}\right)^{\lambda} \leq \left(\frac{3\sqrt{3}}{2}\right)^{\lambda} (1-x^2)^{1-\lambda} \qquad (-1 < x \leq 0), \tag{3.9}$$

or equivalently,

$$\lambda \geq \sup_{-1<x\leq 0} G_1(x), \tag{3.10}$$

where $G_1(x)$ is given by (2.9).

By Lemma 2.6, it follows that $\lambda \geq 1$. Moreover, the second inequality of (1.3) holds when $\lambda_0 = 1$. Thus the second inequality of (1.3) holds and the best constant λ for inequality (1.3) is $\lambda_0 = 1$. The proof of Theorem 1.1 is hence completed. □

J I M P A

Double Inequality in a Triangle
Yu-Dong Wu, Nu-Chun Hu and Wei-Ping Kuang
vol. 10, iss. 1, art. 29, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] O. BOTTEMA, R.Ž. DJORDJEVIĆ, R.R. JANIĆ, D.S. MITRINOVIĆ, AND P.M. VASIĆ. *Geometric Inequality*. Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.

[2] S.-L. CHEN, Inequalities involving R, r, s in acute-angled triangle, *Geometric Inequalties in China*, Jiangsu Educational Press, Nanjing (1996), No. 72-81. (in Chinese)

[3] S.-L. CHEN, The simplified method to prove inequalities in triangle, *Studies of Inequalities*, Tibet People's Press, Lhasa (2000), 3–8. (in Chinese)

[4] S.-L. CHEN, A new method to prove one kind of inequalities-equate substitution method, *Fujian High-School Mathematics*, No. 20-23, 1993(3). (in Chinese)

[5] Y.-D. WU. The best constant for a geometric inequality, *J. Inequal. Pure Appl. Math.*, **6**(4) (2005), Art. 111. [ONLINE http://jipam.vu.edu.au/article.php?sid=585].

[6] X.-Z. YANG AND H.-Y. YIN, The comprehensive investigations of trigonometric inequalities for half-angles of triangle in China, *Studies of Inequalities*, Tibet People's Press, Lhasa (2000), No.123–174. (in Chinese)

vol. 10, iss. 1, art. 29, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756