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ON CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS INVOLVING INTEGRAL OPERATORS

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ABSTRACT. We introduce and study some classes of meromorphic functions defined by using a meromorphic analogue of Noor [also Choi-Saigo-Srivastava] operator for analytic functions. Several inclusion results and some other interesting properties of these classes are investigated.

Key words and phrases: Meromorphic functions, Functions with positive real part, Convolution, Integral operator, Functions with bounded boundary and bounded radius rotation, Quasi-convex and close-to-convex functions.

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1. INTRODUCTION

Let $\mathcal{M}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in $D = \{z : 0 < |z| < 1\}$.

Let $P_k(\beta)$ be the class of analytic functions $p(z)$ defined in unit disc $E = D \cup \{0\}$, satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \beta}{1-\beta} \right| d\theta \leq k\pi, \tag{1.1}$$

where $z = re^{i\theta},\ k \geq 2$ and $0 \leq \beta < 1$. When $\beta = 0$, we obtain the class P_k defined in [14] and for $\beta = 0,\ k = 2$, we have the class P of functions with positive real part.

Also, we can write (1.1) as

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1-2\beta)ze^{-it}}{1 - ze^{-it}} d\mu(t), \tag{1.2}$$

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where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2, \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k. \tag{1.3}$$

From (1.1), we can write, for $p \in P_k(\beta)$,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \tag{1.4}$$

where $p_1, p_2 \in P_2(\beta) = P(\beta),\ z \in E.$

We define the function $\lambda(a, b, z)$ by

$$\lambda(a, b, z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(c)_{n+1}} z^n, \quad z \in D,$$

$c \neq 0, -1, -2, \ldots, a > 0$, where $(a)_n$ is the Pochhamer symbol (or the shifted factorial) defined by

$$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1), \quad n > 1.$$

We note that

$$\lambda(a, c, z) = \frac{1}{z} {}_2F_1(1, a; c, z),$$

${}_2F_1(1, a; c, z)$ is Gauss hypergeometric function.

Let $f \in \mathcal{M}$. Denote by $\tilde{L}(a, c); \mathcal{M} \longrightarrow \mathcal{M}$, the operator defined by

$$\tilde{L}(a, c) f(z) = \lambda(a, c, z) \star f(z), \quad z \in D,$$

where the symbol $\star$ stands for the Hadamard product (or convolution). The operator $\tilde{L}(a, c)$ was introduced and studied in [5]. This operator is closely related to the Carlson-Shaeffer operator [1] defined for the space of analytic and univalent functions in E, see [11, 13].

We now introduce a function $(\lambda(a, c, z))^{(-1)}$ given by

$$\lambda(a, c, z) \star (\lambda(a, c, z))^{(-1)} = \frac{1}{z(1-z)^{\mu}}, \quad (\mu > 0), \quad z \in D.$$

Analogous to $\tilde{L}(a, c)$, a linear operator $I_\mu(a, c)$ on $\mathcal{M}$ is defined as follows, see [2].

$$I_\mu(a, c) f(z) = (\lambda(a, c, z))^{(-1)} \star f(z), \quad (\mu > 0, a > 0, \quad c \neq 0, -1, -2, \ldots, \quad z \in D). \tag{1.5}$$

We note that

$$I_2(2, 1) f(z) = f(z), \quad \text{and} \quad I_2(1, 1) f(z) = z f'(z) + 2 f(z).$$

It can easily be verified that

$$z \left(I_\mu(a+1, c) f(z)\right)' = a I_\mu(a, c) f(z) - (a+1) I_\mu(a+1, c) f(z), \tag{1.6}$$

$$z \left(I_\mu(a, c) f(z)\right)' = \mu I_{\mu+1}(a, c) f(z) - (\mu+1) I_\mu(a, c) f(z). \tag{1.7}$$

We note that the operator $I_\mu(a, c)$ is motivated essentially by the operators defined and studied in [2, 11].

Now, using the operator $I_\mu(a, c)$, we define the following classes of meromorphic functions for $\mu > 0,\ 0 \leq \eta, \beta < 1,\ \alpha \geq 0,\ z \in D.$

We shall assume, unless stated otherwise, that $a \neq 0, -1, -2, \ldots,\ c \neq 0, -1, -2, \ldots$

Definition 1.1. A function $f \in \mathcal{M}$ is said to belong to the class $MR_k(\eta)$ for $z \in D$, $0 \leq \eta < 1$, $k \geq 2$, if and only if

$$-\frac{zf'(z)}{f(z)} \in P_k(\eta)$$

and $f \in MV_k(\eta)$, for $z \in D$, $0 \leq \eta < 1$, $k \geq 2$, if and only if

$$-\frac{(zf'(z))'}{f'(z)} \in P_k(\eta).$$

We call $f \in MR_k(\eta)$, a meromorphic function with bounded radius rotation of order η and $f \in MV_k$ a meromorphic function with bounded boundary rotation.

Definition 1.2. Let $f \in \mathcal{M}$, $0 \leq \eta < 1$, $k \geq 2$, $z \in D$. Then

$$f \in MR_k(\mu, \eta, a, c) \quad \text{if and only if} \quad I_\mu(a,c)f \in MR_k(\eta).$$

Also

$$f \in MV_k(\mu, \eta, a, c) \quad \text{if and only if} \quad I_\mu(a,c)f \in MV_k(\eta), \quad z \in D.$$

We note that, for $z \in D$,

$$f \in MV_k(\mu, \eta, a, c) \quad \Longleftrightarrow \quad -zf' \in MR_k(\mu, \eta, a, c).$$

Definition 1.3. Let $\alpha \geq 0$, $f \in \mathcal{M}$, $0 \leq \eta, \beta < 1, \mu > 0$ and $z \in D$. Then $f \in \mathcal{B}_k^\alpha(\mu, \beta, \eta, a, c)$, if and only if there exists a function $g \in MC(\mu, \eta, a, c)$, such that

$$\left[(1-\alpha)\frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'} + \alpha\left\{-\frac{(z(I_\mu(a,c)f(z))')'}{(I_\mu(a,c)g(z))'}\right\}\right] \in P_k(\beta).$$

In particular, for $\alpha = 0$, $k = a = \mu = 2$, and $c = 1$, we obtain the class of meromorphic close-to-convex functions, see [4]. For $\alpha = 1$, $k = \mu = a = 2$, $c = 1$, we have the class of meromorphic quasi-convex functions defined for $z \in D$. We note that the class $C^\star$ of quasi-convex univalent functions, analytic in E, were first introduced and studied in [7]. See also [9, 12].

The following lemma will be required in our investigation.

Lemma 1.1 ([6]). *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Phi(u,v)$ be a complex-valued function satisfying the conditions:*

(i) $\Phi(u,v)$ *is continuous in a domain $\mathcal{D} \subset \mathcal{C}^2$,*
(ii) $(1,0) \in \mathcal{D}$ *and* $\Phi(1,0) > 0$.
(iii) $\operatorname{Re}\Phi(iu_2, v_1) \leq 0$ *whenever* $(iu_2, v_1) \in \mathcal{D}$ *and* $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + \sum_{m=1}^{\infty} c_m z^m$ is a function, analytic in E, such that $(h(z), zh'(z)) \in \mathcal{D}$ and $\operatorname{Re}(h(z), zh'(z)) > 0$ for $z \in E$, then $\operatorname{Re} h(z) > 0$ in E.

2. MAIN RESULTS

Theorem 2.1.

$$MR_k(\mu+1, \eta, a, c) \subset MR_k(\mu, \beta, a, c) \subset MR_k(\mu, \gamma, a+1, c).$$

Proof. We prove the first part of the result and the second part follows by using similar arguments. Let

$$f \in MR_k(\mu+1, \eta, a, c), \quad z \in D$$

and set

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z)$$
$$= -\left[\frac{z(I_\mu(a,c)f(z))'}{I_\mu(a,c)f(z)}\right], \tag{2.1}$$

where $H(z)$ is analytic in E with $H(0)=1$.

Simple computation together with (2.1) and (1.7) yields

$$-\left[\frac{z\left(I_{\mu+1}(a,c)f(z)\right)'}{I_{\mu+1}(a,c)f(z)}\right] = \left[H(z) + \frac{zH'(z)}{-H(z)+\mu+1}\right] \in P_k(\eta), \quad z \in E. \tag{2.2}$$

Let

$$\Phi_\mu(z) = \frac{1}{\mu+1}\left[\frac{1}{z} + \sum_{k=0}^{\infty} z^k\right] + \frac{\mu}{\mu+1}\left[\frac{1}{z} + \sum_{k=0}^{\infty} kz^k\right],$$

then

$$\begin{aligned}(H(z) \star z\Phi_\mu(z)) &= H(z) + \frac{zH'(z)}{-H(z)+\mu+1} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)(h_1(z)\star z\Phi_\mu(z)) - \left(\frac{k}{4}-\frac{1}{2}\right)(h_2(z)\star z\Phi_\mu(z)) \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z) + \frac{zh_1'(z)}{-h_1(z)+\mu+1}\right] \\ &\quad - \left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z) + \frac{zh_2'(z)}{-h_2(z)+\mu+1}\right].\end{aligned} \tag{2.3}$$

Since $f \in MR_k(\mu+1,\eta,a,c)$, it follows from (2.2) and (2.3) that

$$\left[h_i(z) + \frac{zh_i'(z)}{-h_i(z)+\mu+1}\right] \in P(\eta), \quad i=1,2, \quad z \in E.$$

Let $h_i(z) = (1-\beta)p_i(z)+\beta$. Then

$$\left\{(1-\beta)p_i(z) + \left[\frac{(1-\beta)zp_i'(z)}{-(1-\beta)p_i(z)-\beta+\mu+1}\right] + (\beta-\eta)\right\} \in P, \quad z \in E.$$

We shall show that $p_i \in P,\ i=1,2$.

We form the functional $\Phi(u,v)$ by taking $u=p_i(z),\ v=zp_i'(z)$ with $u=u_1+iu_2,\ v=v_1+iv_2$. The first two conditions of Lemma 1.1 can easily be verified. We proceed to verify the condition (iii).

$$\Phi(u,v) = (1-\beta)u + \frac{(1-\beta)v}{-(1-\beta)u-\beta+\mu+1} + (\beta-\eta),$$

implies that

$$\operatorname{Re}\Phi(iu_2,v_1) = (\beta-\eta) + \frac{(1-\beta)(1+\mu-\beta)v_1}{(1+\mu-\beta)^2+(1-\beta)^2u_2^2}.$$

By taking $v_1 \le -\frac{1}{2}(1+u_2^2)$, we have

$$\operatorname{Re}\Phi(iu_2,v_1) \le \frac{A+Bu_2^2}{2C},$$

where

$$\begin{aligned}A &= 2(\beta-\eta)(1+\mu-\beta)^2-(1-\beta)(1+\mu-\beta),\\ B &= 2(\beta-\eta)(1-\beta)^2-(1-\beta)(1+\mu-\beta),\\ C &= (1+\mu-\beta)^2+(1-\beta)^2u_2^2>0.\end{aligned}$$

We note that $\operatorname{Re}\Phi(iu_2, v_1) \le 0$ if and only if $A \le 0$ and $B \le 0$. From $A \le 0$, we obtain

$$\beta = \frac{1}{4}\left[(3+2\mu+2\eta)-\sqrt{(3+2\mu+2\eta)^2-8}\right], \tag{2.4}$$

and $B \le 0$ gives us $0 \le \beta < 1$.

Now using Lemma 1.1, we see that $p_i \in P$ for $z \in E,\ i=1,2$ and hence $f \in MR_k(\mu,\beta,a,c)$ with β given by (2.4). □

In particular, we note that

$$\beta = \frac{1}{4}\left[(3+2\mu)-\sqrt{4\mu^2+12\mu+1}\right].$$

Theorem 2.2.

$$MV_k(\mu+1,\eta,a,c) \subset MV_k(\mu,\beta,a,,c) \subset MV_k(\mu,\gamma,a+1,c).$$

Proof.

$$\begin{aligned}f \in MV_k(\mu+1,\eta,a,c) \quad &\Longleftrightarrow \quad -zf' \in MR_k(\mu+1,\eta,a,c)\\ &\Rightarrow -zf' \in MR_k(\mu,\beta,a,c)\\ &\Longleftrightarrow f \in MV_k(\mu,\beta,a,c),\end{aligned}$$

where β is given by (2.4).

The second part can be proved with similar arguments. □

Theorem 2.3.

$$\mathcal{B}_k^\alpha(\mu+1,\beta_1,\eta_1,a,c) \subset \mathcal{B}_k^\alpha(\mu,\beta_2,\eta_2,a,c) \subset \mathcal{B}_k^\alpha(\mu,\beta_3,\eta_3,a+1,c),$$

where $\eta_i = \eta_i(\beta_i,\mu),\ i=1,2,3$ are given in the proof.

Proof. We prove the first inclusion of this result and other part follows along similar lines. Let $f \in \mathcal{B}_k^\alpha(\mu+1,\beta_1,\eta_1,a,c)$. Then, by Definition 1.3, there exists a function $g \in MV_2(\mu+1,\eta_1,a,c)$ such that

$$(1-\alpha)\left[\frac{(I_{\mu+1}(a,c)f(z))'}{(I_{\mu+1}(a,c)g(z))'}\right]+\alpha\left[-\frac{(z(I_{\mu+1}(a,c)f(z))')'}{(I_{\mu+1}(a,c)g(z))'}\right] \in P_k(\beta_1). \tag{2.5}$$

Set

$$p(z) = (1-\alpha)\left[\frac{(I_{\mu}(a,c)f(z))'}{(I_{\mu}(a,c)g(z))'}\right]+\alpha\left[-\frac{(z(I_{\mu}(a,c)f(z))')'}{(I_{\mu}(a,c)g(z))'}\right], \tag{2.6}$$

where p is an analytic function in E with $p(0)=1$.

Now, $g \in MV_2(\mu+1,\eta_1,a,c) \subset MV_2(\mu,\eta_2,a,c)$, where η_2 is given by the equation

$$2\eta_2^2+(3+2\mu-2\eta_1)\eta_2-[2\eta_1(1+\mu)+1]=0. \tag{2.7}$$

Therefore,

$$q(z) = \left(-\frac{(z(I_\mu(a,c)g(z))')'}{(I_\mu(a,c)g(z))'}\right) \in P(\eta_2), \quad z \in E.$$

By using (1.7), (2.5), (2.6) and (2.7), we have

$$\left[p(z) + \alpha \frac{zp'(z)}{-q(z)+\mu+1}\right] \in P_k(\beta_1), \quad q \in P(\eta_2), \quad z \in E. \tag{2.8}$$

With

$$p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)[(1-\beta_2)p_1(z)+\beta_2] - \left(\frac{k}{4}-\frac{1}{2}\right)[(1-\beta_2)p_2(z)+\beta_2],$$

(2.8) can be written as

$$\left(\frac{k}{4}+\frac{1}{2}\right)\left[(1-\beta_2)p_1(z) + \alpha\frac{(1-\beta_2)zp_1'(z)}{-q(z)+\mu+1} + \beta_2\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[(1-\beta_2)p_2(z) + \alpha\frac{(1-\beta_2)zp_2'(z)}{-q(z)+\mu+1} + \beta_2\right],$$

where

$$\left[(1-\beta_2)p_i(z) + \alpha\frac{(1-\beta_2)zp_i'(z)}{-q(z)+\mu+1} + \beta_2\right] \in P(\beta_1), \quad z \in E, i = 1, 2.$$

That is

$$\left[(1-\beta_2)p_i(z) + \alpha\frac{(1-\beta_2)zp_i'(z)}{-q(z)+\mu+1} + (\beta_2-\beta_1)\right] \in P, \quad z \in E, \quad i = 1, 2.$$

We form the functional $\Psi(u,v)$ by taking $u = u_1 + iu_2 = p_i,\ v = v_1 + iv_2 = zp_i'$, and

$$\Psi(u,v) = (1-\beta_2)u + \alpha\frac{(1-\beta_2)v}{(-q_1+iq_2)+\mu+1} + (\beta_2-\beta_1), \quad (q = q_1 + iq_2).$$

The first two conditions of Lemma 1.1 are clearly satisfied. We verify (iii), with $v_1 \le -\frac{1}{2}(1+u_2^2)$ as follows

$$\begin{aligned}
\operatorname{Re}\Psi(iu_2, v_1) &= (\beta_2-\beta_1) + \operatorname{Re}\left[\frac{\alpha(1-\beta_2)v_1\{(-q_1+\mu+1)+iq_2\}}{(-q+\mu+1)^2+q_2^2}\right] \\
&\le \frac{2(\beta-2-\beta_1)|-q+\mu+1|^2 - \alpha(1-\beta_2)(-q_1+\mu+1)(1+u_2^2)}{2|-q+\mu+1|^2} \\
&= \frac{A+Bu_2^2}{2C}, \quad C = |-q+\mu+1|^2 > 0 \\
&\le 0, \quad \text{if} \quad A \le 0 \quad \text{and} \quad B \le 0,
\end{aligned}$$

where

$$\begin{aligned}
A &= 2(\beta_2-\beta_1)|-q+\mu+1|^2 - \alpha(1-\beta_2)(-q_1+\mu+1), \\
B &= -\alpha(1-\beta_2)(-q_1+\mu+1) \le 0.
\end{aligned}$$

From $A \le 0$, we get

$$\beta_2 = \frac{2\beta_1|-q+\mu+1|^2 + \alpha\operatorname{Re}(-q(z)+\mu+1)}{2|-q+\mu+1|^2 + \alpha\operatorname{Re}(-q(z)+\mu+1)}. \tag{2.9}$$

Hence, using Lemma 1.1, it follows that $p(z)$, defined by (2.6), belongs to $P_k(\beta_2)$ and thus $f \in \mathcal{B}_k^\alpha(\mu, \beta_2, \eta_2, a, c),\ z \in D$. This completes the proof of the first part. The second part of this result can be obtained by using similar arguments and the relation (1.6). □

Theorem 2.4.

(i) $\mathcal{B}_k^\alpha(\mu,\beta,\eta,a,c) \subset \mathcal{B}_k^0(\mu,\gamma,\eta,a,c)$

(ii) $\mathcal{B}_k^{\alpha_1}(\mu,\beta,\eta,a,c) \subset \mathcal{B}_k^{\alpha_2}(\mu,\beta,\eta,a,c), \quad$ *for* $\quad 0 \le \alpha_2 < \alpha_1.$

Proof. (i). Let

$$h(z) = \frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'},$$

$h(z)$ is analytic in E and $h(0) = 1$. Then

$$(1-\alpha)\left[\frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'}\right] + \alpha\left[-\frac{(z(I_\mu(a,c)f(z))')'}{(I_\mu(a,c)g(z))'}\right] = h(z) + \alpha\frac{zh'(z)}{-h_0(z)}, \tag{2.10}$$

where

$$h_0(z) = -\frac{(z(I_\mu(a,c)g(z))')'}{(I_\mu(a,c)g(z))'} \in P(\eta).$$

Since $f \in \mathcal{B}_k^\alpha(\mu,\beta,\eta,a,c)$, it follows that

$$\left[h(z) + \alpha\frac{zh'(z)}{-h_0(z)}\right] \in P_k(\beta), \quad h_0 \in P(\eta), \quad \text{for} \quad z \in E.$$

Let

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z).$$

Then (2.10) implies that

$$\left[h_i(z) + \alpha\frac{zh_i'(z)}{-h_0(z)}\right] \in P(\beta), \quad z \in E, \quad i = 1,2,$$

and from use of similar arguments, together with Lemma 1.1, it follows that $h_i \in P(\gamma),\ i = 1,2$, where

$$\gamma = \frac{2\beta|h_0|^2 + \alpha \operatorname{Re} h_0}{2|h_0|^2 + \alpha \operatorname{Re} h_0}.$$

Therefore $h \in P_k(\gamma)$, and $f \in \mathcal{B}_k^0(\mu,\gamma,\eta,a,c),\ z \in D$. In particular, it can be shown that $h_i \in P(\beta),\ i = 1,2$. Consequently $h \in P_k(\beta)$ and $f \in \mathcal{B}_k^0(\mu,\beta,\eta,a,c)$ in D.

For $\alpha_2 = 0$, we have (i). Therefore, we let $\alpha_2 > 0$ and $f \in \mathcal{B}_k^{\alpha_1}(\mu,\beta,\eta,a,c)$. There exist two functions $H_1, H_2 \in P_k(\beta)$ such that

$$H_1(z) = (1-\alpha_1)\left[\frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'}\right] + \alpha_1\left[-\frac{(z(I_\mu(a,c)f(z))')'}{(I_\mu(a,c)g(z))'}\right]$$

$$H_2(z) = \frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'}, \quad g \in MV_2(\mu,\eta,a,c).$$

Now

$$\begin{aligned}(1-\alpha_2)\left[\frac{(I_\mu(a,c)f(z))'}{(I_\mu(a,c)g(z))'}\right] &+ \alpha_2\left[-\frac{(z(I_\mu(a,c)f(z))')'}{(I_\mu(a,c)g(z))'}\right] \\ &= \frac{\alpha_2}{\alpha_1}H_1(z) + \left(1 - \frac{\alpha_2}{\alpha_1}\right)H_2(z).\end{aligned} \tag{2.11}$$

Since the class $P_k(\beta)$ is a convex set [10], it follows that the right hand side of (2.11) belongs to $P_k(\beta)$ and this shows that $f \in \mathcal{B}_k^{\alpha_2}(\mu,\beta,\eta,a,c)$ for $z \in D$. This completes the proof. □

Let $f \in \mathcal{M},\ b > 0$ and let the integral operator F_b be defined by

$$F_b(f) = F_b(f)(z) = \frac{b}{z^{b+1}}\int_0^z t^b f(t)dt. \tag{2.12}$$

From (2.12), we note that

$$z\left(I_\mu(a,c)F_b(f)(z)\right)' = bI_\mu(a,c)f(z) - (b+1)I_\mu(a,c)F_b(f)(z). \tag{2.13}$$

Using (2.12), (2.13) with similar techniques used earlier, we can prove the following:

Theorem 2.5. *Let $f \in MR_k(\mu, \beta, a, c)$, or $MV_k(\mu, \beta, a, c)$, or $\mathcal{B}_k^{\alpha}(\mu, \beta, \eta, a, c)$, for $z \in D$. Then $F_b(f)$ defined by (2.12) is also in the same class for $z \in D$.*

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