

## STRONGLY NONLINEAR ELLIPTIC UNILATERAL PROBLEMS IN ORLICZ SPACE AND $L^1$ DATA

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## Abstract

In this paper, we shall be concerned with the existence result of Unilateral problem associated to the equations of the form,

$$Au + g(x, u, \nabla u) = f,$$

where  $A$  is a Leray-Lions operator from its domain  $D(A) \subset W_0^1 L_M(\Omega)$  into  $W^{-1} E_{\overline{M}}(\Omega)$ . On the nonlinear lower order term  $g(x, u, \nabla u)$ , we assume that it is a Carathéodory function having natural growth with respect to  $|\nabla u|$ , and satisfies the sign condition. The right hand side  $f$  belongs to  $L^1(\Omega)$ .

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# 1. Introduction

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^N$ ,  $N \geq 2$ , with segment property. Let us consider the following nonlinear Dirichlet problem

$$(1.1) \quad -\operatorname{div}(a(x, u, \nabla u)) + g(x, u, \nabla u) = f,$$

where  $f \in L^1(\Omega)$ ,  $Au = -\operatorname{div} a(x, u, \nabla u)$  is a Leray-Lions operator defined on its domain  $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$ , with  $M$  an  $N$ -function and where  $g$  is a nonlinearity with the "natural" growth condition:

$$|g(x, s, \xi)| \leq b(|s|)(h(x) + M(|\xi|))$$

and which satisfies the classical sign condition

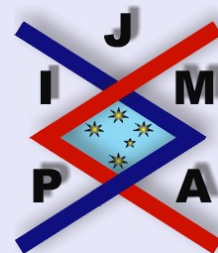
$$g(x, s, \xi) \cdot s \geq 0.$$

In the case where  $f \in W^{-1}E_{\overline{M}}(\Omega)$ , an existence theorem has been proved in [14] with the nonlinearities  $g$  depends only on  $x$  and  $u$ , and in [4] where  $g$  depends also the  $\nabla u$ .

For the case where  $f \in L^1(\Omega)$ , the authors in [5] studied the problem (1.1), with the added assumption of exact natural growth

$$|g(x, s, \xi)| \geq \beta M(|\xi|) \text{ for } |s| \geq \mu$$

and in [6] no coercivity condition is assumed on  $g$  but the result is restricted to the  $N$ -function,  $M$  satisfying a  $\Delta_2$ -condition, while in [11] the authors were concerned about the above problem without assuming a  $\Delta_2$ -condition on  $M$ .



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The purpose of this paper is to prove an existence result for unilateral problems associated to (1.1) without assuming the  $\Delta_2$ -condition in the setting of the Orlicz-Sobolev space.

Further work for the equation (1.1) in the  $L^p$  case where there is no restriction can be found in [17], and in [12, 9, 8] in the case of obstacle problems, see also [18].



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## 2. Preliminaries

Let  $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be an  $N$ -function, i.e.  $M$  is continuous, convex, with  $M(t) > 0$  for  $t > 0$ ,  $\frac{M(t)}{t} \rightarrow 0$  as  $t \rightarrow 0$  and  $\frac{M(t)}{t} \rightarrow \infty$  as  $t \rightarrow \infty$ .

Equivalently,  $M$  admits the representation:  $M(t) = \int_0^t a(s)ds$  where  $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is nondecreasing, right continuous, with  $a(0) = 0$ ,  $a(t) > 0$  for  $t > 0$  and  $a(t)$  tends to  $\infty$  as  $t \rightarrow \infty$ .

The  $N$ -function  $\bar{M}$  conjugate to  $M$  is defined by  $\bar{M} = \int_0^t \bar{a}(s)ds$ , where  $\bar{a} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is given by  $\bar{a}(t) = \sup\{s : a(s) \leq t\}$ .

The  $N$ -function  $M$  is said to satisfy the  $\Delta_2$ -condition if, for some  $k$

$$(2.1) \quad M(2t) \leq kM(t), \quad \forall t \geq 0.$$

When (2.1) holds only for  $t \geq$  some  $t_0 > 0$  then  $M$  is said to satisfy the  $\Delta_2$ -condition near infinity.

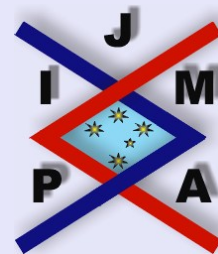
We will extend these  $N$ -functions to even functions on all  $\mathbb{R}$ , i.e.  $M(t) = M(|t|)$  if  $t \leq 0$ .

Moreover, we have the following Young's inequality

$$\forall s, t \geq 0, \quad st \leq M(t) + \bar{M}(s).$$

Let  $P$  and  $Q$  be two  $N$ -functions.  $P \ll Q$  means that  $P$  grows essentially less rapidly than  $Q$ , i.e., for each  $\epsilon > 0$ ,  $\frac{P(t)}{Q(\epsilon t)} \rightarrow 0$  as  $t \rightarrow \infty$ . This is the case if and only if  $\lim_{t \rightarrow \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$ .

Let  $\Omega$  be an open subset of  $\mathbb{R}^N$ . The Orlicz class  $K_M(\Omega)$  (resp. the Orlicz space  $L_M(\Omega)$ ) is defined as the set of (equivalence classes of) real valued



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measurable functions  $u$  on  $\Omega$  such that

$$\int_{\Omega} M(u(x))dx < +\infty \quad \left( \text{resp. } \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0 \right).$$

$L_M(\Omega)$  is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0, \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) dx \leq 1 \right\}$$

and  $K_M(\Omega)$  is a convex subset of  $L_M(\Omega)$ .

The closure in  $L_M(\Omega)$  of the set of bounded measurable functions with compact support in  $\bar{\Omega}$  is denoted by  $E_M(\Omega)$ .

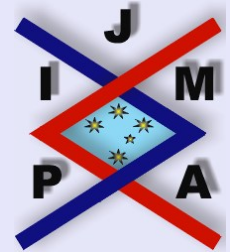
The dual of  $E_M(\Omega)$  can be identified with  $L_{\bar{M}}(\Omega)$  by means of the pairing  $\int_{\Omega} uv \, dx$ , and the dual norm of  $L_{\bar{M}}(\Omega)$  is equivalent to  $\|\cdot\|_{\bar{M},\Omega}$ .

We now turn to the Orlicz-Sobolev space,  $W^1L_M(\Omega)$  (resp.  $W^1E_M(\Omega)$ ) is the space of all functions  $u$  such that  $u$  and its distributional derivatives of order 1 lie in  $L_M(\Omega)$  (resp.  $E_M(\Omega)$ ). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_M.$$

Thus,  $W^1L_M(\Omega)$  and  $W^1E_M(\Omega)$  can be identified with subspaces of the product of  $N + 1$  copies of  $L_M(\Omega)$ . Denoting this product by  $\prod L_M$ , we will use the weak topologies  $\sigma(\prod L_M, \prod E_{\bar{M}})$  and  $\sigma(\prod L_M, \prod L_{\bar{M}})$ .

The space  $W_0^1E_M(\Omega)$  is defined as the (norm) closure of the Schwartz space  $D(\Omega)$  in  $W^1E_M(\Omega)$  and the space  $W_0^1L_M(\Omega)$  as the  $\sigma(\prod L_M, \prod E_{\bar{M}})$  closure of  $D(\Omega)$  in  $W^1L_M(\Omega)$ .



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Let  $W^{-1}L_{\overline{M}}(\Omega)$  (resp.  $W^{-1}E_{\overline{M}}(\Omega)$ ) denote the space of distributions on  $\Omega$  which can be written as sums of derivatives of order  $\leq 1$  of functions in  $L_{\overline{M}}(\Omega)$  (resp.  $E_{\overline{M}}(\Omega)$ ). It is a Banach space under the usual quotient norm (for more details see [1]).

We recall some lemmas introduced in [4] which will be used later.

**Lemma 2.1.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly Lipschitzian, with  $F(0) = 0$ . Let  $M$  be an  $N$ -function and let  $u \in W^1L_M(\Omega)$  (resp.  $W^1E_M(\Omega)$ ). Then  $F(u) \in W^1L_M(\Omega)$  ( resp.  $W^1E_M(\Omega)$ ). Moreover, if the set  $D$  of discontinuity points of  $F'$  is finite, then*

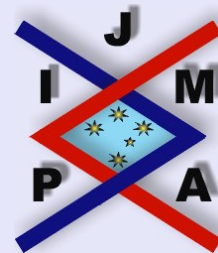
$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

**Lemma 2.2.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly Lipschitzian, with  $F(0) = 0$ . We assume that the set of discontinuity points of  $F'$  is finite. Let  $M$  be an  $N$ -function, then the mapping  $F : W^1L_M(\Omega) \rightarrow W^1L_M(\Omega)$  is sequentially continuous with respect to the weak\* topology  $\sigma(\prod L_M, \prod E_{\overline{M}})$ .*

We give now the following lemma which concerns operators of Nemytskii type in Orlicz spaces (see [4]).

**Lemma 2.3.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^N$  with finite measure. Let  $M, P$  and  $Q$  be  $N$ -functions such that  $Q \ll P$ , and let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that, for a.e.  $x \in \Omega$  and all  $s \in \mathbb{R}$ :*

$$|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2 |s|),$$



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where  $k_1, k_2$  are real constants and  $c(x) \in E_Q(\Omega)$ . Then the Nemytskii operator  $N_f$  defined by  $N_f(u)(x) = f(x, u(x))$  is strongly continuous from

$$\mathcal{P} \left( E_M(\Omega), \frac{1}{k_2} \right) = \left\{ u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2} \right\}$$

into  $E_Q(\Omega)$ .

We define  $\mathcal{T}_0^{1,M}(\Omega)$  to be the set of measurable functions  $u : \Omega \rightarrow \mathbb{R}$  such that  $T_k(u) \in W_0^1 L_M(\Omega)$ , where  $T_k(s) = \max(-k, \min(k, s))$  for  $s \in \mathbb{R}$  and  $k \geq 0$ . We give the following lemma which is a generalization of Lemma 2.1 of [2] in Orlicz spaces.

**Lemma 2.4.** For every  $u \in \mathcal{T}_0^{1,M}(\Omega)$ , there exists a unique measurable function  $v : \Omega \rightarrow \mathbb{R}^N$  such that

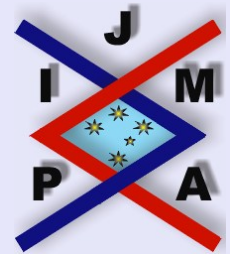
$$\nabla T_k(u) = v \chi_{\{|u| < k\}}, \text{ almost everywhere in } \Omega \text{ for every } k > 0.$$

We will define the gradient of  $u$  as the function  $v$ , and we will denote it by  $v = \nabla u$ .

**Lemma 2.5.** Let  $\lambda \in \mathbb{R}$  and let  $u$  and  $v$  be two measurable functions defined on  $\Omega$  which are finite almost everywhere, and which are such that  $T_k(u)$ ,  $T_k(v)$  and  $T_k(u + \lambda v)$  belong to  $W_0^1 L_M(\Omega)$  for every  $k > 0$  then

$$\nabla(u + \lambda v) = \nabla(u) + \lambda \nabla(v) \quad \text{a.e. in } \Omega$$

where  $\nabla(u)$ ,  $\nabla(v)$  and  $\nabla(u + \lambda v)$  are the gradients of  $u$ ,  $v$  and  $u + \lambda v$  introduced in Lemma 2.4.



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The proof of this lemma is similar to the proof of Lemma 2.12 in [10] for the  $L^p$  case.

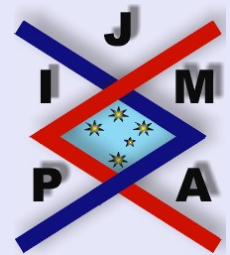
Below, we will use the following technical lemma.

**Lemma 2.6.** *Let  $(f_n), f \in L^1(\Omega)$  such that*

(i)  $f_n \geq 0$  a.e. in  $\Omega$

(ii)  $f_n \rightarrow f$  a.e. in  $\Omega$

(iii)  $\int_{\Omega} f_n(x)dx \rightarrow \int_{\Omega} f(x)dx$   
then  $f_n \rightarrow f$  strongly in  $L^1(\Omega)$ .



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### 3. Main Results

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^N$ ,  $N \geq 2$ , with the segment property.

Given an obstacle function  $\psi : \Omega \rightarrow \mathbb{R}$ , we consider

$$(3.1) \quad K_\psi = \{u \in W_0^1 L_M(\Omega); u \geq \psi \text{ a.e. in } \Omega\},$$

this convex set is sequentially  $\sigma(\Pi L_M, \Pi E_{\overline{M}})$  closed in  $W_0^1 L_M(\Omega)$  (see [15]).

We now state conditions on the differential operator

$$(3.2) \quad Au = -\operatorname{div}(a(x, u, \nabla u))$$

(A<sub>1</sub>)  $a(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a Carathéodory function.

(A<sub>2</sub>) There exist two  $N$ -functions  $M$  and  $P$  with  $P \ll M$ , function  $c(x)$  in  $E_{\overline{M}}(\Omega)$ , constants  $k_1, k_2, k_3, k_4$  such that, for a.e.  $x$  in  $\Omega$  and for all  $s \in \mathbb{R}, \zeta \in \mathbb{R}^N$

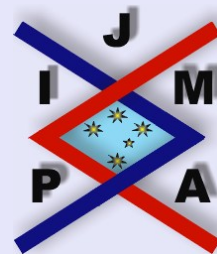
$$|a(x, s, \zeta)| \leq c(x) + k_1 \overline{P}^{-1} M(k_2 |s|) + k_3 \overline{M}^{-1} M(k_4 |\zeta|).$$

(A<sub>3</sub>)  $[a(x, s, \zeta) - a(x, s, \zeta')](\zeta - \zeta') > 0$  for a.e.  $x$  in  $\Omega$ ,  $s$  in  $\mathbb{R}$  and  $\zeta, \zeta'$  in  $\mathbb{R}^N$ , with  $\zeta \neq \zeta'$ .

(A<sub>4</sub>) There exist  $\delta(x)$  in  $L^1(\Omega)$ , strictly positive constant  $\alpha$  such that, for some fixed element  $v_0$  in  $K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ ,

$$a(x, s, \zeta)(\zeta - Dv_0) \geq \alpha M(|\zeta|) - \delta(x)$$

for a.e.  $x$  in  $\Omega$ , and all  $s \in \mathbb{R}, \zeta \in \mathbb{R}^N$ .



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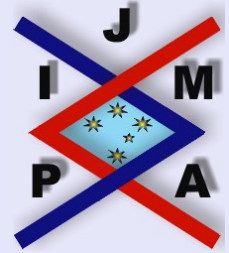


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(A<sub>5</sub>) For each  $v \in K_\psi \cap L^\infty(\Omega)$  there exists a sequence  $v_n \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$  such that  $v_n \rightarrow u$  for the modular convergence.

Furthermore let  $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  be a Carathéodory function such that for a.e.  $x \in \Omega$  and for all  $s \in \mathbb{R}, \zeta \in \mathbb{R}^N$

$$(G_1) \quad g(x, s, \zeta)s \geq 0$$

$$(G_2) \quad |g(x, s, \zeta)| \leq b(|s|) (h(x) + M(|\zeta|)),$$

where  $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous non decreasing function, and  $h$  is a given nonnegative function in  $L^1(\Omega)$ .

Consider the following Dirichlet problem:

$$(3.3) \quad A(u) + g(x, u, \nabla u) = f \quad \text{in } \Omega.$$

**Remark 1.** *The condition (A<sub>5</sub>) holds if one of the following conditions is verified.*

1. *There exist  $\bar{\psi} \in K_\psi$  such that  $\psi - \bar{\psi}$  is continuous in  $\Omega$ , (see [15, Proposition 9]).*
2.  *$\psi \in W_0^1 E_M(\Omega)$ , (see [15, Proposition 10]).*

We shall prove the following existence theorem.

**Theorem 3.1.** Assume that  $(A_1) - (A_5)$ ,  $(G_1)$  and  $(G_2)$  hold and  $f \in L^1(\Omega)$ . Then there exists at least one solution of the following unilateral problem,

$$(P) \begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega), u \geq \psi \text{ a.e. in } \Omega, \\ g(x, u, \nabla u) \in L^1(\Omega) \\ \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - v) dx + \int_{\Omega} g(x, u, \nabla u) T_k(u - v) dx \\ \qquad \qquad \qquad \leq \int_{\Omega} f T_k(u - v) dx, \\ \forall v \in K_{\psi} \cap L^{\infty}(\Omega), \forall k > 0. \end{cases}$$




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## 4. Proof of Theorem 3.1

To prove the existence theorem, we proceed by steps.

**STEP 1:** *Approximate unilateral problems.*

Let us define

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|}$$

and let us consider the approximate unilateral problems:

$$(P_n) \quad \begin{cases} u_n \in K_\psi \cap D(A), \\ \langle Au_n, u_n - v \rangle + \int_\Omega g_n(x, u_n, \nabla u_n)(u_n - v) dx \\ \qquad \qquad \qquad \leq \int_\Omega f_n(u_n - v) dx, \\ \forall v \in K_\psi. \end{cases}$$

where  $f_n$  is a regular function such that  $f_n$  strongly converges to  $f$  in  $L^1(\Omega)$ .

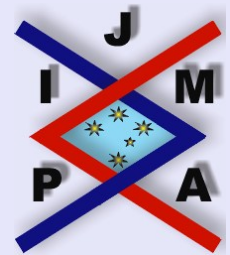
From Gossez and Mustonen ([15, Proposition 5]), the problem  $(P_n)$  has at least one solution.

**STEP 2:** *A priori estimates.*

Let  $k \geq \|v_0\|_\infty$  and let  $\varphi_k(s) = se^{\gamma s^2}$ , where  $\gamma = \left(\frac{b(k)}{\alpha}\right)^2$ .

It is well known that

$$(4.1) \quad \varphi'_k(s) - \frac{b(k)}{\alpha} |\varphi_k(s)| \geq \frac{1}{2}, \quad \forall s \in \mathbb{R} \quad (\text{see [9]}).$$



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Taking  $u_n - \eta\varphi_k(T_l(u_n - v_0))$  as test function in  $(P_n)$ , where  $l = k + \|v_0\|_\infty$ , we obtain,

$$\begin{aligned} \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_l(u_n - v_0) \varphi'_k(T_l(u_n - v_0)) dx \\ + \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_k(T_l(u_n - v_0)) dx \\ \leq \int_{\Omega} f_n \varphi_k(T_l(u_n - v_0)) dx. \end{aligned}$$

Since

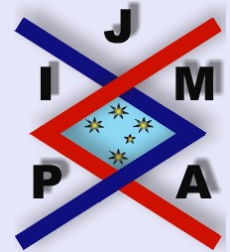
$$g_n(x, u_n, \nabla u_n) \varphi_k(T_l(u_n - v_0)) \geq 0$$

on the subset  $\{x \in \Omega : |u_n(x)| > k\}$ , then

$$\begin{aligned} \int_{\{|u_n - v_0| \leq l\}} a(x, u_n, \nabla u_n) \nabla (u_n - v_0) \varphi'_k(T_l(u_n - v_0)) dx \\ \leq \int_{\{|u_n| \leq k\}} |g_n(x, u_n, \nabla u_n)| |\varphi_k(T_l(u_n - v_0))| dx \\ + \int_{\Omega} f_n \varphi_k(T_l(u_n - v_0)) dx. \end{aligned}$$

By using  $(A_4)$  and  $(G_1)$ , we have

$$\begin{aligned} \alpha \int_{\{|u_n - v_0| \leq l\}} M(|\nabla u_n|) \varphi'_k(T_l(u_n - v_0)) dx \\ \leq b(|k|) \int_{\Omega} (h(x) + M(\nabla T_k(u_n))) |\varphi_k(T_l(u_n - v_0))| dx \end{aligned}$$



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$$+ \int_{\Omega} \delta(x) \varphi'_k(T_l(u_n - v_0)) dx + \int_{\Omega} f_n \varphi_k(T_l(u_n - v_0)) dx.$$

Since

$$\{x \in \Omega, |u_n(x)| \leq k\} \subseteq \{x \in \Omega : |u_n - v_0| \leq l\}$$

and the fact that  $h, \delta \in L^1(\Omega)$ , further  $f_n$  is bounded in  $L^1(\Omega)$ , then

$$\begin{aligned} \int_{\Omega} M(|\nabla T_k(u_n)|) \varphi'_k(T_l(u_n - v_0)) dx \\ \leq \frac{b(k)}{\alpha} \int_{\Omega} M(|\nabla T_k(u_n)|) |\varphi_k(T_l(u_n - v_0))| dx + c_k, \end{aligned}$$

where  $c_k$  is a positive constant depending on  $k$ , which implies that

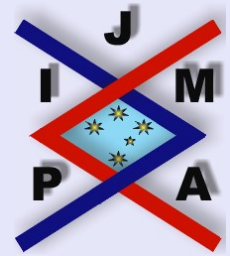
$$\int_{\Omega} M(|\nabla T_k(u_n)|) \left[ \varphi'_k(T_l(u_n - v_0)) - \frac{b(k)}{\alpha} |\varphi_k(T_l(u_n - v_0))| \right] dx \leq c_k.$$

By using (4.1), we deduce,

$$(4.2) \quad \int_{\Omega} M(|\nabla T_k(u_n)|) dx \leq 2c_k.$$

Since  $T_k(u_n)$  is bounded in  $W_0^1 L_M(\Omega)$ , there exists some  $v_k \in W_0^1 L_M(\Omega)$  such that

$$(4.3) \quad \begin{aligned} T_k(u_n) &\rightharpoonup v_k \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\prod L_M, \prod E_{\overline{M}}), \\ T_k(u_n) &\rightarrow v_k \text{ strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \end{aligned}$$



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**STEP 3:** Convergence in measure of  $u_n$

Let  $k_0 \geq \|v_0\|_\infty$  and  $k > k_0$ , taking  $v = u_n - T_k(u_n - v_0)$  as test function in  $(P_n)$  gives,

$$(4.4) \quad \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n - v_0) dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) T_k(u_n - v_0) dx \leq \int_{\Omega} f_n T_k(u_n - v_0) dx,$$

since  $g_n(x, u_n, \nabla u_n) T_k(u_n - v_0) \geq 0$  on the subset  $\{x \in \Omega, |u_n(x)| > k_0\}$ , hence (4.4) implies that,

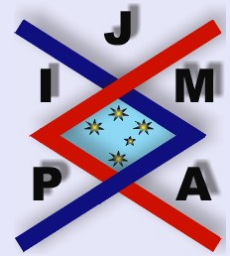
$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n - v_0) dx \leq k \int_{\{|u_n| \leq k_0\}} |g_n(x, u_n, \nabla u_n)| dx + k \|f\|_{L^1(\Omega)}$$

which gives, by using  $(G_1)$ ,

$$(4.5) \quad \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n - v_0) dx \leq kb(k_0) \left[ \int_{\Omega} |h(x)| dx + \int_{\Omega} M(|\nabla T_{k_0}(u_n)|) dx \right] + kc.$$

Combining (4.2) and (4.5), we have,

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n - v_0) dx \leq k[c_{k_0} + c].$$



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By  $(A_4)$ , we obtain,

$$\int_{\{|u_n - v_0| \leq k\}} M(|\nabla u_n|) dx \leq kc_1,$$

where  $c_1$  is independent of  $k$ , since  $k$  is arbitrary, we have

$$\int_{\{|u_n| \leq k\}} M(|\nabla u_n|) dx \leq \int_{\{|u_n - v_0| \leq k + \|v_0\|_\infty\}} M(|\nabla u_n|) dx \leq kc_2,$$

i.e.,

$$(4.6) \quad \int_{\Omega} M(|\nabla T_k(u_n)|) dx \leq kc_2.$$

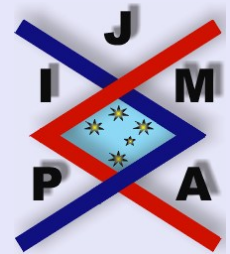
Now, we prove that  $u_n$  converges to some function  $u$  in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). We shall show that  $u_n$  is a Cauchy sequence in measure.

Let  $k > 0$  large enough, by Lemma 5.7 of [13], there exist two positive constants  $c_3$  and  $c_4$  such that

$$(4.7) \quad \int_{\Omega} M(c_3 T_k(u_n)) dx \leq c_4 \int_{\Omega} M(|\nabla T_k(u_n)|) dx \leq kc_5,$$

then, we deduce, by using (4.7) that

$$M(c_3 k) \text{ meas}\{|u_n| > k\} = \int_{\{|u_n| > k\}} M(c_3 T_k(u_n)) dx \leq c_5 k,$$



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hence

$$(4.8) \quad \text{meas}(|u_n| > k) \leq \frac{c_5 k}{M(c_3 k)} \quad \forall n, \forall k.$$

Letting  $k$  to infinity, we deduce that,  $\text{meas}\{|u_n| > k\}$  tends to 0 as  $k$  tends to infinity.

For every  $\lambda > 0$ , we have

$$(4.9) \quad \text{meas}(\{|u_n - u_m| > \lambda\}) \leq \text{meas}(\{|u_n| > k\}) + \text{meas}(\{|u_m| > k\}) \\ + \text{meas}(\{|T_k(u_n) - T_k(u_m)| > \lambda\}).$$

Consequently, by (4.3) we can assume that  $T_k(u_n)$  is a Cauchy sequence in measure in  $\Omega$ .

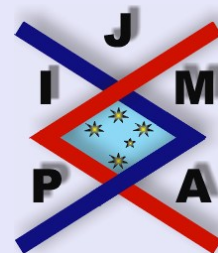
Let  $\epsilon > 0$  then, by (4.9) there exists some  $k(\epsilon) > 0$  such that  $\text{meas}(\{|u_n - u_m| > \lambda\}) < \epsilon$  for all  $n, m \geq h_0(k(\epsilon), \lambda)$ . This proves that  $(u_n)$  is a Cauchy sequence in measure in  $\Omega$ , thus converges almost everywhere to some measurable function  $u$ . Then

$$(4.10) \quad T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\prod L_M, \prod E_{\overline{M}}), \\ T_k(u_n) \rightarrow T_k(u) \text{ strongly in } E_M(\Omega) \text{ and a.e. in } \Omega.$$

**Step 4: Boundedness of  $(a(x, T_k(u_n)), \nabla T_k(u_n))_n$  in  $(L_{\overline{M}}(\Omega))^N$ .**

Let  $w \in (E_M(\Omega))^N$  be arbitrary, by  $(A_3)$  we have

$$(a(x, u_n, \nabla u_n) - a(x, u_n, w))(\nabla u_n - w) \geq 0,$$



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which implies that

$$a(x, u_n, \nabla u_n)(w - \nabla v_0) \leq a(x, u_n, \nabla u_n)(\nabla u_n - \nabla v_0) - a(x, u_n, w)(\nabla u_n - w)$$

and integrating on the subset  $\{x \in \Omega, |u_n - v_0| \leq k\}$ , we obtain,

$$(4.11) \quad \int_{\{|u_n - v_0| \leq k\}} a(x, u_n, \nabla u_n)(w - \nabla v_0) dx \\ \leq \int_{\{|u_n - v_0| \leq k\}} a(x, u_n, \nabla u_n)(\nabla u_n - \nabla v_0) dx \\ + \int_{\{|u_n - v_0| \leq k\}} a(x, u_n, w)(w - \nabla u_n) dx.$$

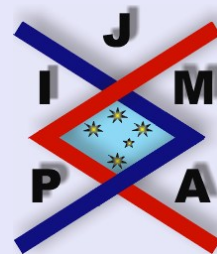
We claim that,

$$(4.12) \quad \int_{\{|u_n - v_0| \leq k\}} a(x, u_n, \nabla u_n)(\nabla u_n - v_0) dx \leq c_{10},$$

where  $c_{10}$  is a positive constant depending on  $k$ .

Indeed, if we take  $v = u_n - T_k(u_n - v_0)$  as test function in  $(P_n)$ , we get,

$$\int_{\{|u_n - v_0| \leq k\}} a(x, u_n, \nabla u_n)(\nabla u_n - \nabla v_0) dx \\ + \int_{\Omega} g_n(x, u_n, \nabla u_n) T_k(u_n - v_0) dx \\ \leq \int_{\Omega} f_n T_k(u_n - v_0) dx.$$



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Since  $g_n(x, u_n, \nabla u_n)T_k(u_n - v_0) \geq 0$  on the subset  $\{x \in \Omega, |u_n| \geq \|v_0\|_\infty\}$ , which implies

$$(4.13) \quad \int_{\{|u_n - v_0| \leq k\}} a(x, u_n, \nabla u_n)(\nabla u_n - \nabla v_0) dx \leq b(\|v_0\|_\infty) \left( \int_{\Omega} h(x) dx + \int_{\Omega} M(\nabla T_{\|v_0\|_\infty}(u_n)) dx + k\|f\|_{L^1(\Omega)} \right).$$

Combining (4.2) and (4.13), we deduce (4.12).

On the other hand, for  $\lambda$  large enough, we have by using  $(A_2)$

$$(4.14) \quad \int_{\{|u_n - v_0| \leq k\}} \overline{M} \left( \frac{a(x, u_n, w)}{\lambda} \right) dx \leq \overline{M} \left( \frac{c(x)}{\lambda} \right) + \frac{k_3}{\lambda} M(k_4|w|) + c \leq c_{11},$$

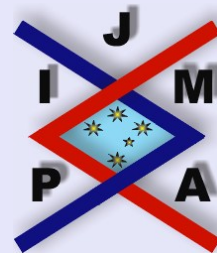
hence,  $|a(x, u_n, w)|$  bounded in  $L_{\overline{M}}(\Omega)$ , which implies that the second term of the right hand side of (4.11) is bounded

Consequently, we obtain,

$$(4.15) \quad \int_{\{|u_n - v_0| \leq k\}} a(x, u_n, \nabla u_n)(w - \nabla v_0) dx \leq c_{12},$$

with  $c_{12}$  is positive constant depending of  $k$ .

Hence, by the Theorem of Banach–Steinhaus, the sequence  $(a(x, u_n, \nabla u_n))_{\chi_{\{|u_n - v_0| \leq k\}}}_n$  remains bounded in  $(L_{\overline{M}}(\Omega))^N$ . Since  $k$  is arbitrary, we



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deduce that  $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$  is also bounded in  $(L_{\overline{M}}(\Omega))^N$ , which implies that, for all  $k > 0$  there exists a function  $h_k \in (L_{\overline{M}}(\Omega))^N$ , such that,

$$(4.16) \quad a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N$$

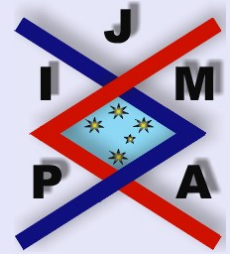
for  $\sigma(\Pi L_{\overline{M}}(\Omega), \Pi E_M(\Omega))$ .

**STEP 5:** *Almost everywhere convergence of the gradient.*

We fix  $k > \|v_0\|_\infty$ . Let  $\Omega_r = \{x \in \Omega, |\nabla T_k(u(x))| \leq r\}$  and denote by  $\chi_r$  the characteristic function of  $\Omega_r$ . Clearly,  $\Omega_r \subset \Omega_{r+1}$  and  $\text{meas}(\Omega \setminus \Omega_r) \rightarrow 0$  as  $r \rightarrow \infty$ .

Fix  $r$  and let  $s \geq r$ , we have,

$$(4.17) \quad 0 \leq \int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\ \times [\nabla T_k(u_n) - \nabla T_k(u)] dx \\ \leq \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\ \times [\nabla T_k(u_n) - \nabla T_k(u)] dx \\ = \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \\ \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] dx \\ \leq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \\ \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] dx.$$



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By  $(A_5)$  there exists a sequence  $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$  which converges to  $T_k(u)$  for the modular converge in  $W_0^1 L_M(\Omega)$ .

Here, we define

$$\begin{aligned} w_{n,j}^h &= T_{2k}(u_n - v_0 - T_h(u_n - v_0) + T_k(u_n) - T_k(v_j)), \\ w_j^h &= T_{2k}(u - v_0 - T_h(u - v_0) + T_k(u) - T_k(v_j)) \end{aligned}$$

and

$$w^h = T_{2k}(u - v_0 - T_h(u - v_0)),$$

where  $h > 2k > 0$ .

For  $\eta = \exp(-4\gamma k^2)$ , we defined the following function as

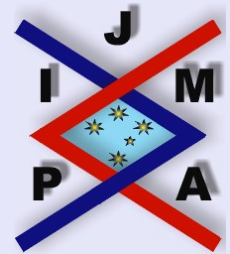
$$(4.18) \quad v_{n,j}^h = u_n - \eta \varphi_k(w_{n,j}^h).$$

We take  $v_{n,j}^h$  as test function in  $(P_n)$ , we obtain,

$$\langle A(u_n), \eta \varphi_k(w_{n,j}^h) \rangle + \int_{\Omega} g_n(x, u_n, \nabla u_n) \eta \varphi_k(w_{n,j}^h) dx \leq \int_{\Omega} f_n \eta \varphi_k(w_{n,j}^h) dx.$$

Which, implies that

$$(4.19) \quad \begin{aligned} \langle A(u_n), \varphi_k(w_{n,j}^h) \rangle + \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h) dx \\ \leq \int_{\Omega} f_n \varphi_k(w_{n,j}^h) dx. \end{aligned}$$



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It follows that

$$(4.20) \quad \int_{\Omega} a(x, u_n, \nabla u_n) \nabla w_{n,j}^h \varphi_k'(w_{n,j}^h) dx \\ + \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h) dx \leq \int_{\Omega} f_n \varphi_k(w_{n,j}^h) dx.$$

Note that,  $\nabla w_{n,j}^h = 0$  on the set where  $|u_n| > h + 5k$ , therefore, setting  $m = 5k + h$ , and denoting by  $\epsilon(n, j, h)$  any quantity such that

$$\lim_{h \rightarrow +\infty} \lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \epsilon(n, j, h) = 0.$$

If the quantity we consider does not depend on one parameter among  $n, j$  and  $h$ , we will omit the dependence on the corresponding parameter: as an example,  $\epsilon(n, h)$  is any quantity such that

$$\lim_{h \rightarrow +\infty} \lim_{n \rightarrow +\infty} \epsilon(n, h) = 0.$$

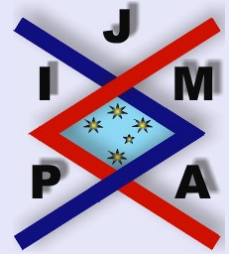
Finally, we will denote (for example) by  $\epsilon_h(n, j)$  a quantity that depends on  $n, j, h$  and is such that

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \epsilon_h(n, j) = 0$$

for any fixed value of  $h$ .

We get, by (4.20),

$$\int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_{n,j}^h \varphi_k'(w_{n,j}^h) dx \\ + \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h) dx \leq \int_{\Omega} f_n \varphi_k(w_{n,j}^h) dx,$$



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In view of (4.10), we have  $\varphi_k(w_{n,j}^h) \rightarrow \varphi_k(w_j^h)$  weakly\* as  $n \rightarrow +\infty$  in  $L^\infty(\Omega)$ , and then

$$\int_{\Omega} f_n \varphi_k(w_{n,j}^h) dx \rightarrow \int_{\Omega} f \varphi_k(w_j^h) dx \quad \text{as } n \rightarrow +\infty.$$

Again tends  $j$  to infinity, we get

$$\int_{\Omega} f \varphi_k(w_j^h) dx \rightarrow \int_{\Omega} f \varphi_k(w^h) dx \quad \text{as } j \rightarrow +\infty,$$

finally letting  $h$  the infinity, we deduce by using the Lebesgue Theorem that  $\int_{\Omega} f \varphi_k(w^h) dx \rightarrow 0$ .

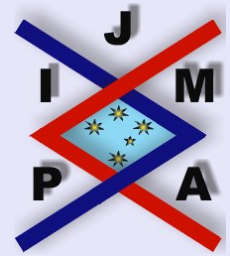
So that

$$\int_{\Omega} f_n \varphi_k(w_{n,j}^h) dx = \epsilon(n, j, h).$$

Since in the set  $\{x \in \Omega, |u_n(x)| > k\}$ , we have  $g(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h) \geq 0$ , we deduce from (4.20) that

$$(4.21) \quad \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_{n,j}^h \varphi_k'(w_{n,j}^h) dx + \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h) dx \leq \epsilon(n, j, h).$$

Splitting the first integral on the left hand side of (4.21) where  $|u_n| \leq k$  and



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$|u_n| > k$ , we can write,

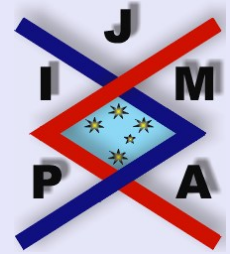
$$\begin{aligned}
 (4.22) \quad & \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_{n,j}^h \varphi'_k(w_{n,j}^h) dx \\
 &= \int_{\{|u_n| \leq k\}} a(x, T_m(u_n), \nabla T_m(u_n)) [\nabla T_k(u_n) - \nabla T_k(v_j)] \varphi'_k(w_{n,j}^h) dx \\
 &\quad + \int_{\{|u_n| > k\}} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_{n,j}^h \varphi'_k(w_{n,j}^h) dx.
 \end{aligned}$$

The first term of the right hand side of the last inequality can write as

$$\begin{aligned}
 (4.23) \quad & \int_{\{|u_n| \leq k\}} a(x, T_m(u_n), \nabla T_m(u_n)) [\nabla T_k(u_n) - \nabla T_k(v_j)] \varphi'_k(w_{n,j}^h) dx \\
 &\geq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(u_n) - \nabla T_k(v_j)] \varphi'_k(w_{n,j}^h) dx \\
 &\quad - \varphi'_k(2k) \int_{\{|u_n| > k\}} |a(x, T_k(u_n), 0)| |\nabla T_k(v_j)| dx.
 \end{aligned}$$

Recalling that,  $|a(x, T_k(u_n), 0)| \chi_{\{|u_n| > k\}}$  converges to  $|a(x, T_k(u), 0)| \chi_{\{|u| > k\}}$  strongly in  $L_{\overline{M}}(\Omega)$ , moreover, since  $|\nabla T_k(v_j)|$  modular converges to  $|\nabla T_k(u)|$ , then

$$-\varphi'_k(2k) \int_{\{|u_n| > k\}} |a(x, T_k(u_n), 0)| |\nabla T_k(v_j)| dx = \epsilon(n, j).$$



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For the second term of the right hand side of (4.14) we can write, using (A<sub>3</sub>)

$$(4.24) \quad \int_{\{|u_n|>k\}} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_{n,j}^h \varphi'_k(w_{n,j}^h) dx$$

$$\geq -\varphi'_k(2k) \int_{\{|u_n|>k\}} |a(x, T_m(u_n), \nabla T_m(u_n))| |\nabla T_k(v_j)| dx$$

$$- \varphi'(2k) \int_{\{|u_n-v_0|>h\}} \delta(x) dx.$$

Since  $|a(x, T_m(u_n), \nabla T_m(u_n))|$  is bounded in  $L_{\overline{M}}(\Omega)$ , we have, for a subsequence

$$|a(x, T_m(u_n), \nabla T_m(u_n))| \rightharpoonup l_m$$

weakly in  $L_{\overline{M}}(\Omega)$  in  $\sigma(L_{\overline{M}}, E_M)$  as  $n$  tends to infinity, and since

$$|\nabla T_k(v_j)| \chi_{\{|u_n|>k\}} \rightarrow |\nabla T_k(v_j)| \chi_{\{|u|>k\}}$$

strongly in  $E_M(\Omega)$  as  $n$  tends to infinity, we have

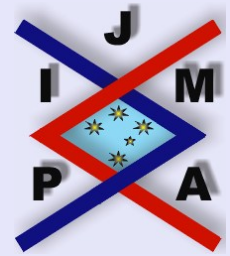
$$- \varphi'_k(2k) \int_{\{|u_n|>k\}} |a(x, T_m(u_n), \nabla T_m(u_n))| |\nabla T_k(v_j)| dx$$

$$\rightarrow -\varphi'_k(2k) \int_{\{|u|>k\}} l_m |\nabla T_k(v_j)| dx$$

as  $n$  tends to infinity.

Using now, the modular convergence of  $(v_j)$ , we get

$$-\varphi'_k(2k) \int_{\{|u|>k\}} l_m |\nabla T_k(v_j)| dx \rightarrow -\varphi'_k(2k) \int_{\{|u|>k\}} l_m |\nabla T_k(u)| dx = 0$$



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as  $j$  tends to infinity.

Finally

$$(4.25) \quad -\varphi'_k(2k) \int_{\{u_n > k\}} |a(x, T_m(u_n), \nabla T_m(u_n))| |\nabla T_k(v_j)| dx = \epsilon_h(n, j).$$

On the other hand, since  $\delta \in L^1(\Omega)$  it is easy to see that

$$(4.26) \quad -\varphi'_k(2k) \int_{\{u_n - v_0 > h\}} \delta(x) dx = \epsilon(n, h).$$

Combining (4.23) – (4.26), we deduce

$$(4.27) \quad \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_{n,j}^h \varphi'_k(w_{n,j}^h) dx \\ \geq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(u_n) - \nabla T_k(v_j)] \varphi'_k(w_{n,j}^h) dx \\ + \epsilon(n, h) + \epsilon(n, j) + \epsilon_h(n, j),$$

which implies that

$$(4.28) \quad \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_{n,j}^h \varphi'_k(w_{n,j}^h) dx \\ \geq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j)] \\ \times [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^j] \varphi'_k(w_{n,j}^h) dx$$



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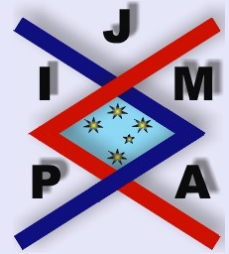


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$$\begin{aligned}
 &+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j) [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^j] \varphi'_k(w_{n,j}^h) dx \\
 &\quad - \int_{\Omega \setminus \Omega_s^j} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \varphi'_k(w_{n,j}^h) dx \\
 &\qquad\qquad\qquad + \epsilon(n, h) + \epsilon(n, j) + \epsilon_h(n, j),
 \end{aligned}$$

where  $\chi_s^j$  denotes the characteristic function of the subset  $\Omega_s^j = \{x \in \Omega : |\nabla T_k(v_j)| \leq s\}$ .

By (4.16) and the fact that  $\nabla T_k(v_j) \chi_{\Omega \setminus \Omega_s^j} \varphi'_k(w_{n,j}^h)$  tends to  $\nabla T_k(v_j) \chi_{\Omega \setminus \Omega_s^j} \varphi'_k(w_j^h)$  strongly in  $(E_M(\Omega))^N$ , the third term on the right hand side of (4.28) tends to the quantity

$$\int_{\Omega \setminus \Omega_s^j} h_k \nabla T_k(v_j) \varphi'_k(w_j^h) dx$$

as  $n$  tends to infinity.

Letting now  $j$  tend to infinity, by using the modular convergence of  $v_j$ , we have

$$\int_{\Omega} h_k \nabla T_k(v_j) \chi_{\Omega \setminus \Omega_s^j} \varphi'_k(w_j^h) dx \rightarrow \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) \varphi'_k(w^h) dx$$

as  $j$  tends to infinity.

Finally

$$(4.29) \quad - \int_{\Omega \setminus \Omega_s^j} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \varphi'_k(w_{n,j}^h) dx$$

$$= - \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) \varphi'_k(w^h) dx + \epsilon_h(n, j).$$

Concerning the second term on the right hand side of (4.28) we can write

$$(4.30) \quad \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j) [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^j] \varphi'_k(w_{n,j}^h) dx$$

$$= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j) \nabla T_k(u_n) \varphi'_k(T_k(u_n) - T_k(v_j)) dx$$

$$- \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j) \nabla T_k(v_j) \chi_s^j \varphi'_k(w_{n,j}^h) dx.$$

The first term on the right hand side of (4.30) tends to the quantity

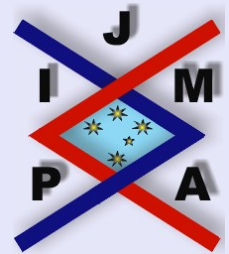
$$\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_s^j) \nabla T_k(u) \varphi'_k(T_k(u) - T_k(v_j)) dx \text{ as } n \rightarrow \infty$$

since

$$a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j) \varphi'_k(T_k(u_n) - T_k(v_j))$$

$$\rightarrow a(x, T_k(u), \nabla T_k(v_j) \chi_s^j) \varphi'_k(T_k(u) - T_k(v_j))$$

strongly in  $(E_{\overline{M}}(\Omega))^N$  by Lemma 2.3 and  $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$  weakly in  $(L_M(\Omega))^N$  for  $\sigma(\prod L_M, \prod E_{\overline{M}})$ .



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For the second term on the right hand side of (4.30) it is easy to see that

$$(4.31) \quad \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j) \nabla T_k(v_j)\chi_s^j \varphi'_k(w_{n,j}^h) dx \\ \longrightarrow \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j)\chi_s^j) \nabla T_k(v_j)\chi_s^j \varphi'_k(w_j^h) dx.$$

as  $n \rightarrow \infty$ .

Consequently, we have

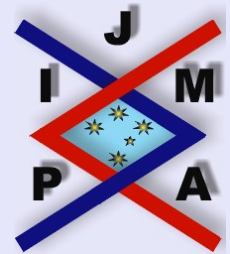
$$(4.32) \quad \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j) [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] \varphi'_k(w_{n,j}^h) dx \\ = \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j)\chi_s^j) [\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j] \varphi'_k(w_j^h) dx \\ + \epsilon_{j,h}(n)$$

since

$$\nabla T_k(v_j)\chi_s^j \varphi'_k(w_j^h) \rightarrow \nabla T_k(u)\chi_s \varphi'_k(w^h)$$

strongly in  $(E_M(\Omega))^N$  as  $j \rightarrow +\infty$ , it is easy to see that

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j)\chi_s^j) [\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j] \varphi'_k(w_j^h) dx \\ \longrightarrow \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi'_k(w^h) dx$$



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as  $j \rightarrow +\infty$ , thus

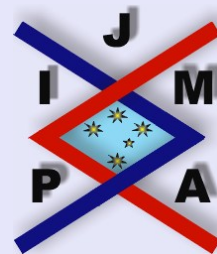
$$(4.33) \quad \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j) [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] \varphi'_k(w_{n,j}^h) dx \\ = \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi'_k(0) dx + \epsilon(n, j).$$

Combining (4.28), (4.29) and (4.32), we get

$$(4.34) \quad \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_{n,j}^h \varphi'_k(w_{n,j}^h) dx \\ \geq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)] \\ \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] \varphi'_k(w_{n,j}^h) dx \\ - \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) \varphi'_k(0) dx \\ + \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi'_k(0) dx + \epsilon(n, j, h).$$

We now, turn to the second term on the left hand side of (4.21), we have

$$(4.35) \quad \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h) dx \\ \leq b(k) \int_{\Omega} (h(x) + M(|\nabla T_k(u_n)|)) |\varphi_k(w_{n,j}^h)| dx$$



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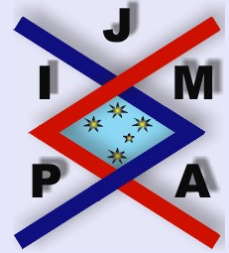


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$$\begin{aligned}
 &\leq b(k) \int_{\Omega} h(x) |\varphi_k(w_{n,j}^h)| dx + \frac{b(k)}{\alpha} \int_{\Omega} \delta(x) |\varphi_k(w_{n,j}^h)| dx \\
 &\quad + \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\varphi_k(w_{n,j}^h)| dx \\
 &\quad - \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla v_0 |\varphi_k(w_{n,j}^h)| dx \\
 &\leq \epsilon(n, j, h) \\
 &\quad + \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\varphi_k(w_{n,j}^h)| dx.
 \end{aligned}$$

The last term on the last side of this inequality reads as

$$\begin{aligned}
 (4.36) \quad &\frac{b(k)}{\alpha} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j)] \\
 &\times [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^j] |\varphi_k(w_{n,j}^h)| dx \\
 &\quad + \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j) \\
 &\quad \times [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^j] |\varphi_k(w_{n,j}^h)| dx \\
 &\quad + \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_s^j |\varphi_k(w_{n,j}^h)| dx
 \end{aligned}$$

and reasoning as above, it is easy to see that

$$\begin{aligned}
 &\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j) [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^j] |\varphi_k(w_{n,j}^h)| dx \\
 &= \epsilon(n, j)
 \end{aligned}$$



and

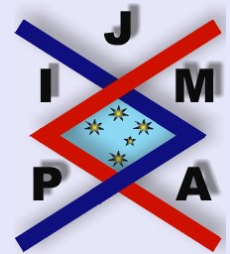
$$-\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_s^j |\varphi_k(w_{n,j}^h)| dx = \epsilon(n, j, h).$$

So that

$$(4.37) \quad \left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h) dx \right| \leq \frac{b(k)}{\alpha} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j)] \times [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^j] |\varphi_k(w_{n,j}^h)| dx + \epsilon(n, j, h).$$

Combining (4.21), (4.34) and (4.37), we obtain

$$(4.38) \quad \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^j)] \times [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^j] \left( \varphi_k'(w_{n,j}^h) - \frac{b(k)}{\alpha} |\varphi_k(w_{n,j}^h)| \right) dx \leq \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) \varphi_k'(0) dx + \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi_k'(0) dx + \epsilon(n, j, h),$$



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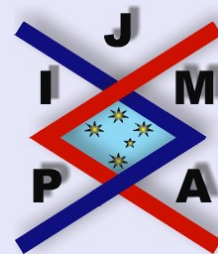
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which implies that, by using (4.1)

$$\begin{aligned}
 (4.39) \quad & \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)] \\
 & \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] dx \\
 & \leq 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) \varphi'_k(0) dx \\
 & \quad + 2 \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi'_k(0) dx + \epsilon(n, j, h).
 \end{aligned}$$

Now, remark that

$$\begin{aligned}
 (4.40) \quad & \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \\
 & \quad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] dx \\
 & = \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)] \\
 & \quad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] dx \\
 & + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j) [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] dx \\
 & - \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi_s) [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] dx \\
 & \quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(v_j)\chi_s^j - \nabla T_k(u)\chi_s] dx.
 \end{aligned}$$



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We shall pass to the limit in  $n$  and  $j$  in the last three terms of the right hand side of the last inequality, we get

$$\begin{aligned} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j) [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] dx \\ = \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) dx + \epsilon(n, j), \end{aligned}$$

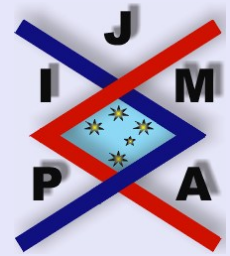
$$\begin{aligned} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi_s) [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] dx \\ = \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) dx + \epsilon(n), \end{aligned}$$

and

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(v_j)\chi_s^j - \nabla T_k(u)\chi_s] dx = \epsilon(n, j),$$

which implies that

$$\begin{aligned} (4.41) \quad & \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \\ & \quad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] dx \\ & = \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)] \\ & \quad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j] dx + \epsilon(n, j). \end{aligned}$$



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Combining (4.17), (4.39) and (4.41), we have

$$\begin{aligned}
 (4.42) \quad & \int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\
 & \quad \times [\nabla T_k(u_n) - \nabla T_k(u)] dx \\
 & \leq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u) \chi_s)] \\
 & \quad \times [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] dx \\
 & \leq 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) \varphi'_k(0) dx \\
 & \quad + 2 \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi'_k(0) dx + \epsilon(n, j, h).
 \end{aligned}$$

By passing to the lim sup over  $n$ , and letting  $j, h, s$  tend to infinity, we obtain

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} \int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\
 \times [\nabla T_k(u_n) - \nabla T_k(u)] dx = 0,
 \end{aligned}$$

thus implies by the same method used in [5] that

$$(4.43) \quad \nabla u \rightarrow \nabla u_n \text{ a.e. in } \Omega.$$

**Step 6: Modular convergence of the truncation:**

By (4.16) and (4.43), we have  $h_k = a(x, T_k(u), \nabla T_k(u))$ , which implies by



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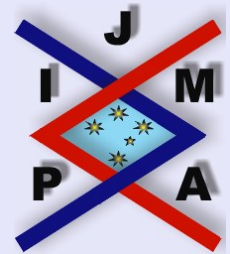
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using (4.42)

$$\begin{aligned}
 (4.44) \quad & \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + \delta(x)] dx \\
 & \leq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u)\chi_s - \nabla v_0) + \delta(x)] dx \\
 & \quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s) dx \\
 & \quad + 2 \int_{\Omega \setminus \Omega_s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \varphi'_k(0) dx \\
 & \quad + 2 \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi'_k(0) dx + \epsilon(n, j, h),
 \end{aligned}$$

which implies, by using Fatou's Lemma,

$$\begin{aligned}
 (4.45) \quad & \int_{\Omega} [a(x, T_k(u), \nabla T_k(u))(\nabla T_k(u) - \nabla v_0) + \delta(x)] dx \\
 & \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + \delta(x)] dx \\
 & \leq \limsup_{n \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + \delta(x)] dx \\
 & \leq \limsup_{n \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u)\chi_s - \nabla v_0) + \delta(x)] dx \\
 & \quad + \limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s) dx
 \end{aligned}$$



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$$\begin{aligned}
& + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) \varphi'_k(0) dx \\
& + 2 \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi'_k(0) dx + \epsilon(n, j, h).
\end{aligned}$$

Reasoning as above, we have

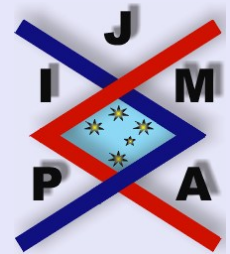
$$\begin{aligned}
(4.46) \quad \limsup_{n \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u) \chi_s - \nabla v_0) + \delta(x)] dx \\
= \int_{\Omega} [a(x, T_k(u), \nabla T_k(u)) (\nabla T_k(u) \chi_s - \nabla v_0) + \delta(x)] dx,
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u) \chi_s) (\nabla T_k(u_n) - \nabla T_k(u) \chi_s) dx \\
= \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) dx.
\end{aligned}$$

Which implies that

$$\begin{aligned}
(4.47) \quad \int_{\Omega} [a(x, T_k(u), \nabla T_k(u)) (\nabla T_k(u) - \nabla v_0) + \delta(x)] dx \\
\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla v_0) + \delta(x)] dx \\
\leq \limsup_{n \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla v_0) + \delta(x)] dx
\end{aligned}$$



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$$\begin{aligned} &\leq \int_{\Omega} [a(x, T_k(u), \nabla T_k(u))(\nabla T_k(u)\chi_s - \nabla v_0) + \delta(x)] dx \\ &\quad + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) \varphi'_k(0) dx + \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) dx \\ &\quad + 2 \int_{\Omega \setminus \Omega_s} a(x, T_k(u), 0) \nabla T_k(u) \varphi'_k(0) dx. \end{aligned}$$

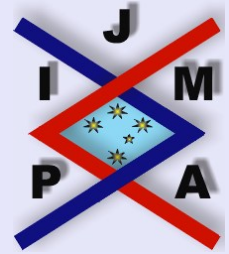
Using the fact that

$$[a(x, T_k(u), \nabla T_k(u))(\nabla T_k(u)\chi_s - \nabla v_0) + \delta(x)], h_k \nabla T_k(u) \varphi'_k(0) \quad \text{and}$$

$$a(x, T_k(u), 0) \nabla T_k(u) \varphi'_k(0) \quad \text{in } L^1(\Omega)$$

and letting  $s \rightarrow +\infty$ , we get, since  $\text{meas}(\Omega \setminus \Omega_s) \rightarrow 0$ ,

$$\begin{aligned} (4.48) \quad &\int_{\Omega} [a(x, T_k(u), \nabla T_k(u))(\nabla T_k(u) - \nabla v_0) + \delta(x)] dx \\ &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + \delta(x)] dx \\ &\leq \limsup_{n \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + \delta(x)] dx \\ &\leq \int_{\Omega} [a(x, T_k(u), \nabla T_k(u))(\nabla T_k(u) - \nabla v_0) + \delta(x)] dx. \end{aligned}$$



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Finally, we have

$$(4.49) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + \delta(x)] dx \\ = \int_{\Omega} [a(x, T_k(u), \nabla T_k(u))(\nabla T_k(u) - \nabla v_0) + \delta(x)] dx$$

and by using  $(A_3)$ , one obtains, by Lemma 2.6

$$(4.50) \quad M(\nabla T_k(u_n)) \rightarrow M(\nabla T_k(u)) \text{ in } L^1(\Omega).$$

### Step 7: *Equi-integrability of the nonlinearities.*

We need to prove that

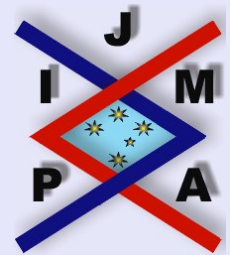
$$(4.51) \quad g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ strongly in } L^1(\Omega),$$

in particular it is enough to prove the equi-integrability of  $g_n(x, u_n, \nabla u_n)$ . To this purpose. We take  $u_n - T_1(u_n - v_0 - T_h(u_n - v_0))$  as test function in  $(P_n)$ , we obtain

$$\int_{\{|u_n - v_0| > h+1\}} |g_n(x, u_n, \nabla u_n)| dx \leq \int_{\{|u_n - v_0| > h\}} (|f_n| + \delta(x)) dx.$$

Let  $\varepsilon > 0$ , then there exists  $h(\varepsilon) \geq 1$  such that

$$(4.52) \quad \int_{\{|u_n - v_0| > h(\varepsilon)\}} |g(x, u_n, \nabla u_n)| dx < \frac{\varepsilon}{2}.$$



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For any measurable subset  $E \subset \Omega$ , we have

$$\begin{aligned} & \int_E |g_n(x, u_n, \nabla u_n)| dx \\ & \leq \int_E b(h(\varepsilon) + \|v_0\|_\infty)(h(x) + M(\nabla T_{h(\varepsilon) + \|v_0\|_\infty}(u_n))) dx \\ & \quad + \int_{\{|u_n - v_0| > h(\varepsilon)\}} |g(x, u_n, \nabla u_n)| dx. \end{aligned}$$

In view of (4.50) there exists  $\eta(\varepsilon) > 0$  such that

$$(4.53) \quad \int_E b(h(\varepsilon) + \|v_0\|_\infty)(h(x) + M(\nabla T_{h(\varepsilon) + \|v_0\|_\infty}(u_n))) dx < \frac{\varepsilon}{2}$$

for all  $E$  such that  $\text{meas}(E) < \eta(\varepsilon)$ .

Finally, combining (4.52) and (4.53), one easily has

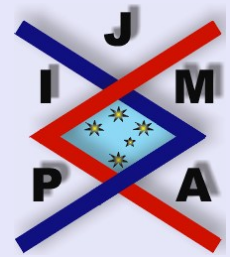
$$\int_E |g_n(x, u_n, \nabla u_n)| dx < \varepsilon \text{ for all } E \text{ such that } \text{meas}(E) < \eta(\varepsilon),$$

which implies (4.51).

**Step 8: Passing to the limit.**

Let  $v \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ , we take  $u_n - T_k(u_n - v)$  as test function in  $(P_n)$ , we can write

$$(4.54) \quad \int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - v) dx + \int_\Omega g(x, u_n, \nabla u_n) T_k(u_n - v) dx \\ \leq \int_\Omega f_n T_k(u_n - v) dx,$$



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which implies that

$$(4.55) \quad \int_{\{|u_n-v|\leq k\}} a(x, u_n, \nabla u_n) \nabla(u_n - v_0) dx \\ + \int_{\{|u_n-v|\leq k\}} a(x, T_{k+\|v\|_\infty} u_n, \nabla T_{k+\|v\|_\infty}(u_n)) \nabla(v_0 - v) dx \\ + \int_{\Omega} g(x, u_n, \nabla u_n) T_k(u_n - v) dx \leq \int_{\Omega} f_n T_k(u_n - v) dx.$$

By Fatou's lemma and the fact that

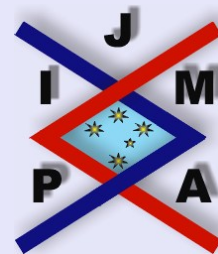
$$a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))$$

weakly in  $(L_{\overline{M}}(\Omega))^N$  for  $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$  one can easily see that

$$(4.56) \quad \int_{\{|u-v|\leq k\}} a(x, u, \nabla u) \nabla(u - v_0) dx \\ + \int_{\{|u-v|\leq k\}} a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u)) \nabla(v_0 - v) dx \\ + \int_{\Omega} g(x, u, \nabla u) T_k(u - v) dx \leq \int_{\Omega} f T_k(u - v) dx.$$

Hence

$$(4.57) \quad \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - v) dx + \int_{\Omega} g(x, u, \nabla u) T_k(u - v) dx \\ \leq \int_{\Omega} f T_k(u - v) dx.$$



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Now, let  $v \in K_\psi \cap L^\infty(\Omega)$ , by the condition  $(A_5)$  there exists  $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$  such that  $v_j$  converges to  $v$  modular, let  $h > \|v_0\|_\infty$ , taking  $v = T_h(v_j)$  in (4.57), we have

$$(4.58) \quad \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - T_h(v_j)) dx + \int_{\Omega} g(x, u, \nabla u) T_k(u - T_h(v_j)) dx \leq \int_{\Omega} f T_k(u - T_h(v_j)) dx.$$

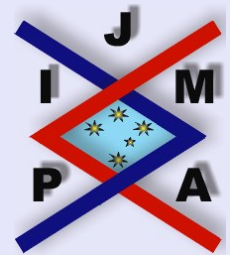
We can easily pass to the limit as  $j \rightarrow +\infty$  to get

$$(4.59) \quad \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - T_h(v)) dx + \int_{\Omega} g(x, u, \nabla u) T_k(u - T_h(v)) dx \leq \int_{\Omega} f T_k(u - T_h(v)) dx \quad \forall v \in K_\psi \cap L^\infty(\Omega),$$

the same, we pass to the limit as  $h \rightarrow +\infty$ , we deduce

$$(4.60) \quad \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - v) dx + \int_{\Omega} g(x, u, \nabla u) T_k(u - v) dx \leq \int_{\Omega} f T_k(u - v) dx \quad \forall v \in K_\psi \cap L^\infty(\Omega), \forall k > 0.$$

Thus, the proof of the theorem is now complete.  $\square$



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- [1] R. ADAMS, *Sobolev Spaces*, Academic Press, New York, (1975).
- [2] P. BÉNILAN, L. BOCCARDO, T. GALLOUET, R. GARIEPY, M. PIERRE AND J.L. VÁZQUEZ, An  $L^1$ -theory of existence and uniqueness of nonlinear elliptic equations., *Ann. Scuola Norm. Sup. Pisa*, **22** (1995), 240–273.
- [3] A. BENKIRANE, Approximation de type de Hedberg dans les espaces  $W^m LlogL(\Omega)$  et application, *Ann. Fac. Sci. Toulouse*, **11**(4) (1990), 67–78.
- [4] A. BENKIRANE AND A. ELMAHI, An existence theorem for a strongly nonlinear elliptic problems in Orlicz spaces, *Nonlinear Anal. T. M. A.*, **36** (1999), 11–24.
- [5] A. BENKIRANE AND A. ELMAHI, A strongly nonlinear elliptic equation having natural growth terms and  $L^1$  data, *Nonlinear Anal. T. M. A.*, **39** (2000), 403–411.
- [6] A. BENKIRANE, A. ELMAHI AND D. MESKINE, An existence theorem for a class of elliptic problems in  $L^1$ , *Applicationes Mathematicae*, **29**(4) (2002), 439–457.
- [7] A. BENKIRANE AND J.P. GOSSEZ, An approximation theorem for higher order Orlicz-Sobolev spaces, *Studia Math.*, **92** (1989), 231–255.
- [8] L. BOCCARDO AND T. GALLOUET, Problèmes unilatéraux avec données dans  $L^1$ , *C.R. Acad. Sci. Paris*, **311** (1990), 617–619.



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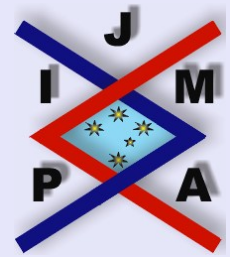
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- [9] L. BOCCARDO, F. MURAT AND J.P. PUEL, Existence of bounded solutions for nonlinear elliptic unilateral problems, *Ann. Mat. Pura e Appl.*, **152** (1988), 183–196.
- [10] G. DALMASO, F. MURAT, L. ORSINA AND A. PRIGNET, Renormalized solutions of elliptic equations with general measure data, *Ann. Scuola Norm. Sup Pisa Cl. Sci.*, **12**(4) (1999), 741–808.
- [11] A. ELMAHI AND D. MESKINE, Nonlinear elliptic problems having natural growth and  $L^1$  data in Orlicz spaces, to appear in *Anali di Matematica*.
- [12] A. ELMAHI AND D. MESKINE, Unilateral elliptic problems in  $L^1$  with natural growth terms, *J. Nonlinear and Convex Analysis*, **5**(1) (2004), 97–112
- [13] J.P. GOSSEZ, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, *Trans. Amer. Math. Soc.*, **190** (1974), 163–205.
- [14] J.P. GOSSEZ, A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces, *Proc. AMS. symp. Pure. Math.*, **45** (1986), 163–205.
- [15] J.P. GOSSEZ AND V. MUSTONEN, Variational inequalities in Orlicz-Sobolev spaces, *Nonlinear Anal.*, **11** (1987), 379–492.
- [16] M.A. KRASNOSEL'SKII AND Y.B. RUTIKII, *Convex Functions and Orlicz Spaces*, Noordhoff Groningen, (1969).




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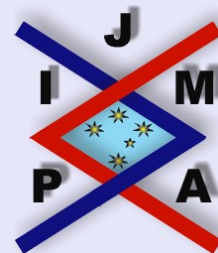
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- [17] A. PORRETTA, Existence for elliptic equations in  $L^1$  having lower order terms with natural growth, *Portugal. Math.*, **57** (2000), 179–190.
- [18] R.M. POSTERARO, Estimates for solutions of nonlinear variational inequalities, *Ann. Inst. H. Poincaré*, **12** (1995), 577–597.



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