

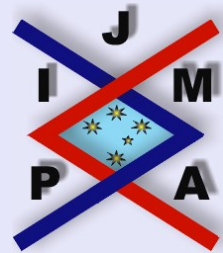
## SOME CONVEXITY PROPERTIES FOR A GENERAL INTEGRAL OPERATOR

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volume 7, issue 5, article 177,  
2006.

*Received 12 September, 2006;  
accepted 18 October, 2006.*

*Communicated by: G. Kohr*

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## Abstract

In this paper we consider the classes of starlike functions, starlike functions of order  $\alpha$ , convex functions, convex functions of order  $\alpha$  and the classes of the univalent functions denoted by  $SH(\beta)$ ,  $SP$  and  $SP(\alpha, \beta)$ . On these classes we study the convexity and  $\alpha$ - order convexity for a general integral operator.

*2000 Mathematics Subject Classification:* 30C45.

*Key words:* Univalent function, Integral operator, Convex function, Analytic function, Starlike function.

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# 1. Introduction

Let  $U = \{z \in \mathbb{C}, |z| < 1\}$  be the unit disc of the complex plane and denote by  $H(U)$ , the class of the holomorphic functions in  $U$ . Consider

$$A = \{f \in H(U), f(z) = z + a_2 z^2 + a_3 z^3 + \dots, z \in U\}$$

the class of analytic functions in  $U$  and  $S = \{f \in A : f \text{ is univalent in } U\}$ . We denote by  $S^*$  the class of starlike functions that are defined as holomorphic functions in the unit disc with the properties  $f(0) = f'(0) - 1 = 0$  and

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > 0, \quad z \in U.$$

A function  $f \in A$  is a starlike function by the order  $\alpha$ ,  $0 \leq \alpha < 1$  if  $f$  satisfies the inequality

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > \alpha, \quad z \in U.$$

We denote this class by  $S^*(\alpha)$ . Also, we denote by  $K$  the class of convex functions that are defined as holomorphic functions in the unit disc with the properties  $f(0) = f'(0) - 1 = 0$  and

$$\operatorname{Re} \left\{ \frac{z f''(z)}{f'(z)} + 1 \right\} > 0, \quad z \in U.$$

A function  $f \in A$  is a convex function by the order  $\alpha$ ,  $0 \leq \alpha < 1$  if  $f$  verifies the inequality

$$\operatorname{Re} \left\{ \frac{z f''(z)}{f'(z)} + 1 \right\} > \alpha, \quad z \in U.$$



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We denote this class by  $K(\alpha)$ .

In the paper [5] J. Stankiewicz and A. Wisniowska introduced the class of univalent functions,  $SH(\beta)$ ,  $\beta > 0$  defined by:

$$(1.1) \quad \left| \frac{zf'(z)}{f(z)} - 2\beta(\sqrt{2}-1) \right| < \operatorname{Re} \left\{ \sqrt{2} \frac{zf'(z)}{f(z)} \right\} + 2\beta(\sqrt{2}-1), \quad f \in S,$$

for all  $z \in U$ .

Also, in the paper [3] F. Ronning introduced the class of univalent functions,  $SP$ , defined by

$$(1.2) \quad \operatorname{Re} \frac{zf'(z)}{f(z)} > \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad f \in S,$$

for all  $z \in U$ . The geometric interpretation of the relation (1.2) is that the class  $SP$  is the class of all functions  $f \in S$  for which the expression  $zf'(z)/f(z)$ ,  $z \in U$  takes all values in the parabolic region

$$\Omega = \{\omega : |\omega - 1| \leq \operatorname{Re} \omega\} = \{\omega = u + iv : v^2 \leq 2u - 1\}.$$

In the paper [3] F. Ronning introduced the class of univalent functions  $SP(\alpha, \beta)$ ,  $\alpha > 0$ ,  $\beta \in [0, 1)$ , as the class of all functions  $f \in S$  which have the property:

$$(1.3) \quad \left| \frac{zf'(z)}{f(z)} - (\alpha + \beta) \right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)} + \alpha - \beta,$$




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for all  $z \in U$ . Geometric interpretation:  $f \in SP(\alpha, \beta)$  if and only if  $zf'(z)/f(z)$ ,  $z \in U$  takes all values in the parabolic region

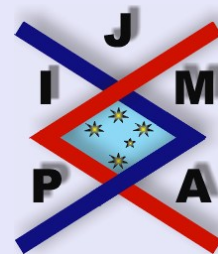
$$\begin{aligned}\Omega_{\alpha, \beta} &= \{\omega : |\omega - (\alpha + \beta)| \leq \operatorname{Re} \omega + \alpha - \beta\} \\ &= \{\omega = u + iv : v^2 \leq 4\alpha(u - \beta)\}.\end{aligned}$$

We consider the integral operator  $F_n$ , defined by:

$$(1.4) \quad F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt$$

and we study its properties.

**Remark 1.** We observe that for  $n = 1$  and  $\alpha_1 = 1$  we obtain the integral operator of Alexander,  $F(z) = \int_0^z \frac{f(t)}{t} dt$ .




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## 2. Main Results

**Theorem 2.1.** Let  $\alpha_i, i \in \{1, \dots, n\}$  be real numbers with the properties  $\alpha_i > 0$  for  $i \in \{1, \dots, n\}$  and

$$\sum_{i=1}^n \alpha_i \leq n + 1.$$

We suppose that the functions  $f_i, i = \{1, \dots, n\}$  are the starlike functions by order  $\frac{1}{\alpha_i}, i \in \{1, \dots, n\}$ , that is  $f_i \in S^*\left(\frac{1}{\alpha_i}\right)$  for all  $i \in \{1, \dots, n\}$ . In these conditions the integral operator defined in (1.4) is convex.

*Proof.* We calculate for  $F_n$  the derivatives of the first and second order. From (1.4) we obtain:

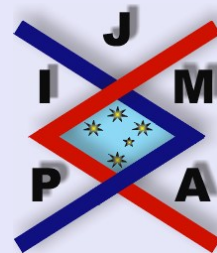
$$F'_n(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \dots \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F''_n(z) = \sum_{i=1}^n \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i-1} \left(\frac{zf'_i(z) - f_i(z)}{zf_i(z)}\right) \prod_{\substack{j=1 \\ j \neq i}}^n \left(\frac{f_j(z)}{z}\right)^{\alpha_j}.$$

$$\frac{F''_n(z)}{F'_n(z)} = \alpha_1 \left(\frac{zf'_1(z) - f_1(z)}{zf_1(z)}\right) + \dots + \alpha_n \left(\frac{zf'_n(z) - f_n(z)}{zf_n(z)}\right),$$

$$(2.1) \quad \frac{F''_n(z)}{F'_n(z)} = \alpha_1 \left(\frac{f'_1(z)}{f_1(z)} - \frac{1}{z}\right) + \dots + \alpha_n \left(\frac{f'_n(z)}{f_n(z)} - \frac{1}{z}\right).$$



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By multiplying the relation (2.1) with  $z$  we obtain:

$$(2.2) \quad \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^n \alpha_i \left( \frac{zf_i'(z)}{f_i(z)} - 1 \right) = \sum_{i=1}^n \alpha_i \frac{zf_i'(z)}{f_i(z)} - \alpha_1 - \dots - \alpha_n.$$

The relation (2.2) is equivalent with

$$(2.3) \quad \frac{zF_n''(z)}{F_n'(z)} + 1 = \alpha_1 \frac{zf_1'(z)}{f_1(z)} + \dots + \alpha_n \frac{zf_n'(z)}{f_n(z)} - \alpha_1 - \dots - \alpha_n + 1.$$

From (2.3) we obtain that:

$$(2.4) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \alpha_1 \operatorname{Re} \frac{zf_1'(z)}{f_1(z)} + \dots + \alpha_n \operatorname{Re} \frac{zf_n'(z)}{f_n(z)} - \alpha_1 - \dots - \alpha_n + 1.$$

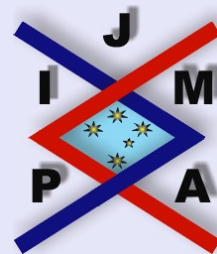
But  $f_i \in S^* \left( \frac{1}{\alpha_i} \right)$ , for all  $i \in \{1, \dots, n\}$ , so  $\operatorname{Re} \frac{zf_i'(z)}{f_i(z)} > \frac{1}{\alpha_i}$ , for all  $i \in \{1, \dots, n\}$ . We apply this affirmation in the equality (2.4) and obtain:

$$(2.5) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \alpha_1 \frac{1}{\alpha_1} + \dots + \alpha_n \frac{1}{\alpha_n} - \alpha_1 - \dots - \alpha_n + 1 = n + 1 - \sum_{i=1}^n \alpha_i.$$

But, in accordance with the hypothesis, we obtain:

$$\operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > 0$$

so,  $F_n$  is a convex function. □




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**Theorem 2.2.** Let  $\alpha_i, i \in \{1, \dots, n\}$ , be real numbers with the properties  $\alpha_i > 0$  for  $i \in \{1, \dots, n\}$  and

$$\sum_{i=1}^n \alpha_i \leq 1.$$

We suppose that the functions  $f_i, i = \{1, \dots, n\}$ , are the starlike functions. Then the integral operator defined in (1.4) is convex by order,  $1 - \sum_{i=1}^n \alpha_i$ .

*Proof.* Following the same steps as in Theorem 2.1, we obtain:

$$(2.6) \quad \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^n \alpha_i \left( \frac{zf_i'(z)}{f_i(z)} - 1 \right) = \sum_{i=1}^n \alpha_i \frac{zf_i'(z)}{f_i(z)} - \alpha_1 - \dots - \alpha_n.$$

The relation (2.6) is equivalent with

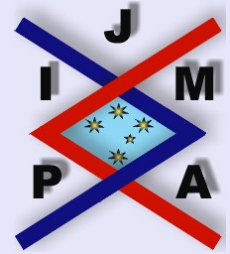
$$(2.7) \quad \frac{zF_n''(z)}{F_n'(z)} + 1 = \alpha_1 \frac{zf_1'(z)}{f_1(z)} + \dots + \alpha_n \frac{zf_n'(z)}{f_n(z)} - \alpha_1 - \dots - \alpha_n + 1.$$

From (2.7) we obtain that:

$$(2.8) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \alpha_1 \operatorname{Re} \frac{zf_1'(z)}{f_1(z)} + \dots + \alpha_n \operatorname{Re} \frac{zf_n'(z)}{f_n(z)} - \alpha_1 - \dots - \alpha_n + 1.$$

But  $f_i \in S^*$  for all  $i \in \{1, \dots, n\}$ , so  $\operatorname{Re} \frac{zf_i'(z)}{f_i(z)} > 0$  for all  $i \in \{1, \dots, n\}$ . We apply this affirmation in the equality (2.8) and obtain that:

$$(2.9) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \alpha_1 \cdot 0 + \dots + \alpha_n \cdot 0 - \alpha_1 - \dots - \alpha_n + 1 = 1 - \sum_{i=1}^n \alpha_i.$$



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But in accordance with the inequality (2.9), obtain that

$$\operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > 1 - \sum_{i=1}^n \alpha_i$$

so,  $F_n$  is a convex function by order  $1 - \sum_{i=1}^n \alpha_i$ . □

**Theorem 2.3.** Let  $\alpha_i, i \in \{1, \dots, n\}$ , be real numbers with the properties  $\alpha_i > 0$ , for  $i \in \{1, \dots, n\}$  and

$$(2.10) \quad \sum_{i=1}^n \alpha_i \leq \frac{\sqrt{2}}{2\beta(\sqrt{2}-1) + \sqrt{2}}.$$

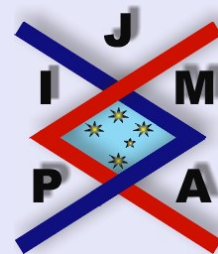
We suppose that  $f_i \in SH(\beta)$ , for  $i = \{1, \dots, n\}$  and  $\beta > 0$ . In these conditions, the integral operator defined in (1.4) is convex.

*Proof.* Following the same steps as in Theorem 2.1, we obtain that:

$$(2.11) \quad \frac{zF_n''(z)}{F_n'(z)} + 1 = \sum_{i=1}^n \alpha_i \frac{zf_i'(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i + 1.$$

We multiply the relation (2.11) with  $\sqrt{2}$  and obtain:

$$(2.12) \quad \sqrt{2} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \sum_{i=1}^n \sqrt{2}\alpha_i \frac{zf_i'(z)}{f_i(z)} - \sqrt{2} \sum_{i=1}^n \alpha_i + \sqrt{2}.$$




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The equality (2.12) is equivalent with:

$$\sqrt{2} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \sum_{i=1}^n \left( \alpha_i \sqrt{2} \frac{zf_i'(z)}{f_i(z)} + 2\alpha_i\beta (\sqrt{2} - 1) \right) - \sum_{i=1}^n 2\alpha_i\beta (\sqrt{2} - 1) - \sqrt{2} \sum_{i=1}^n \alpha_i + \sqrt{2}.$$

We calculate the real part from both terms of the above equality and obtain:

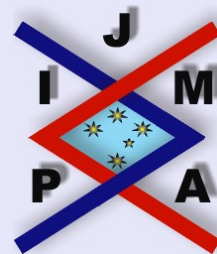
$$\sqrt{2} \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \sum_{i=1}^n \left( \alpha_i \left( \operatorname{Re} \left\{ \sqrt{2} \frac{zf_i'(z)}{f_i(z)} \right\} + 2\beta (\sqrt{2} - 1) \right) \right) - \sum_{i=1}^n 2\alpha_i\beta (\sqrt{2} - 1) - \sqrt{2} \sum_{i=1}^n \alpha_i + \sqrt{2}.$$

Because  $f_i \in SH(\beta)$  for  $i = \{1, \dots, n\}$ , we apply in the above relation the inequality (1.1) and obtain:

$$\sqrt{2} \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \sum_{i=1}^n \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - 2\beta (\sqrt{2} - 1) \right| - \sum_{i=1}^n 2\alpha_i\beta (\sqrt{2} - 1) - \sqrt{2} \sum_{i=1}^n \alpha_i + \sqrt{2}.$$

Because  $\alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - 2\beta (\sqrt{2} - 1) \right| > 0$ , for all  $i \in \{1, \dots, n\}$ , we obtain that

$$(2.13) \quad \sqrt{2} \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > - \sum_{i=1}^n 2\alpha_i\beta (\sqrt{2} - 1) - \sqrt{2} \sum_{i=1}^n \alpha_i + \sqrt{2}.$$



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Using the hypothesis (2.10), we have:

$$(2.14) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > 0,$$

so,  $F_n$  is a convex function.  $\square$

**Corollary 2.4.** Let  $\alpha$  be real numbers with the properties  $0 < \alpha \leq \frac{\sqrt{2}}{2\beta(\sqrt{2}-1)+\sqrt{2}}$ ,  $\beta > 0$ . We suppose that the functions  $f \in SH(\beta)$ . In these conditions the integral operator,  $F(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\alpha dt$  is convex.

*Proof.* In Theorem 2.3, we consider  $n = 1$ ,  $\alpha_1 = \alpha$  and  $f_1 = f$ .  $\square$

**Theorem 2.5.** Let  $\alpha_i, i \in \{1, \dots, n\}$  be real numbers with the properties  $\alpha_i > 0$  for  $i \in \{1, \dots, n\}$ ,

$$(2.15) \quad \sum_{i=1}^n \alpha_i < 1$$

and  $1 - \sum_{i=1}^n \alpha_i \in [0, 1)$ . We consider the functions  $f_i, f_i \in SP$  for  $i = \{1, \dots, n\}$ . In these conditions, the integral operator defined in (1.4) is convex by  $1 - \sum_{i=1}^n \alpha_i$  order.

*Proof.* Following the same steps as in Theorem 2.1, we have:

$$(2.16) \quad \frac{zF_n''(z)}{F_n'(z)} + 1 = \sum_{i=1}^n \alpha_i \frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i + 1.$$



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We calculate the real part from both terms of the above equality and obtain:

$$(2.17) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \sum_{i=1}^n \alpha_i \operatorname{Re} \left( \frac{zf_i'(z)}{f_i(z)} \right) - \sum_{i=1}^n \alpha_i + 1.$$

Because  $f_i \in SP$  for  $i = \{1, \dots, n\}$  we apply in the above relation the inequality (1.2) and obtain:

$$(2.18) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \sum_{i=1}^n \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| - \sum_{i=1}^n \alpha_i + 1.$$

Because  $\alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| > 0$ , for all  $i \in \{1, \dots, n\}$ , we get

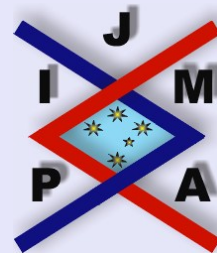
$$(2.19) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > 1 - \sum_{i=1}^n \alpha_i.$$

Using the hypothesis, we obtain that  $F_n$  is a convex function by  $1 - \sum_{i=1}^n \alpha_i$  order.  $\square$

**Remark 2.** If  $\sum_{i=1}^n \alpha_i = 1$  then

$$(2.20) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > 0,$$

so,  $F_n$  is a convex function.



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**Corollary 2.6.** Let  $\gamma$  be a real number with the property  $0 < \gamma < 1$ . We suppose that  $f \in SP$ . In these conditions the integral operator  $F(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$  is convex of  $1 - \gamma$  order.

*Proof.* In Theorem 2.5, we consider  $n = 1$ ,  $\alpha_1 = \gamma$  and  $f_1 = f$ . □

**Theorem 2.7.** We suppose that  $f \in SP$ . In this condition, the integral operator of Alexander, defined by

$$(2.21) \quad F_1(z) = \int_0^z \frac{f(t)}{t} dt,$$

is convex.

*Proof.* We have:

$$(2.22) \quad \operatorname{Re} \left( \frac{zF_1''(z)}{F_1'(z)} + 1 \right) = \operatorname{Re} \frac{zf'(z)}{f(z)} > \left| \frac{zf'(z)}{f(z)} - 1 \right| > 0.$$

So, the relation (2.22) implies that the Alexander operator is convex. □

**Remark 3.** Theorem 2.7 can be obtained from Corollary 2.6, for  $\gamma = 1$ .

**Theorem 2.8.** Let  $\alpha_i, i \in \{1, \dots, n\}$  be real numbers with the properties  $\alpha_i > 0$  for  $i \in \{1, \dots, n\}$ ,

$$(2.23) \quad \sum_{i=1}^n \alpha_i < \frac{1}{\alpha - \beta + 1}, \quad \alpha > 0, \beta \in [0, 1)$$



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and  $(\beta - \alpha - 1) \sum_{i=1}^n \alpha_i + 1 \in (0, 1)$ . We suppose that  $f_i \in SP(\alpha, \beta)$ , for  $i = \{1, \dots, n\}$ . In these conditions, the integral operator defined in (1.4) is convex by  $(\beta - \alpha - 1) \sum_{i=1}^n \alpha_i + 1$  order.

*Proof.* Following the same steps as in Theorem 2.1, we obtain that:

$$(2.24) \quad \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^n \alpha_i \left( \frac{zf_i'(z)}{f_i(z)} + \alpha - \beta \right) + (\beta - \alpha - 1) \sum_{i=1}^n \alpha_i.$$

and

$$(2.25) \quad \frac{zF_n''(z)}{F_n'(z)} + 1 = \sum_{i=1}^n \alpha_i \left( \frac{zf_i'(z)}{f_i(z)} + \alpha - \beta \right) + (\beta - \alpha - 1) \sum_{i=1}^n \alpha_i + 1.$$

We calculate the real part from both terms of the above equality and get:

$$(2.26) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \operatorname{Re} \left\{ \sum_{i=1}^n \alpha_i \left( \frac{zf_i'(z)}{f_i(z)} + \alpha - \beta \right) \right\} + (\beta - \alpha - 1) \sum_{i=1}^n \alpha_i + 1.$$

Because  $f_i \in SP(\alpha, \beta)$  for  $i = \{1, \dots, n\}$  we apply in the above relation the inequality (1.3) and obtain:

$$(2.27) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) \geq \sum_{i=1}^n \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - (\alpha + \beta) \right| + (\beta - \alpha - 1) \sum_{i=1}^n \alpha_i + 1.$$



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Since  $\alpha_i \left| \frac{zf'_i(z)}{f_i(z)} - (\alpha + \beta) \right| > 0$ , for all  $i \in \{1, \dots, n\}$ , using the inequality (1.3), we have

$$(2.28) \quad \operatorname{Re} \left( \frac{zF''_n(z)}{F'_n(z)} + 1 \right) \geq (\beta - \alpha - 1) \sum_{i=1}^n \alpha_i + 1 > 0.$$

From (2.28), since  $(\beta - \alpha - 1) \sum_{i=1}^n \alpha_i + 1 \in (0, 1)$ , we obtain that the integral operator defined in (1.4) is convex by  $(\beta - \alpha - 1) \sum_{i=1}^n \alpha_i + 1$  order.  $\square$

**Corollary 2.9.** *Let  $\gamma$  be a real number with the property  $0 < \gamma < \frac{1}{\alpha - \beta + 1}$ ,  $\alpha > 0$ ,  $\beta \in [0, 1)$ . We suppose that  $f \in SP(\alpha, \beta)$ . In these conditions, the integral operator  $F(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\gamma dt$  is convex.*

*Proof.* In Theorem 2.8, we consider  $n = 1$ ,  $\alpha_1 = \gamma$  and  $f_1 = f$ .

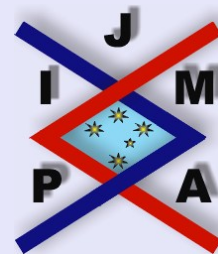
For  $\alpha = \beta \in (0, 1)$  we obtain the class  $S(\alpha, \alpha)$  that is characterized by the property

$$(2.29) \quad \left| \frac{zf'(z)}{f(z)} - 2\alpha \right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)}.$$

$\square$

**Corollary 2.10.** *Let  $\alpha_i$ ,  $i \in \{1, \dots, n\}$  be real numbers with the properties  $\alpha_i > 0$  for  $i \in \{1, \dots, n\}$  and*

$$(2.30) \quad 1 - \sum_{i=1}^n \alpha_i \in [0, 1).$$



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We consider the functions  $f_i, f_i \in SP(\alpha, \alpha), i = \{1, \dots, n\}, \alpha \in (0, 1)$ . In these conditions, the integral operator defined in (1.4) is convex by  $1 - \sum_{i=1}^n \alpha_i$  order.

*Proof.* From (1.4) we obtain

$$(2.31) \quad \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^n \alpha_i \frac{zf_i'(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i,$$

which is equivalent with

$$(2.32) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \sum_{i=1}^n \alpha_i \operatorname{Re} \frac{zf_i'(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i + 1.$$

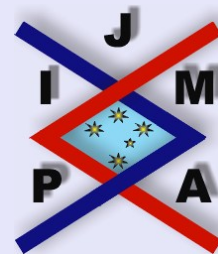
From (2.31) and (2.32), we have:

$$(2.33) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \sum_{i=1}^n \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - 2\alpha \right| + 1 - \sum_{i=1}^n \alpha_i.$$

Since  $\sum_{i=1}^n \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - 2\alpha \right| > 0$ , for all  $i \in \{1, \dots, n\}$ , from (2.33), we get:

$$(2.34) \quad \operatorname{Re} \left( \frac{zF_n''(z)}{F_n'(z)} + 1 \right) > 1 - \sum_{i=1}^n \alpha_i.$$

Now, from (2.34) we obtain that the operator defined in (1.4) is convex by  $1 - \sum_{i=1}^n \alpha_i$  order.  $\square$



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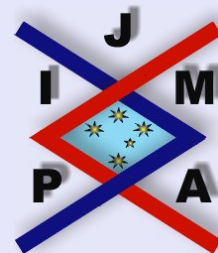
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