ON A DISCRETE OPIAL-TYPE INEQUALITY

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Abstract: The main purpose of the present paper is to establish a new discrete Opial-type inequality. Our result provide a new estimates on such type of inequality.

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1. Introduction

In 1960, Z. Opial [14] established the following integral inequality:

Theorem A. *Suppose* $f \in C^1[0,h]$ *satisfies* $f(0) = f(h) = 0$ *and* $f(x) > 0$ *for all* $x \in (0,h)$. *Then the following integral inequality holds*

$$\int_0^h |f(x)f'(x)|\,dx \leq \frac{h}{4}\int_0^h (f')^2\,dx, \tag{1.1}$$

where the constant $\frac{h}{4}$ *is best possible.*

Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [1, 2, 3, 10, 12]. In recent years, inequality (1.1) has received further attention and a large number of papers dealing with new proofs, extensions, generalizations and variants of Opial's inequality have appeared in the literature [4] – [9], [13], [15], [16], [18] – [20]. For an extensive survey on these inequalities, see [1, 12].

For discrete analogues of Opial-type inequalities, good accounts of the recent works in this aspect are given in [1, 12], etc. In particular, an inequality involving two sequences was established by Pachpatte in [17] as follows:

Theorem B. *Let* x_i *and* y_i $(i = 0, 1, \ldots, \tau)$ *be non-decreasing sequences of non-negative numbers, and* $x_0 = y_0 = 0$. *Then, the following inequality holds*

$$\sum_{i=0}^{\tau-1} [x_i \Delta y_i + y_{i+1}\Delta x_i] \leq \frac{\tau}{2}\sum_{i=0}^{\tau-1}\left[(\Delta x_i)^2 + (\Delta y_i)^2\right]. \tag{1.2}$$

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The main purpose of the present paper is to establish a new discrete Opial-type inequality involving two sequences as follows.

Theorem 1.1. *Let* $\{x_{i,j}\}$ *and* $\{y_{i,j}\}$ *be non-decreasing sequences of non-negative numbers defined for* $i = 0, 1, \ldots, \tau$, $j = 0, 1, \ldots, \sigma$, *where* τ, σ *are natural numbers, and* $x_{0,j} = x_{i,0} = 0$, $y_{0,j} = y_{i,0} = 0$ $(i = 0, 1, \ldots, \tau;\ j = 0, 1, \ldots, \sigma)$. *Let*

$$\Delta_1 x_{i,j} = x_{i+1,j} - x_{i,j}, \quad \Delta_2 x_{i,j} = x_{i,j+1} - x_{i,j},$$

then

$$\begin{aligned}(1.3)\quad \sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Big[x_{i,j}\cdot\Delta_2\Delta_1 y_{i,j} + \Delta_1 y_{i,j+1}\cdot\Delta_2 x_{i+1,j} \\ + y_{i,j}\cdot\Delta_2\Delta_1 x_{i,j} + \Delta_1 x_{i,j+1}\cdot\Delta_2 y_{i+1,j+1}\Big] \\ \leq \frac{\sigma\tau}{2}\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Big[(\Delta_2\Delta_1 x_{i,j})^2 + (\Delta_2\Delta_1 y_{i,j})^2\Big].\end{aligned}$$

Our result in special cases yields some of the recent results on Opial's inequality and provides a new estimate on such types of inequalities.

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2. Main Results

Theorem 2.1. *Let* $\{x_{i,j}\}$ *and* $\{y_{i,j}\}$ *be non-decreasing sequences of non-negative numbers defined for* $i = 0, 1, \ldots, \tau$, $j = 0, 1, \ldots, \sigma$, *where* τ, σ *are natural numbers, with* $x_{0,j} = x_{i,0} = 0$, $y_{0,j} = y_{i,0} = 0$ $(i = 0, 1, \ldots, \tau; j = 0, 1, \ldots, \sigma)$. *Let* $\frac{1}{p} + \frac{1}{q} = 1, p > 1$, *and*

$$\Delta_1 x_{i,j} = x_{i+1,j} - x_{i,j}, \quad \Delta_2 x_{i,j} = x_{i,j+1} - x_{i,j},$$

then

$$\begin{aligned}(2.1)\quad \sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Big[&x_{i,j} \cdot \Delta_2\Delta_1 y_{i,j} + \Delta_1 y_{i,j+1} \cdot \Delta_2 x_{i+1,j} \\ &+ y_{i,j} \cdot \Delta_2\Delta_1 x_{i,j} + \Delta_1 x_{i,j+1} \cdot \Delta_2 y_{i+1,j+1}\Big] \\ &\leq \frac{1}{p}(\sigma\tau)^{p/q}\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}(\Delta_2\Delta_1 x_{i,j})^p + \frac{1}{q}(\sigma\tau)^{q/p}\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}(\Delta_2\Delta_1 y_{i,j})^q.\end{aligned}$$

Proof. We have

$$\begin{aligned}\Delta_2\Delta_1(x_{ij}y_{ij}) &= \Delta_2(x_{i,j}\Delta_1 y_{i,j} + y_{i+1,j}\Delta_1 x_{i,j}) \\ &= \Delta_2(x_{i,j}\Delta_1 y_{i,j}) + \Delta_2(y_{i+1,j}\Delta_1 x_{i,j}) \\ &= x_{i,j} \cdot \Delta_2\Delta_1 y_{i,j} + \Delta_1 y_{i,j+1}\Delta_2 x_{i,j} \\ &\qquad + y_{i+1,j} \cdot \Delta_2\Delta_1 x_{i,j} + \Delta_1 x_{i,j+1}\Delta_2 y_{i+1,j+1}.\end{aligned}$$

On the other hand, in view of $x_{0,j} = x_{i,0} = 0$, $y_{0,j} = y_{i,0} = 0$ $(i = 0, 1, \ldots, \tau;\ j =$

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$0, 1, \ldots, \sigma)$, it follows that

$$\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\left[x_{i,j}\cdot\Delta_2\Delta_1 y_{i,j}+\Delta_1 y_{i,j+1}\cdot\Delta_2 x_{i+1,j}+y_{i,j}\cdot\Delta_2\Delta_1 x_{i,j}+\Delta_1 x_{i,j+1}\cdot\Delta_2 y_{i+1,j+1}\right]$$
$$= x_{\tau,\sigma}\cdot y_{\tau,\sigma}.$$

Now, using the elementary inequality

$$ab \le \frac{a^p}{p}+\frac{b^q}{q}, \quad \frac{1}{p}+\frac{1}{q}=1, \quad p>1,$$

the facts that

$$x_{\tau,\sigma}=\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Delta_2\Delta_1 x_{i,j},$$
$$y_{\tau,\sigma}=\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Delta_2\Delta_1 y_{i,j},$$

and Hölder's inequality, we obtain

$$\begin{aligned}&\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\left[x_{i,j}\cdot\Delta_2\Delta_1 y_{i,j}+\Delta_1 y_{i,j+1}\cdot\Delta_2 x_{i+1,j}+y_{i,j}\cdot\Delta_2\Delta_1 x_{i,j}+\Delta_1 x_{i,j+1}\cdot\Delta_2 y_{i+1,j+1}\right]\\ &\le \frac{x^p_{\tau,\sigma}}{p}+\frac{y^q_{\tau,\sigma}}{q}\\ &=\frac{1}{p}\left(\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Delta_2\Delta_1 x_{i,j}\right)^p+\frac{1}{q}\left(\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Delta_2\Delta_1 y_{i,j}\right)^q\end{aligned}$$

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$$\leq \frac{1}{p}(\sigma\tau)^{p/q}\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}(\Delta_2\Delta_1 x_{i,j})^p + \frac{1}{q}(\sigma\tau)^{q/p}\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}(\Delta_2\Delta_1 y_{i,j})^q.$$

$\square$

Remark 1. Taking $p = q = 2$, Theorem 2.1 reduces to Theorem 1.1.

Furthermore, by reducing $\{x_{i,j}\}$ and $\{y_{i,j}\}$ to $\{x_i\}$ and $\{y_i\}$ $(i = 0, 1, \ldots, \tau)$, respectively, and with suitable changes, we have

$$\sum_{i=0}^{\tau-1}\left[x_i \Delta y_i + y_{i+1}\Delta x_i\right] \leq \frac{\tau}{2}\sum_{i=0}^{\tau-1}\left[(\Delta x_i)^2 + (\Delta y_i)^2\right].$$

This result was given by Pachpatte in [17].

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