

SOME NEW HILBERT-PACHPATTE'S INEQUALITIES

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Abstract: Some new Hilbert-Pachpatte discrete inequalities and their integral analogues are established in this paper. Other inequalities are also given in remarks.



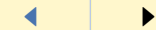
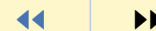
Hilbert-Pachpatte's Inequalities

Wengui Yang

vol. 10, iss. 1, art. 26, 2009

[Title Page](#)

[Contents](#)



Page 1 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

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Contents

| | | |
|---|------------------------------------|----|
| 1 | Introduction | 3 |
| 2 | Main Results | 6 |
| 3 | Integral Analogues | 17 |



Hilbert-Pachpatte's Inequalities

Wengui Yang

vol. 10, iss. 1, art. 26, 2009

[Title Page](#)

[Contents](#)



Page 2 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
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mathematics

issn: 1443-5756



Title Page

Contents



Page 3 of 28

Go Back

Full Screen

Close

1. Introduction

Let $p \geq 1$, $q \geq 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \dots, k$ and $n = 1, 2, \dots, r$, where k and r are natural numbers and define $A_m = \sum_{s=1}^m a_s$ and $B_n = \sum_{t=1}^n b_t$. Then

$$(1.1) \quad \sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{m+n} \leq C(p, q, k, r) \left\{ \sum_{m=1}^k (k-m+1)(A_m^{p-1} a_m)^2 \right\}^{\frac{1}{2}} \\ \times \left\{ \sum_{n=1}^r (r-n+1)(B_n^{q-1} b_n)^2 \right\}^{\frac{1}{2}},$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(p, q, k, r) = \frac{1}{2}pq\sqrt{kr}$.

An integral analogue of (1.1) is given in the following result.

Let $p \geq 1$, $q \geq 1$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$ for $\sigma \in (0, x)$, $\tau \in (0, y)$, where x, y are positive real numbers and define $F(s) = \int_0^s f(\sigma)d\sigma$ and $G(t) = \int_0^t g(\tau)d\tau$, for $s \in (0, x)$, $t \in (0, y)$. Then

$$(1.2) \quad \int_0^x \int_0^y \frac{F^p(s)G^q(t)dsdt}{s+t} \leq D(p, q, x, y) \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^2 ds \right\}^{\frac{1}{2}} \\ \times \left\{ \int_0^y (y-t)(G^{q-1}(t)g(t))^2 dt \right\}^{\frac{1}{2}},$$

unless $f(\sigma) \equiv 0$ or $g(\tau) \equiv 0$, where $D(p, q, x, y) = \frac{1}{2}pq\sqrt{xy}$.

Inequalities (1.1) and (1.2) are the well known Hilbert-Pachpatte inequalities [1], which gave new estimates on Hilbert type inequalities [2]. It is well known that the Hilbert-Pachpatte inequalities play a dominant role in analysis, so the literature on such inequalities and their applications is vast [3] – [8].



Title Page

Contents



Page 4 of 28

Go Back

Full Screen

Close

Young-Ho Kim [9] gave new inequalities similar to the Hilbert-Pachpatte inequalities as follows.

Let $p \geq 1$, $q \geq 1$, $\alpha > 0$, and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \dots, k$ and $n = 1, 2, \dots, r$, where k and r are natural numbers and define $A_m = \sum_{s=1}^m a_s$ and $B_n = \sum_{t=1}^n b_t$. Then

$$(1.3) \quad \sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{(m^\alpha + n^\alpha)^{\frac{1}{\alpha}}} \leq C(p, q, k, r; \alpha) \left\{ \sum_{m=1}^k (k - m + 1)(A_m^{p-1} a_m)^2 \right\}^{\frac{1}{2}} \\ \times \left\{ \sum_{n=1}^r (r - n + 1)(B_n^{q-1} b_n)^2 \right\}^{\frac{1}{2}},$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(p, q, k, r; \alpha) = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} pq\sqrt{kr}$.

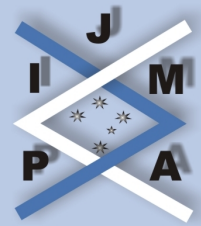
An integral analogue of (1.3) is given in the following result.

Let $p \geq 1$, $q \geq 1$, $\alpha > 0$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$ for $\sigma \in (0, x)$, $\tau \in (0, y)$, where x, y are positive real numbers and define $F(s) = \int_0^s f(\sigma) d\sigma$ and $G(t) = \int_0^t g(\tau) d\tau$, for $s \in (0, x)$, $t \in (0, y)$. Then

$$(1.4) \quad \int_0^x \int_0^y \frac{F^p(s) G^q(t) ds dt}{(s^\alpha + t^\alpha)^{\frac{1}{\alpha}}} \\ \leq D(p, q, x, y; \alpha) \left\{ \int_0^x (x - s)(F^{p-1}(s) f(s))^2 ds \right\}^{\frac{1}{2}} \\ \times \left\{ \int_0^y (y - t)(G^{q-1}(t) g(t))^2 dt \right\}^{\frac{1}{2}},$$

unless $f(\sigma) \equiv 0$ or $g(\tau) \equiv 0$, where $D(p, q, x, y; \alpha) = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} pq\sqrt{xy}$.

The purpose of the present paper is to derive some new generalized inequalities (1.1) and (1.2) that are similar to (1.3) and (1.4). By applying an elementary inequality, we also obtain some new inequalities similar to some results in [1, 9].



Hilbert-Pachpatte's Inequalities

Wengui Yang

vol. 10, iss. 1, art. 26, 2009

[Title Page](#)

[Contents](#)



Page 5 of 28

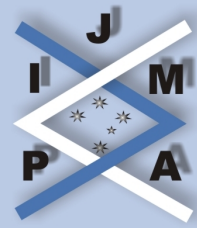
[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
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2. Main Results

Now we give our results as follows in this paper.

Theorem 2.1. Let $p \geq 1$, $q \geq 1$, $\alpha > 1$, $\gamma > 1$ and $\{a_m\}$ and $\{b_n\}$ be two non-negative sequences of real numbers defined for $m = 1, 2, \dots, k$ and $n = 1, 2, \dots, r$, where k and r are natural numbers and define $A_m = \sum_{s=1}^m a_s$ and $B_n = \sum_{t=1}^n b_t$. Then

$$(2.1) \quad \sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq C(p, q, k, r; \alpha, \gamma) \\ \times \left\{ \sum_{m=1}^k (k-m+1)(A_m^{p-1} a_m)^\alpha \right\}^{\frac{1}{\alpha}} \left\{ \sum_{n=1}^r (r-n+1)(B_n^{q-1} b_n)^\gamma \right\}^{\frac{1}{\gamma}},$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(p, q, k, r; \alpha, \gamma) = \frac{pq}{\alpha+\gamma} k^{\frac{\alpha-1}{\alpha}} r^{\frac{\gamma-1}{\gamma}}$.

Proof. The idea for the proof Theorem 2.1 comes from Theorem 1 of [1] and Theorem 2.1 of [9]. From the hypotheses of Theorem 2.1 and using the following inequality (see [10, 11]),

$$(2.2) \quad \left\{ \sum_{m=1}^n z_m \right\}^\beta \leq \beta \sum_{m=1}^n z_m \left\{ \sum_{k=1}^m z_k \right\}^{\beta-1},$$

where $\beta \geq 1$ is a constant and $z_m \geq 0$, ($m = 1, 2, \dots, n$), it is easy to observe that

$$(2.3) \quad A_m^p \leq p \sum_{s=1}^m A_s^{p-1} a_s, \quad m = 1, 2, \dots, k, \\ B_n^q \leq q \sum_{t=1}^n B_t^{q-1} b_t, \quad n = 1, 2, \dots, r.$$



Title Page

Contents



Page 7 of 28

Go Back

Full Screen

Close

From (2.3) and Hölder's inequality, we have

$$(2.4) \quad \sum_{s=1}^m A_s^{p-1} a_s \leq m^{\frac{\alpha-1}{\alpha}} \left\{ \sum_{s=1}^m (A_s^{p-1} a_s)^\alpha \right\}^{\frac{1}{\alpha}}, \quad m = 1, 2, \dots, k,$$

and

$$(2.5) \quad \sum_{t=1}^n B_t^{q-1} b_t \leq n^{\frac{\gamma-1}{\gamma}} \left\{ \sum_{t=1}^n (B_t^{q-1} b_t)^\gamma \right\}^{\frac{1}{\gamma}}, \quad n = 1, 2, \dots, r.$$

Using the inequality of means [12]

$$(2.6) \quad \left\{ \prod_{i=1}^n s_i^{\omega_i} \right\}^{\frac{1}{\Omega_n}} \leq \left\{ \frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^r \right\}^{\frac{1}{r}}$$

for $r > 0$, $\omega_i > 0$, $\sum_{i=1}^n \omega_i = \Omega_n$, we observe that

$$(2.7) \quad (s_1^{\omega_1} s_2^{\omega_2})^{r/(\omega_1+\omega_2)} \leq \frac{1}{\omega_1 + \omega_2} (\omega_1 s_1^r + \omega_2 s_2^r).$$

Let $s_1 = m^{\alpha-1}$, $s_2 = n^{\gamma-1}$, $\omega_1 = \frac{1}{\alpha}$, $\omega_2 = \frac{1}{\gamma}$ and $r = \omega_1 + \omega_2$, from (2.3) – (2.5) and (2.7), we have

$$(2.8) \quad A_m^p B_n^q \leq pqm^{\frac{\alpha-1}{\alpha}} n^{\frac{\gamma-1}{\gamma}} \left\{ \sum_{s=1}^m (A_s^{p-1} a_s)^\alpha \right\}^{\frac{1}{\alpha}} \left\{ \sum_{t=1}^n (B_t^{q-1} b_t)^\gamma \right\}^{\frac{1}{\gamma}} \\ \leq \frac{pq\alpha\gamma}{\alpha + \gamma} \left\{ \frac{m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}}{\alpha} + \frac{n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}{\gamma} \right\}$$



[Title Page](#)

[Contents](#)



Page 8 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

$$\times \left\{ \sum_{s=1}^m (A_s^{p-1} a_s)^\alpha \right\}^{\frac{1}{\alpha}} \left\{ \sum_{t=1}^n (B_t^{q-1} b_t)^\gamma \right\}^{\frac{1}{\gamma}},$$

for $m = 1, 2, \dots, k, n = 1, 2, \dots, r$. From (2.8), we observe that

$$(2.9) \quad \frac{A_m^p B_n^q}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq \frac{pq}{\alpha + \gamma} \left\{ \sum_{s=1}^m (A_s^{p-1} a_s)^\alpha \right\}^{\frac{1}{\alpha}} \left\{ \sum_{t=1}^n (B_t^{q-1} b_t)^\gamma \right\}^{\frac{1}{\gamma}},$$

for $m = 1, 2, \dots, k, n = 1, 2, \dots, r$. Taking the sum on both sides of (2.9) first over n from 1 to r and then over m from 1 to k of the resulting inequality and using Hölder's inequality with indices $\alpha, \alpha/(\alpha - 1)$ and $\gamma, \gamma/(\gamma - 1)$ and interchanging the order of summations, we observe that

$$\begin{aligned} & \sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \\ & \leq \frac{pq}{\alpha + \gamma} \left[\sum_{m=1}^k \left\{ \sum_{s=1}^m (A_s^{p-1} a_s)^\alpha \right\}^{\frac{1}{\alpha}} \right] \left[\sum_{n=1}^r \left\{ \sum_{t=1}^n (B_t^{q-1} b_t)^\gamma \right\}^{\frac{1}{\gamma}} \right] \\ & \leq \frac{pq}{\alpha + \gamma} k^{\frac{\alpha-1}{\alpha}} \left\{ \sum_{m=1}^k \sum_{s=1}^m (A_s^{p-1} a_s)^\alpha \right\}^{\frac{1}{\alpha}} r^{\frac{\gamma-1}{\gamma}} \left\{ \sum_{n=1}^r \sum_{t=1}^n (B_t^{q-1} b_t)^\gamma \right\}^{\frac{1}{\gamma}} \end{aligned}$$

$$= \frac{pq}{\alpha + \gamma} k^{\frac{\alpha-1}{\alpha}} r^{\frac{\gamma-1}{\gamma}} \left\{ \sum_{m=1}^k (k - m + 1) (A_m^{p-1} a_m)^\alpha \right\}^{\frac{1}{\alpha}} \\ \times \left\{ \sum_{n=1}^r (r - n + 1) (B_n^{q-1} b_n)^\gamma \right\}^{\frac{1}{\gamma}}.$$

□

Remark 1. In Theorem 2.1, setting $\alpha = \gamma = 2$, we have (1.1). In Theorem 2.1, setting $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$, we have

$$\sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{\gamma m^{\alpha-1} + \alpha n^{\gamma-1}} \leq C(p, q, k, r; \alpha, \gamma) \\ \times \left\{ \sum_{m=1}^k (k - m + 1) (A_m^{p-1} a_m)^\alpha \right\}^{\frac{1}{\alpha}} \left\{ \sum_{n=1}^r (r - n + 1) (B_n^{q-1} b_n)^\gamma \right\}^{\frac{1}{\gamma}},$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(p, q, k, r; \alpha, \gamma) = \frac{pq}{\alpha + \gamma} k^{\frac{\alpha-1}{\alpha}} r^{\frac{\gamma-1}{\gamma}}$.

Remark 2. In Theorem 2.1, setting $p = q = 1$, we have

$$(2.10) \quad \sum_{m=1}^k \sum_{n=1}^r \frac{A_m B_n}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \\ \leq C(1, 1, k, r; \alpha, \gamma) \left\{ \sum_{m=1}^k (k - m + 1) a_m^\alpha \right\}^{\frac{1}{\alpha}} \left\{ \sum_{n=1}^r (r - n + 1) b_n^\gamma \right\}^{\frac{1}{\gamma}},$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(1, 1, k, r; \alpha, \gamma) = \frac{1}{\alpha + \gamma} k^{\frac{\alpha-1}{\alpha}} r^{\frac{\gamma-1}{\gamma}}$.



Title Page

Contents



Page 9 of 28

Go Back

Full Screen

Close



[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 10 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

In the following theorem we give a further generalization of the inequality (2.10) obtained in Remark 2. Before we give our result, we point out that $\{p_m\}$ and $\{q_n\}$ should be two positive sequences for $m = 1, 2, \dots, k$ and $n = 1, 2, \dots, r$ in Theorem 2.3 of [9].

Theorem 2.2. *Let $\alpha > 1, \gamma > 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers and $\{p_m\}$ and $\{q_n\}$ be positive sequences defined for $m = 1, 2, \dots, k$ and $n = 1, 2, \dots, r$, where k and r are natural numbers and define $A_m = \sum_{s=1}^m a_s$, $B_n = \sum_{t=1}^n b_t$, $P_m = \sum_{s=1}^m p_s$ and $Q_n = \sum_{t=1}^n q_t$. Let Φ and Ψ be two real-valued, nonnegative, convex, and submultiplicative functions defined on $\mathbb{R}_+ = [0, \infty)$. Then*

$$(2.11) \quad \sum_{m=1}^k \sum_{n=1}^r \frac{\Phi(A_m)\Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq M(k, r; \alpha, \gamma) \left\{ \sum_{m=1}^k (k-m+1) \left[p_m \Phi\left(\frac{a_m}{p_m}\right) \right]^\alpha \right\}^{\frac{1}{\alpha}} \times \left\{ \sum_{n=1}^r (r-n+1) \left[q_n \Psi\left(\frac{b_n}{q_n}\right) \right]^\gamma \right\}^{\frac{1}{\gamma}},$$

where

$$M(k, r; \alpha, \gamma) = \frac{1}{\alpha + \gamma} \left\{ \sum_{m=1}^k \left[\frac{\Phi(P_m)}{P_m} \right]^{\frac{\alpha}{\alpha-1}} \right\}^{\frac{\alpha-1}{\alpha}} \left\{ \sum_{n=1}^r \left[\frac{\Psi(Q_n)}{Q_n} \right]^{\frac{\gamma}{\gamma-1}} \right\}^{\frac{\gamma-1}{\gamma}}.$$

Proof. From the hypotheses of Φ and Ψ and by using Jensen's inequality and Hölder's



Title Page

Contents



Page 11 of 28

Go Back

Full Screen

Close

inequality, it is easy to observe that

$$\begin{aligned} (2.12) \quad \Phi(A_m) &= \Phi\left(\frac{P_m \sum_{s=1}^m p_s a_s / p_s}{\sum_{s=1}^m p_s}\right) \\ &\leq \Phi(P_m) \Phi\left(\frac{\sum_{s=1}^m p_s a_s / p_s}{\sum_{s=1}^m p_s}\right) \leq \frac{\Phi(P_m)}{P_m} \sum_{s=1}^m p_s \Phi\left(\frac{a_s}{p_s}\right) \\ &\leq \frac{\Phi(P_m)}{P_m} m^{\frac{\alpha-1}{\alpha}} \left\{ \sum_{s=1}^m \left[p_s \Phi\left(\frac{a_s}{p_s}\right) \right]^\alpha \right\}^{\frac{1}{\alpha}}, \end{aligned}$$

and similarly,

$$(2.13) \quad \Psi(B_n) \leq \frac{\Psi(Q_n)}{Q_n} n^{\frac{\gamma-1}{\gamma}} \left\{ \sum_{t=1}^n \left[q_t \Psi\left(\frac{b_t}{q_t}\right) \right]^\gamma \right\}^{\frac{1}{\gamma}}.$$

Let $s_1 = m^{\alpha-1}$, $s_2 = n^{\gamma-1}$, $\omega_1 = \frac{1}{\alpha}$, $\omega_2 = \frac{1}{\gamma}$ and $r = \omega_1 + \omega_2$, from (2.7), (2.12) and (2.13), we have

$$\begin{aligned} (2.14) \quad \Phi(A_m) \Psi(B_n) &\leq m^{\frac{\alpha-1}{\alpha}} n^{\frac{\gamma-1}{\gamma}} \left[\frac{\Phi(P_m)}{P_m} \left\{ \sum_{s=1}^m \left[p_s \Phi\left(\frac{a_s}{p_s}\right) \right]^\alpha \right\}^{\frac{1}{\alpha}} \right] \\ &\quad \times \left[\frac{\Psi(Q_n)}{Q_n} \left\{ \sum_{t=1}^n \left[q_t \Psi\left(\frac{b_t}{q_t}\right) \right]^\gamma \right\}^{\frac{1}{\gamma}} \right] \\ &\leq \frac{\alpha\gamma}{\alpha + \gamma} \left\{ \frac{m^{\frac{(\alpha-1)(\alpha+\gamma)}}{\alpha\gamma}}{\alpha} + \frac{n^{\frac{(\gamma-1)(\alpha+\gamma)}}{\alpha\gamma}}{\gamma} \right\} \end{aligned}$$



[Title Page](#)

[Contents](#)



Page 12 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

$$\times \left[\frac{\Phi(P_m)}{P_m} \left\{ \sum_{s=1}^m \left[p_s \Phi \left(\frac{a_s}{p_s} \right) \right]^\alpha \right\}^{\frac{1}{\alpha}} \right] \left[\frac{\Psi(Q_n)}{Q_n} \left\{ \sum_{t=1}^n \left[q_t \Psi \left(\frac{b_t}{q_t} \right) \right]^\gamma \right\}^{\frac{1}{\gamma}} \right]$$

for $m = 1, 2, \dots, k, n = 1, 2, \dots, r$. From (2.14), we observe that

$$(2.15) \quad \frac{\Phi(A_m)\Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq \frac{1}{\alpha + \gamma} \left[\frac{\Phi(P_m)}{P_m} \left\{ \sum_{s=1}^m \left[p_s \Phi \left(\frac{a_s}{p_s} \right) \right]^\alpha \right\}^{\frac{1}{\alpha}} \right] \times \left[\frac{\Psi(Q_n)}{Q_n} \left\{ \sum_{t=1}^n \left[q_t \Psi \left(\frac{b_t}{q_t} \right) \right]^\gamma \right\}^{\frac{1}{\gamma}} \right]$$

for $m = 1, 2, \dots, k, n = 1, 2, \dots, r$. Taking the sum on both sides of (2.15) first over n from 1 to r and then over m from 1 to k of the resulting inequality and using Hölder's inequality with indices $\alpha, \alpha/(\alpha - 1)$ and $\gamma, \gamma/(\gamma - 1)$ and interchanging the order of summations, we observe that

$$\begin{aligned} & \sum_{m=1}^k \sum_{n=1}^r \frac{\Phi(A_m)\Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \\ & \leq \frac{1}{\alpha + \gamma} \left[\sum_{m=1}^k \frac{\Phi(P_m)}{P_m} \left\{ \sum_{s=1}^m \left[p_s \Phi \left(\frac{a_s}{p_s} \right) \right]^\alpha \right\}^{\frac{1}{\alpha}} \right] \left[\sum_{n=1}^r \frac{\Psi(Q_n)}{Q_n} \left\{ \sum_{t=1}^n \left[q_t \Psi \left(\frac{b_t}{q_t} \right) \right]^\gamma \right\}^{\frac{1}{\gamma}} \right] \\ & \leq \frac{1}{\alpha + \gamma} \left\{ \sum_{m=1}^k \left[\frac{\Phi(P_m)}{P_m} \right]^{\frac{\alpha}{\alpha-1}} \right\}^{\frac{\alpha-1}{\alpha}} \left\{ \sum_{m=1}^k \sum_{s=1}^m \left[p_s \Phi \left(\frac{a_s}{p_s} \right) \right]^\alpha \right\}^{\frac{1}{\alpha}} \end{aligned}$$



Title Page

Contents



Page 13 of 28

Go Back

Full Screen

Close

$$\begin{aligned} & \times \left\{ \sum_{n=1}^r \left[\frac{\Psi(Q_n)}{Q_n} \right]^{\frac{\gamma}{\gamma-1}} \right\}^{\frac{\gamma-1}{\gamma}} \left\{ \sum_{n=1}^r \sum_{t=1}^n \left[q_t \Psi \left(\frac{b_t}{q_t} \right) \right]^{\gamma} \right\}^{\frac{1}{\gamma}} \\ & = \frac{1}{\alpha + \gamma} \left\{ \sum_{m=1}^k \left[\frac{\Phi(P_m)}{P_m} \right]^{\frac{\alpha}{\alpha-1}} \right\}^{\frac{\alpha-1}{\alpha}} \left\{ \sum_{n=1}^r \left[\frac{\Psi(Q_n)}{Q_n} \right]^{\frac{\gamma}{\gamma-1}} \right\}^{\frac{\gamma-1}{\gamma}} \\ & \times \left\{ \sum_{m=1}^k (k - m + 1) \left[p_s \Phi \left(\frac{a_s}{p_s} \right) \right]^{\alpha} \right\}^{\frac{1}{\alpha}} \left\{ \sum_{n=1}^r (r - n + 1) \left[q_t \Psi \left(\frac{b_t}{q_t} \right) \right]^{\gamma} \right\}^{\frac{1}{\gamma}} . \end{aligned}$$

□

Remark 3. From the inequality (2.7), we obtain

$$(2.16) \quad s_1^{\omega_1} s_2^{\omega_2} \leq \frac{1}{\omega_1 + \omega_2} (\omega_1 s_1^{\omega_1 + \omega_2} + \omega_2 s_2^{\omega_1 + \omega_2})$$

for $\omega_1 > 0$, $\omega_2 > 0$. If we apply the elementary inequality (2.16) on the right-hand sides of (2.1) in Theorem 2.1 and (2.11) in Theorem 2.2, then we get the following inequalities

$$\begin{aligned} & \sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \\ & \leq \frac{\alpha\gamma C(p, q, k, r; \alpha, \gamma)}{\alpha + \gamma} \left[\frac{1}{\alpha} \left\{ \sum_{m=1}^k (k - m + 1) (A_m^{p-1} a_m)^\alpha \right\}^{\frac{\alpha+\gamma}{\alpha\gamma}} \right. \\ & \quad \left. + \frac{1}{\gamma} \left\{ \sum_{n=1}^r (r - n + 1) (B_n^{q-1} b_n)^\gamma \right\}^{\frac{\alpha+\gamma}{\alpha\gamma}} \right], \end{aligned}$$



[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 14 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

where $C(p, q, k, r; \alpha, \gamma) = \frac{pq}{\alpha + \gamma} k^{\frac{\alpha-1}{\alpha}} r^{\frac{\gamma-1}{\gamma}}$. Also,

$$\begin{aligned} & \sum_{m=1}^k \sum_{n=1}^r \frac{\Phi(A_m) \Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \\ & \leq \frac{\alpha\gamma M(k, r; \alpha, \gamma)}{\alpha + \gamma} \left[\frac{1}{\alpha} \left\{ \sum_{m=1}^k (k - m + 1) \left[p_m \Phi \left(\frac{a_m}{p_m} \right) \right]^\alpha \right\}^{\frac{\alpha+\gamma}{\alpha\gamma}} \right. \\ & \quad \left. + \frac{1}{\gamma} \left\{ \sum_{n=1}^r (r - n + 1) \left[q_n \Psi \left(\frac{b_n}{q_n} \right) \right]^\gamma \right\}^{\frac{\alpha+\gamma}{\alpha\gamma}} \right], \end{aligned}$$

where

$$M(k, r; \alpha, \gamma) = \frac{1}{\alpha + \gamma} \left\{ \sum_{m=1}^k \left[\frac{\Phi(P_m)}{P_m} \right]^{\frac{\alpha}{\alpha-1}} \right\}^{\frac{\alpha-1}{\alpha}} \left\{ \sum_{n=1}^r \left[\frac{\Psi(Q_n)}{Q_n} \right]^{\frac{\gamma}{\gamma-1}} \right\}^{\frac{\gamma-1}{\gamma}}.$$

The following theorems deal with slight variants of the inequality (2.11) given in Theorem 2.2.

Theorem 2.3. Let $\alpha > 1, \gamma > 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \dots, k$ and $n = 1, 2, \dots, r$, where k and r are natural numbers and define $A_m = \frac{1}{m} \sum_{s=1}^m a_s$ and $B_n = \frac{1}{n} \sum_{t=1}^n b_t$. Let Φ and Ψ be two real-valued, nonnegative, convex functions defined on $\mathbb{R}_+ = [0, \infty)$. Then

$$\sum_{m=1}^k \sum_{n=1}^r \frac{mn \Phi(A_m) \Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq C(1, 1, k, r; \alpha, \gamma)$$

$$\times \left\{ \sum_{m=1}^k (k-m+1) \Phi^\alpha(a_m) \right\}^{\frac{1}{\alpha}} \left\{ \sum_{n=1}^r (r-n+1) \Psi^\gamma(b_n) \right\}^{\frac{1}{\gamma}},$$

where $C(1, 1, k, r; \alpha, \gamma) = \frac{1}{\alpha+\gamma} k^{\frac{\alpha-1}{\alpha}} r^{\frac{\gamma-1}{\gamma}}$.

Proof. From the hypotheses and by using Jensen's inequality and Hölder's inequality, it is easy to observe that

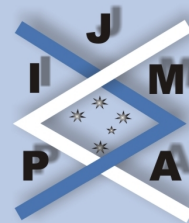
$$\Phi(A_m) = \Phi\left(\frac{1}{m} \sum_{s=1}^m a_s\right) \leq \frac{1}{m} \sum_{s=1}^m \Phi(a_s) \leq \frac{1}{m} m^{\frac{\alpha-1}{\alpha}} \left\{ \sum_{s=1}^m \Phi^\alpha(a_s) \right\}^{\frac{1}{\alpha}},$$

$$\Psi(B_n) = \Psi\left(\frac{1}{n} \sum_{t=1}^n b_t\right) \leq \frac{1}{n} \sum_{t=1}^n \Psi(b_t) \leq \frac{1}{n} n^{\frac{\gamma-1}{\gamma}} \left\{ \sum_{t=1}^n \Psi^\gamma(b_t) \right\}^{\frac{1}{\gamma}}.$$

The rest of the proof can be completed by following the same steps as in the proofs of Theorems 2.1 and 2.2 with suitable changes and hence we omit the details. \square

Theorem 2.4. Let $\alpha > 1, \gamma > 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers and $\{p_m\}$ and $\{q_n\}$ be positive sequences defined for $m = 1, 2, \dots, k$ and $n = 1, 2, \dots, r$, where k and r are natural numbers and define $P_m = \sum_{s=1}^m p_s$, $Q_n = \sum_{t=1}^n q_t$, $A_m = \frac{1}{P_m} \sum_{s=1}^m p_s a_s$ and $B_n = \frac{1}{Q_n} \sum_{t=1}^n q_n b_t$. Let Φ and Ψ be two real-valued, nonnegative, convex functions defined on $\mathbb{R}_+ = [0, \infty)$. Then

$$\sum_{m=1}^k \sum_{n=1}^r \frac{P_m Q_n \Phi(A_m) \Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}$$



[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 15 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

$$\leq C(1, 1, k, r; \alpha, \gamma) \left\{ \sum_{m=1}^k (k - m + 1) [p_m \Phi(a_m)]^\alpha \right\}^{\frac{1}{\alpha}} \\ \times \left\{ \sum_{n=1}^r (r - n + 1) [q_n \Psi(b_n)]^\gamma \right\}^{\frac{1}{\gamma}},$$

where $C(1, 1, k, r; \alpha, \gamma) = \frac{1}{\alpha + \gamma} k^{\frac{\alpha-1}{\alpha}} r^{\frac{\gamma-1}{\gamma}}$.

Proof. From the hypotheses and by using Jensen's inequality and Hölder's inequality, it is easy to observe that

$$\Phi(A_m) = \Phi \left(\frac{1}{P_m} \sum_{s=1}^m p_s a_s \right) \\ \leq \frac{1}{P_m} \sum_{s=1}^m p_s \Phi(a_s) \leq \frac{1}{P_m} m^{\frac{\alpha-1}{\alpha}} \left\{ \sum_{s=1}^m [p_s \Phi(a_s)]^\alpha \right\}^{\frac{\alpha-1}{\alpha}}, \\ \Psi(B_n) = \Psi \left(\frac{1}{Q_n} \sum_{t=1}^n q_t b_t \right) \\ \leq \frac{1}{Q_n} \sum_{t=1}^n q_t \Psi(b_t) \leq \frac{1}{Q_n} n^{\frac{\gamma-1}{\gamma}} \left\{ \sum_{t=1}^n [q_t \Psi(b_t)]^\gamma \right\}^{\frac{\gamma-1}{\gamma}}.$$

The rest of the proof can be completed by following the same steps as in the proofs of Theorems 2.1 and 2.2 with suitable changes and hence we omit the details. \square



Title Page

Contents



Page 16 of 28

Go Back

Full Screen

Close



3. Integral Analogues

Now we give the integral analogues of the inequalities in Theorems 2.1 – 2.4.

An integral analogue of Theorem 2.1 is given in the following theorem.

Theorem 3.1. *Let $p \geq 0, q \geq 0, \alpha > 1, \gamma > 1$ and $f(\sigma) \geq 0, g(\tau) \geq 0$ for $\sigma \in (0, x), \tau \in (0, y)$, where x, y are positive real numbers, define $F(s) = \int_0^s f(\sigma) d\sigma, G(t) = \int_0^t g(\tau) d\tau$ for $s \in (0, x), t \in (0, y)$. Then*

$$(3.1) \quad \int_0^x \int_0^y \frac{F^p(s)G^q(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} ds dt$$

$$\leq D(p, q, x, y; \alpha, \gamma) \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^\alpha ds \right\}^{\frac{1}{\alpha}}$$

$$\times \left\{ \int_0^y (y-t)(G^{q-1}(t)g(t))^\gamma dt \right\}^{\frac{1}{\gamma}},$$

unless $f(\sigma) \equiv 0$ or $g(\tau) \equiv 0$, where $D(p, q, x, y; \alpha, \gamma) = \frac{pq}{\alpha+\gamma} x^{\frac{\alpha-1}{\alpha}} y^{\frac{\gamma-1}{\gamma}}$.

Proof. From the hypotheses of $F(s)$ and $G(t)$, it is easy to observe that

$$(3.2) \quad F^p(s) = p \int_0^s F^{p-1}(\sigma) f(\sigma) d\sigma, \quad s \in (0, x),$$

$$G^q(t) = q \int_0^t G^{q-1}(\tau) g(\tau) d\tau, \quad t \in (0, y).$$

From (3.2) and Hölder's inequality, we have

$$(3.3) \quad \int_0^x F^{p-1}(\sigma) f(\sigma) d\sigma \leq s^{\frac{\alpha-1}{\alpha}} \left\{ \int_0^s (F^{p-1}(\sigma) f(\sigma))^\alpha d\sigma \right\}^{\frac{1}{\alpha}}, \quad s \in (0, x),$$



[Title Page](#)

[Contents](#)



Page 18 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

and

$$(3.4) \quad \int_0^y G^{q-1}(t)g(t)dt \leq t^{\frac{\gamma-1}{\gamma}} \left\{ \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau \right\}^{\frac{1}{\gamma}}, \quad t \in (0, t).$$

Let $s_1 = s^{\alpha-1}$, $s_2 = t^{\gamma-1}$, $\omega_1 = \frac{1}{\alpha}$, $\omega_2 = \frac{1}{\gamma}$, $r = \omega_1 + \omega_2$, from (3.2) – (3.4) and (2.7), we observe that

$$(3.5) \quad \begin{aligned} & F^p(s)G^q(t) \\ & \leq pqs^{\frac{\alpha-1}{\alpha}}t^{\frac{\gamma-1}{\gamma}} \left\{ \int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma \right\}^{\frac{1}{\alpha}} \left\{ \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau \right\}^{\frac{1}{\gamma}} \\ & \leq \frac{pq\alpha\gamma}{\alpha + \gamma} \left\{ m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}} \right\} \\ & \quad \times \left\{ \int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma \right\}^{\frac{1}{\alpha}} \left\{ \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau \right\}^{\frac{1}{\gamma}} \end{aligned}$$

for $s \in (0, x)$, $t \in (0, y)$. From (3.5), we observe that

$$(3.6) \quad \begin{aligned} & \frac{F^p(s)G^q(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \\ & \leq \frac{pq}{\alpha + \gamma} \left\{ \int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma \right\}^{\frac{1}{\alpha}} \left\{ \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau \right\}^{\frac{1}{\gamma}} \end{aligned}$$

for $s \in (0, x)$, $t \in (0, y)$. Taking the integral on both sides of (3.6) first over t from 0 to y and then over s from 0 to x of the resulting inequality and using Hölder's inequality with indices α , $\alpha/(\alpha - 1)$ and γ , $\gamma/(\gamma - 1)$ and interchanging the order



Title Page

Contents



Page 19 of 28

Go Back

Full Screen

Close

of integrals, we observe that

$$\begin{aligned} & \int_0^x \int_0^y \frac{F^p(s)G^q(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} ds dt \\ & \leq \frac{pq}{\alpha + \gamma} \left[\int_0^x \left\{ \int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma \right\}^{\frac{1}{\alpha}} ds \right] \left[\int_0^y \left\{ \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau \right\}^{\frac{1}{\gamma}} dt \right] \\ & \leq \frac{pq}{\alpha + \gamma} x^{\frac{\alpha-1}{\alpha}} \left\{ \int_0^x \int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma ds \right\}^{\frac{1}{\alpha}} y^{\frac{\gamma-1}{\gamma}} \left\{ \int_0^y \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau dt \right\}^{\frac{1}{\gamma}} \\ & = \frac{pq}{\alpha + \gamma} x^{\frac{\alpha-1}{\alpha}} y^{\frac{\gamma-1}{\gamma}} \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^\alpha ds \right\}^{\frac{1}{\alpha}} \left\{ \int_0^t (y-t)(G^{q-1}(t)g(t))^\gamma dt \right\}^{\frac{1}{\gamma}}. \end{aligned}$$

□

Remark 4. In Theorem 3.1, setting $\alpha = \gamma = 2$, we have (1.2). In Theorem 3.1, setting $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$, we have

$$\begin{aligned} & \int_0^x \int_0^y \frac{F^p(s)G^q(t)}{\gamma s^{\alpha-1} + \alpha t^{\gamma-1}} ds dt \\ & \leq D(p, q, x, y; \alpha, \gamma) \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^\alpha ds \right\}^{\frac{1}{\alpha}} \\ & \quad \times \left\{ \int_0^y (y-t)(G^{q-1}(t)g(t))^\gamma dt \right\}^{\frac{1}{\gamma}}, \end{aligned}$$

unless $f(\sigma) \equiv 0$ or $g(\tau) \equiv 0$, where $D(p, q, x, y; \alpha, \gamma) = \frac{pq}{\alpha+\gamma} x^{\frac{\alpha-1}{\alpha}} y^{\frac{\gamma-1}{\gamma}}$.



[Title Page](#)

[Contents](#)



Page 20 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

Remark 5. In Theorem 3.1, setting $p = q = 1$, we have

$$(3.7) \quad \int_0^x \int_0^y \frac{F(s)G(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} ds dt$$

$$\leq D(1, 1, x, y; \alpha, \gamma) \left\{ \int_0^x (x-s) f^\alpha(s) ds \right\}^{\frac{1}{\alpha}} \left\{ \int_0^y (y-t) g^\gamma(t) dt \right\}^{\frac{1}{\gamma}},$$

unless $f(\sigma) \equiv 0$ or $g(\tau) \equiv 0$, where $D(1, 1, x, y; \alpha, \gamma) = \frac{1}{\alpha+\gamma} x^{\frac{\alpha-1}{\alpha}} y^{\frac{\gamma-1}{\gamma}}$.

In the following theorem we give a further generalization of the inequality (3.7) obtained in Remark 5.

Theorem 3.2. Let $\alpha > 1$, $\gamma > 1$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$, $p(\sigma) > 0$ and $q(\tau) > 0$ for $\sigma \in (0, x)$, $\tau \in (0, y)$, where x, y are positive real numbers. Define $F(s) = \int_0^s f(\sigma) d\sigma$ and $G(t) = \int_0^t g(\tau) d\tau$, $P(s) = \int_0^s p(\sigma) d\sigma$ and $Q(t) = \int_0^t q(\tau) d\tau$ for $s \in (0, x)$, $t \in (0, y)$. Let Φ and Ψ be two real-valued, nonnegative, convex, and submultiplicative functions defined on $\mathbb{R}_+ = [0, \infty)$. Then

$$(3.8) \quad \int_0^x \int_0^y \frac{\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} ds dt$$

$$\leq L(x, y; \alpha, \gamma) \left\{ \int_0^x (x-s) \left[p(s)\Phi\left(\frac{f(s)}{p(s)}\right) \right]^\alpha ds \right\}^{\frac{1}{\alpha}}$$

$$\times \left\{ \int_0^y (y-t) \left[q(t)\Psi\left(\frac{g(t)}{q(t)}\right) \right]^\gamma dt \right\}^{\frac{1}{\gamma}},$$

where

$$L(x, y; \alpha, \gamma) = \frac{1}{\alpha + \gamma} \left\{ \int_0^x \left[\frac{\Phi(P(s))}{P(s)} \right]^{\frac{\alpha}{\alpha-1}} ds \right\}^{\frac{\alpha-1}{\alpha}} \left\{ \int_0^y \left[\frac{\Psi(Q(t))}{Q(t)} \right]^{\frac{\gamma}{\gamma-1}} dt \right\}^{\frac{\gamma-1}{\gamma}}.$$



[Title Page](#)

[Contents](#)



Page 21 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

Proof. From the hypotheses of Φ and Ψ and by using Jensen's inequality and Hölder's inequality, it is easy to see that

$$\begin{aligned}
 (3.9) \quad \Phi(F(s)) &= \Phi \left(\frac{P(s) \int_0^s p(\sigma) \left(\frac{f(\sigma)}{p(\sigma)} \right) d\sigma}{\int_0^s p(\sigma) d\sigma} \right) \\
 &\leq \Phi(P(s)) \Phi \left(\frac{\int_0^s p(\sigma) \left(\frac{f(\sigma)}{p(\sigma)} \right) d\sigma}{\int_0^s p(\sigma) d\sigma} \right) \\
 &\leq \frac{\Phi(P(s))}{P(s)} \int_0^s p(\sigma) \Phi \left(\frac{f(\sigma)}{p(\sigma)} \right) d\sigma \\
 &\leq \frac{\Phi(P(s))}{P(s)} s^{\frac{\alpha-1}{\alpha}} \left\{ \int_0^s \left[p(\sigma) \Phi \left(\frac{f(\sigma)}{p(\sigma)} \right) \right]^\alpha d\sigma \right\}^{\frac{1}{\alpha}},
 \end{aligned}$$

and similarly,

$$(3.10) \quad \Psi(G(t)) \leq \frac{\Psi(Q(t))}{Q(t)} t^{\frac{\gamma-1}{\gamma}} \left\{ \int_0^t \left[q(\tau) \Psi \left(\frac{g(\tau)}{q(\tau)} \right) \right]^\gamma d\tau \right\}^{\frac{1}{\gamma}}.$$

Let $s_1 = s^{\alpha-1}$, $s_2 = t^{\gamma-1}$, $\omega_1 = \frac{1}{\alpha}$, $\omega_2 = \frac{1}{\gamma}$, $r = \omega_1 + \omega_2$, from (3.9), (3.10) and (2.7), we observe that

$$\begin{aligned}
 (3.11) \quad \Phi(F(s)) \Psi(G(t)) &\leq s^{\frac{\alpha-1}{\alpha}} t^{\frac{\gamma-1}{\gamma}} \left[\frac{\Phi(P(s))}{P(s)} \left\{ \int_0^s \left[p(\sigma) \Phi \left(\frac{f(\sigma)}{p(\sigma)} \right) \right]^\alpha d\sigma \right\}^{\frac{1}{\alpha}} \right] \\
 &\quad \times \left[\frac{\Psi(Q(t))}{Q(t)} \left\{ \int_0^t \left[q(\tau) \Psi \left(\frac{g(\tau)}{q(\tau)} \right) \right]^\gamma d\tau \right\}^{\frac{1}{\gamma}} \right]
 \end{aligned}$$



[Title Page](#)

[Contents](#)



Page 22 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

$$\begin{aligned} &\leq \frac{\alpha\gamma}{\alpha+\gamma} \left\{ \frac{m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}}{\alpha} + \frac{n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}{\gamma} \right\} \\ &\quad \times \left[\frac{\Phi(P(s))}{P(s)} \left\{ \int_0^s \left[p(\sigma)\Phi\left(\frac{f(\sigma)}{p(\sigma)}\right) \right]^\alpha d\sigma \right\}^{\frac{1}{\alpha}} \right] \\ &\quad \times \left[\frac{\Psi(Q(t))}{Q(t)} \left\{ \int_0^t \left[q(\tau)\Psi\left(\frac{g(\tau)}{q(\tau)}\right) \right]^\gamma d\tau \right\}^{\frac{1}{\gamma}} \right] \end{aligned}$$

for $s \in (0, x)$, $t \in (0, y)$. From (3.11), we observe that

$$\begin{aligned} (3.12) \quad &\frac{\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \\ &\leq \frac{1}{\alpha+\gamma} \left[\frac{\Phi(P(s))}{P(s)} \left\{ \int_0^s \left[p(\sigma)\Phi\left(\frac{f(\sigma)}{p(\sigma)}\right) \right]^\alpha d\sigma \right\}^{\frac{1}{\alpha}} \right] \\ &\quad \times \left[\frac{\Psi(Q(t))}{Q(t)} \left\{ \int_0^t \left[q(\tau)\Psi\left(\frac{g(\tau)}{q(\tau)}\right) \right]^\gamma d\tau \right\}^{\frac{1}{\gamma}} \right] \end{aligned}$$

for $s \in (0, x)$, $t \in (0, y)$. Taking the integral on both sides of (3.12) first over t from 0 to y and then over s from 0 to x of the resulting inequality and using Hölder's inequality with indices α , $\alpha/(\alpha-1)$ and γ , $\gamma/(\gamma-1)$ and interchanging the order of integrals, we observe that

$$\int_0^x \int_0^y \frac{\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} ds dt$$



[Title Page](#)

[Contents](#)



Page 23 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

$$\begin{aligned}
 &\leq \frac{1}{\alpha + \gamma} \left[\int_0^x \frac{\Phi(P(s))}{P(s)} \left\{ \int_0^s \left[p(\sigma) \Phi \left(\frac{f(\sigma)}{p(\sigma)} \right) \right]^\alpha d\sigma \right\}^{\frac{1}{\alpha}} ds \right] \\
 &\quad \times \left[\int_0^y \frac{\Psi(Q(t))}{Q(t)} \left\{ \int_0^t \left[q(\tau) \Psi \left(\frac{g(\tau)}{q(\tau)} \right) \right]^\gamma d\tau \right\}^{\frac{1}{\gamma}} dt \right] \\
 &\leq \frac{1}{\alpha + \gamma} \left\{ \int_0^x \left[\frac{\Phi(P(s))}{P(s)} \right]^{\frac{\alpha}{\alpha-1}} ds \right\}^{\frac{\alpha-1}{\alpha}} \left\{ \int_0^x \int_0^s \left[p(\sigma) \Phi \left(\frac{f(\sigma)}{p(\sigma)} \right) \right]^\alpha d\sigma ds \right\}^{\frac{1}{\alpha}} \\
 &\quad \times \left\{ \int_0^y \left[\frac{\Psi(Q(t))}{Q(t)} \right]^{\frac{\gamma}{\gamma-1}} dt \right\}^{\frac{\gamma-1}{\gamma}} \left\{ \int_0^y \int_0^t \left[q(\tau) \Psi \left(\frac{g(\tau)}{q(\tau)} \right) \right]^\gamma d\tau dt \right\}^{\frac{1}{\gamma}} \\
 &= \frac{1}{\alpha + \gamma} \left\{ \int_0^x \left[\frac{\Phi(P(s))}{P(s)} \right]^{\frac{\alpha}{\alpha-1}} ds \right\}^{\frac{\alpha-1}{\alpha}} \left\{ \int_0^y \left[\frac{\Psi(Q(t))}{Q(t)} \right]^{\frac{\gamma}{\gamma-1}} dt \right\}^{\frac{\gamma-1}{\gamma}} \\
 &\quad \times \left\{ \int_0^x (x-s) \left[p(s) \Phi \left(\frac{f(s)}{p(s)} \right) \right]^\alpha ds \right\}^{\frac{1}{\alpha}} \left\{ \int_0^y (y-t) \left[q(t) \Psi \left(\frac{g(t)}{q(t)} \right) \right]^\gamma dt \right\}^{\frac{1}{\gamma}}.
 \end{aligned}$$

□

Remark 6. From the inequality (2.7), we obtain

$$(3.13) \quad s_1^{\omega_1} s_2^{\omega_2} \leq \frac{1}{\omega_1 + \omega_2} (\omega_1 s_1^{\omega_1 + \omega_2} + \omega_2 s_2^{\omega_1 + \omega_2})$$

for $\omega_1 > 0, \omega_2 > 0$. If we apply the elementary inequality (3.13) on the right-hand sides of (3.1) in Theorem 3.1 and (3.8) in Theorem 3.2, then we get the following



Title Page

Contents



Page 24 of 28

Go Back

Full Screen

Close

inequalities

$$(3.14) \quad \int_0^x \int_0^y \frac{F^p(s)G^q(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} ds dt$$

$$\leq \frac{\alpha\gamma D(p, q, x, y; \alpha, \gamma)}{\alpha + \gamma} \left[\frac{1}{\alpha} \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^\alpha ds \right\}^{\frac{\alpha+\gamma}{\alpha\gamma}} \right.$$

$$\left. + \frac{1}{\gamma} \left\{ \int_0^y (y-t)(G^{q-1}(t)g(t))^\gamma dt \right\}^{\frac{\alpha+\gamma}{\alpha\gamma}} \right],$$

where $D(p, q, x, y; \alpha, \gamma) = \frac{pq}{\alpha+\gamma} x^{\frac{\alpha-1}{\alpha}} y^{\frac{\gamma-1}{\gamma}}$. Also,

$$\int_0^x \int_0^y \frac{\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} ds dt$$

$$\leq \frac{\alpha\gamma L(x, y; \alpha, \gamma)}{\alpha + \gamma} \left[\frac{1}{\alpha} \left\{ \int_0^x (x-s) \left[p(s)\Phi\left(\frac{f(s)}{p(s)}\right) \right]^\alpha ds \right\}^{\frac{\alpha+\gamma}{\alpha\gamma}} \right.$$

$$\left. + \frac{1}{\gamma} \left\{ \int_0^y (y-t) \left[q(t)\Psi\left(\frac{g(t)}{q(t)}\right) \right]^\gamma dt \right\}^{\frac{\alpha+\gamma}{\alpha\gamma}} \right],$$

where

$$L(x, y; \alpha, \gamma) = \frac{1}{\alpha + \gamma} \left\{ \int_0^x \left[\frac{\Phi(P(s))}{P(s)} \right]^{\frac{\alpha}{\alpha-1}} ds \right\}^{\frac{\alpha-1}{\alpha}} \left\{ \int_0^y \left[\frac{\Psi(Q(t))}{Q(t)} \right]^{\frac{\gamma}{\gamma-1}} dt \right\}^{\frac{\gamma-1}{\gamma}}.$$

The following theorems deal with slight variants of (3.8) given in Theorem 3.2. Before we state our next theorem, we point out that “ $F(s) = \int_0^s f(\sigma)d\sigma$ and $G(t) =$



[Title Page](#)

[Contents](#)



Page 25 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

$\int_0^t g(\tau)d\tau$ ” are replaced by “ $F(s) = \frac{1}{s} \int_0^s f(\sigma)d\sigma$ and $G(t) = \frac{1}{t} \int_0^t g(\tau)d\tau$ ” in Theorem 3.4 in [9].

Theorem 3.3. Let $\alpha > 1, \gamma > 1$ and $f(\sigma) \geq 0, g(\tau) \geq 0$ for $\sigma \in (0, x), \tau \in (0, y)$, where x, y are positive real numbers. Define $F(s) = \frac{1}{s} \int_0^s f(\sigma)d\sigma, G(t) = \frac{1}{t} \int_0^t g(\tau)d\tau$ for $s \in (0, x), t \in (0, y)$. Let Φ and Ψ be two real-valued, nonnegative, convex functions defined on $\mathbb{R}_+ = [0, \infty)$. Then

$$\int_0^x \int_0^y \frac{st\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} dsdt$$

$$\leq D(1, 1, x, y; \alpha, \gamma) \left\{ \int_0^x (x-s)\Phi^\alpha(f(s))ds \right\}^{\frac{1}{\alpha}} \left\{ \int_0^y (y-t)\Psi^\gamma(g(t))dt \right\}^{\frac{1}{\gamma}},$$

where $D(1, 1, x, y; \alpha, \gamma) = \frac{1}{\alpha+\gamma} x^{\frac{\alpha-1}{\alpha}} y^{\frac{\gamma-1}{\gamma}}$.

Theorem 3.4. Let $\alpha > 1, \gamma > 1$ and $f(\sigma) \geq 0, g(\tau) \geq 0, p(\sigma) > 0$ and $q(\tau) > 0$ for $\sigma \in (0, x), \tau \in (0, y)$, where x, y are positive real numbers. Define $P(s) = \int_0^s p(\sigma)d\sigma, Q(t) = \int_0^t q(\tau)d\tau, F(s) = \frac{1}{P(s)} \int_0^s p(\sigma)f(\sigma)d\sigma$ and $G(t) = \frac{1}{Q(t)} \int_0^t q(\tau)g(\tau)d\tau$ for $s \in (0, x), t \in (0, y)$. Let Φ and Ψ be two real-valued, nonnegative, convex functions defined on $\mathbb{R}_+ = [0, \infty)$. Then

$$\int_0^x \int_0^y \frac{P(s)Q(t)\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} dsdt$$

$$\leq D(1, 1, x, y; \alpha, \gamma) \left\{ \int_0^x (x-s) [p(s)\Phi(f(s))]^\alpha ds \right\}^{\frac{1}{\alpha}}$$

$$\times \left\{ \int_0^y (y-t) [q(t)\Psi(g(t))]^\gamma dt \right\}^{\frac{1}{\gamma}},$$

where $D(1, 1, k, r; \alpha, \gamma) = \frac{1}{\alpha+\gamma} x^{\frac{\alpha-1}{\alpha}} y^{\frac{\gamma-1}{\gamma}}$.

The proofs of Theorems 3.3 and 3.4 are similar to the proof of Theorem 3.2 and similar to the proofs of Theorems 2.3 and 2.4. Hence, we leave out the details.



Hilbert-Pachpatte's Inequalities

Wengui Yang

vol. 10, iss. 1, art. 26, 2009

[Title Page](#)

[Contents](#)



Page 26 of 28

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
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issn: 1443-5756

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Hilbert-Pachpatte's Inequalities

Wengui Yang

vol. 10, iss. 1, art. 26, 2009

Title Page

Contents



Page 27 of 28

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



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Hilbert-Pachpatte's Inequalities

Wengui Yang

vol. 10, iss. 1, art. 26, 2009

Title Page

Contents



Page 28 of 28

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756