ON EQUIVALENCE OF COEFFICIENT CONDITIONS. II

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1. Introduction

In the papers [2], [3] and [4] we have studied the relations of the following sums:

$$S_1 := \sum_{n=1}^{\infty} c_n^q \, \mu_n,$$

$$S_2 := \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=n}^{\infty} c_k^q \right)^{\frac{p}{q}}, \qquad S_2^* := \sum_{n=1}^{\infty} \lambda_n \left(\mu_n^{-1} \sum_{k=1}^{n} \lambda_k \right)^{\frac{p}{q-p}},$$

$$S_3 := \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n} c_k^q \right)^{\frac{p}{q}}, \qquad S_3^* := \sum_{n=1}^{\infty} \lambda_n \left(\mu_n^{-1} \sum_{k=n}^{\infty} \lambda_k \right)^{\frac{p}{q-p}},$$

$$S_4 := \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} c_k^q \right)^{\frac{p}{q}}, \qquad S_4^* := \sum_{n=1}^{\infty} \lambda_n \left(\frac{\lambda_n}{\mu_{\nu_n}} \right)^{\frac{p}{q-p}},$$

where $0 < p < q$, $\lambda := \{\lambda_n\}$ and $\mathbf{c} := \{c_n\}$ are sequences of nonnegative numbers, $\nu := \{\nu_m\}$ is a subsequence of natural numbers, and $\mu := \{\mu_n\}$ is a certain nondecreasing sequence of positive numbers.

In [2] we verified that $S_2 < \infty$ if and only if there exists a μ satisfying the conditions $S_1 < \infty$ and $S_2^* < \infty$. Similarly $S_3 < \infty$ if and only if $S_1 < \infty$ and $S_3^* < \infty$.

In [3] we showed that $S_4 < \infty$ if and only if there exists a μ such that $S_1 < \infty$ and $S_4^* < \infty$.

Recently, in [4], we proved that if

$$\mu_n := \Lambda_n^{(1)} \, C_n^{p-q}, \quad \text{where} \quad C_n := \left(\sum_{k=n}^{\infty} c_k^q \right)^{1/q} \quad \text{and} \quad \Lambda_n^{(1)} := \sum_{k=1}^{n} \lambda_k,$$

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then the sums S_1, S_2 and S_2^* are already equiconvergent.

Furthermore if

$$\mu_n := \Lambda_n^{(2)} \tilde{C}_n^{p-q}, \quad \text{where} \quad \tilde{C}_n := \left(\sum_{k=1}^{n} c_k^q \right)^{1/q} \quad \text{and} \quad \Lambda_n^{(2)} := \sum_{k=n}^{\infty} \lambda_k,$$

then the sums S_1, S_3 and S_3^* are equiconvergent.

Comparing the results proved in [4] and that of [2] and [3], we can observe that in the former one the explicit sequences $\{\mu_n\}$ are determined, herewith they state more than the outcomes of [2] and [3], where only the existence of a sequence $\{\mu_n\}$ is proved.

Furthermore, in [4] the equiconvergence of these concrete sums are guaranteed, too.

However the equiconvergence in [4] is proved only in connection with the sums S_2 and S_3, but not for S_4. This is a gap or shortcoming at these investigations.

The aim of this note is closing this gap. Unfortunately we cannot give a complete solution, namely our result to be verified requires an additional assumption on the sequence λ. In particular, λ should be quasi geometrically increasing, that is, we assume that there exist a natural number N and $K \geq 1$ such that $\lambda_{n+N} \geq 2\lambda_n$ and $\lambda_n \leq K\lambda_{n+1}$ hold for all n.

Then we can give an explicit sequence μ such that the sums S_1, S_4 and S_4^* are already equiconvergent. We also show that without some additional requirement on λ the equiconvergence does not hold. See the last part. Thus the following open problem can be raised: *What is the weakest additional assumption on sequence λ which ensures the equiconvergence of these sums*?

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2. Result

Theorem 2.1. *If* $0 < p < q$, $\mathbf{c} := \{c_n\}$ *is a sequence of nonnegative numbers,* $\nu := \{\nu_m\}$ *is a subsequence of natural numbers, and* $\lambda := \{\lambda_n\}$ *is a quasi geometrically increasing sequence, and for* $\nu_m \le n < \nu_{m+1}$

$$\mu_n := \lambda_m \left(\sum_{k=\nu_m}^{\infty} c_k^q \right)^{\frac{p}{q}-1}, \quad m = 0, 1, \dots,$$

then the sums S_1, S_4 *and* S_4^* *are equiconvergent.*

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3. Lemma

In order to verify our theorem, first we shall prove a lemma regarding the equiconvergence of two special series.

Lemma 3.1. *Let* $0 < \alpha < 1$, $\mathbf{a} := \{a_n\}$ *be a sequence of nonnegative numbers,* $\nu := \{\nu_m\}$ *be a subsequence of natural numbers, and* $\kappa := \{\kappa_m\}$ *be a quasi geometrically increasing sequence. Furthermore let* $A_k := \sum_{n=k}^{\infty} a_n$, *and for* $\nu_m \leq n < \nu_{m+1}$ *let*

$$\mu_n := \kappa_m \, A_{\nu_m}^{\alpha-1}, \quad m = 0, 1, \dots .$$

Then

$$\sigma_1 := \sum_{n=1}^{\infty} a_n \, \mu_n < \infty \tag{3.1}$$

holds if and only if

$$\sigma_2 := \sum_{m=1}^{\infty} \kappa_m \, A_{\nu_m}^{\alpha} < \infty. \tag{3.2}$$

Proof of Lemma 3.1. Before starting the proofs we note that the following inequality

$$\sum_{n=1}^{m} \kappa_n \leq K \, \kappa_m \tag{3.3}$$

holds for all m, subsequent to the fact that κ is a quasi geometrically increasing sequence (see e.g. [1, Lemma 1]). Here and later on K denotes a constant that is independent of the parameters.

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Furthermore we verify a useful inequality. If $0 \le a < b,\ 0 < \alpha < 1$ and

$$\frac{b^\alpha - a^\alpha}{b-a} = \alpha\, \xi^{\alpha-1}, \tag{3.4}$$

then

$$\xi \ge \alpha^{1/(1-\alpha)} b =: \xi_0,$$

namely if $a = 0$ then $\xi = \xi_0$. Hence we get that

$$\alpha\, \xi^{\alpha-1} \le b^{\alpha-1}. \tag{3.5}$$

Now we show that (3.1) implies (3.2). Since $A_n \searrow 0$, thus, by (3.3),

$$\begin{aligned} \sum_{m=1}^\infty \kappa_m A_{\nu_m}^\alpha &= \sum_{m=1}^\infty \kappa_m \sum_{n=m}^\infty (A_{\nu_n}^\alpha - A_{\nu_{n+1}}^\alpha) \\ &= \sum_{n=1}^\infty (A_{\nu_n}^\alpha - A_{\nu_{n+1}}^\alpha) \sum_{m=1}^n \kappa_m \\ &\le K \sum_{n=1}^\infty \kappa_n (A_{\nu_n}^\alpha - A_{\nu_{n+1}}^\alpha). \end{aligned} \tag{3.6}$$

Using the relations (3.4) and (3.5) we obtain that

$$A_{\nu_n}^\alpha - A_{\nu_{n+1}}^\alpha = \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} a_k \right) \alpha\, \xi^{\alpha-1} \le \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} a_k \right) A_{\nu_n}^{\alpha-1}.$$

This and (3.6) yield that

$$\sum_{m=1}^\infty \kappa_m A_{\nu_m}^\alpha \le K \sum_{n=1}^\infty \kappa_n A_{\nu_n}^{\alpha-1} \sum_{k=\nu_n}^{\nu_{n+1}-1} a_k = K \sum_{n=1}^\infty \sum_{k=\nu_n}^{\nu_{n+1}-1} a_k\, \mu_k.$$

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Herewith the implication (3.1) $\Rightarrow$ (3.2) is proved.

The proof of (3.2) $\Rightarrow$ (3.1) is very easy. Namely

$$\begin{aligned}\sum_{n=\nu_1}^{\infty} a_n\,\mu_n &= \sum_{m=1}^{\infty}\sum_{n=\nu_m}^{\nu_{m+1}-1} a_n\,\mu_n \\ &= \sum_{m=1}^{\infty}\kappa_m\,A_{\nu_m}^{\alpha-1}\sum_{n=\nu_m}^{\nu_{m+1}-1} a_n \\ &\le \sum_{m=1}^{\infty}\kappa_m\,A_{\nu_m}^{\alpha},\end{aligned}$$

that is, (3.2) $\Rightarrow$ (3.1) is verified.

Thus the proof is complete. $\square$

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4. Proof of Theorem 2.1

We shall use the result of Lemma 3.1 with $\alpha = \frac{p}{q}$, $a_n = c_n^q$ and $\kappa_m = \lambda_m$. Then $A_n = \sum_{k=n}^{\infty} c_k^q$ and for $\nu_m \leq n < \nu_{m+1}$

$$\mu_n = \mu_{\nu_m} = \lambda_m \left(\sum_{k=\nu_m}^{\infty} c_k^q \right)^{\frac{p-q}{q}} . \tag{4.1}$$

Then $\sigma_1 = S_1$, thus by Lemma 3.1, $S_1 < \infty$ implies that $\sigma_2 < \infty$, that is,

$$S_4 = \sum_{m=1}^{\infty} \lambda_m \left(\sum_{n=\nu_m}^{\nu_{m+1}-1} c_n^q \right)^{\frac{p}{q}} \leq \sum_{m=1}^{\infty} \lambda_m \left(\sum_{n=\nu_m}^{\infty} c_n^q \right)^{\frac{p}{q}} = \sigma_2 . \tag{4.2}$$

Moreover, by (4.1),

$$S_4^* = \sum_{n=1}^{\infty} \lambda_n \left\{ \left(\sum_{k=\nu_n}^{\infty} c_k^q \right)^{\frac{q-p}{q}} \right\}^{\frac{p}{q-p}} = \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=\nu_n}^{\infty} c_k^q \right)^{\frac{p}{q}} = \sigma_2,$$

thus $S_1 < \infty$ implies that both $S_4 < \infty$ and $S_4^* < \infty$ hold.

Conversely, if $S_4 < \infty$, then it suffices to show that $\sigma_2 = S_4^* < \infty$ also holds.

Applying the inequality

$$\left(\sum a_k \right)^{\alpha} \leq \sum a_k^{\alpha}, \quad 0 < \alpha \leq 1, \; a_k \geq 0,$$

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and (3.3), we obtain that

$$
\begin{aligned}
\sigma_2 = \sum_{m=1}^{\infty} \lambda_m\, A_{\nu_m}^{p/q} &\le \sum_{m=1}^{\infty} \lambda_m \sum_{n=m}^{\infty} \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} c_k^q \right)^{\frac{p}{q}} \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} c_k^q \right)^{\frac{p}{q}} \sum_{m=1}^{n} \lambda_m \\
&\le K \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=\nu_n}^{\nu_{n+1}-1} c_k^q \right)^{\frac{p}{q}} \\
&= K S_4 < \infty.
\end{aligned}
$$

This, (4.2) and, by Lemma 3.1, the implication $\sigma_2 < \infty \Rightarrow \sigma_1 = S_1 < \infty$ complete the proof of Theorem 2.1.

Proof of the necessity of some additional assumption on λ. Let $p = 1,\ q = 2,\ \lambda_n = \log n,\ \nu_n = n$ and

$$
c_n := \begin{cases} m^{-3} & \text{if} \quad n = 2^m, \\ 0 & \text{otherwise.} \end{cases}
$$

Then

$$
S_4 = \sum_{m=2}^{\infty} \frac{\log 2^m}{m^3} < \infty,
$$

but $S_1 < \infty$ and $S_4^* < \infty$ cannot be fulfilled simultaneously. Namely, then with a nondecreasing sequence $\{\mu_n\}$ the conditions

$$
S_1 = \sum_{m=1}^{\infty} m^{-6}\, \mu_{2^m} < \infty
$$

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and

$$S_4^* = \sum_{m=2}^{\infty} \frac{\log^2 m}{\mu_m} < \infty$$

yield a trivial contradiction. □

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