journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 4, Article 95, 10 pp.

COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS

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Received 09 August, 2007; accepted 12 September, 2007
Communicated by Th.M. Rassias

ABSTRACT. For functions $f(z)$ which are starlike of order α, convex of order α, and λ-spiral-like of order α in the open unit disk $\mathbb{U}$, some interesting sufficient conditions involving coefficient inequalities for $f(z)$ are discussed. Several (known or new) special cases and consequences of these coefficient inequalities are also considered.

Key words and phrases: Coefficient inequalities, Analytic functions, Univalent functions, Spiral-like functions, Starlike functions, Convex functions.

2000 *Mathematics Subject Classification.* Primary 30A10, 30C45; Secondary 26D07.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}_0$ be the class of functions $f(z)$ of the form:

$$f(z) = a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

260-07

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

If $f(z) \in \mathcal{A}_0$ is given by (1.1), together with the following normalization:

$$a_0 = 0 \quad \text{and} \quad a_1 = 1,$$

then we say that $f(z) \in \mathcal{A}$.

If $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1), \tag{1.2}$$

then $f(z)$ is said to be starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are starlike of order α in $\mathbb{U}$. Similarly, we say that $f(z)$ is in the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.3}$$

It is easily observed from (1.2) and (1.3) that (see, for details, [3])

$$f(z) \in \mathcal{K}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha) \qquad (0 \leqq \alpha < 1).$$

As usual, in our present investigation, we write

$$\mathcal{S}^* := \mathcal{S}^*(0) \quad \text{and} \quad \mathcal{K} := \mathcal{K}(0).$$

Furthermore, we let $\mathcal{B}$ denote the class of functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic in $\mathbb{U}$.

Each of the following lemmas will be needed in our present investigation.

Lemma 1. *A function $p(z) \in \mathcal{B}$ satisfies the following condition*:

$$\Re[p(z)] > 0 \qquad (z \in \mathbb{U})$$

if and only if

$$p(z) \neq \frac{\zeta - 1}{\zeta + 1} \qquad (z \in \mathbb{U};\ \zeta \in \mathbb{C};\ |\zeta| = 1).$$

Proof. For the sake of completeness, we choose to give a proof of Lemma 1, even though it is fairly obvious that the following bilinear (or Möbius) transformation:

$$w = \frac{z-1}{z+1}$$

maps the unit circle $\partial\mathbb{U}$ onto the imaginary axis $\Re(w) = 0$. Indeed, for all ζ such that $|\zeta| = 1$ $(\zeta \in \mathbb{C})$, we set

$$w = \frac{\zeta - 1}{\zeta + 1} \qquad (\zeta \in \mathbb{C};\ |\zeta| = 1).$$

Then

$$|\zeta| = \left|\frac{1+w}{1-w}\right| = 1,$$

which shows that

$$\Re(w) = \Re\left(\frac{\zeta - 1}{\zeta + 1}\right) = 0 \qquad (\zeta \in \mathbb{C};\ |\zeta| = 1).$$

Moreover, by noting that $p(0) = 1$ for $p(z) \in \mathcal{B}$, we know that

$$p(z) \neq \frac{\zeta - 1}{\zeta + 1} \qquad (z \in \mathbb{U};\ \zeta \in \mathbb{C};\ |\zeta| = 1).$$

This evidently completes the proof of Lemma 1. □

Lemma 2. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*(\alpha)$ if and only if*

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0, \tag{1.4}$$

where

$$A_n = \frac{n + 1 - 2\alpha + (n-1)\zeta}{2 - 2\alpha}\ a_n.$$

Proof. Upon setting

$$p(z) = \frac{\frac{zf'(z)}{f(z)} - \alpha}{1 - \alpha} \qquad \big(f(z) \in \mathcal{S}^*(\alpha)\big),$$

we find that

$$p(z) \in \mathcal{B} \quad \text{and} \quad \Re[p(z)] > 0 \qquad (z \in \mathbb{U}).$$

Using Lemma 1, we have

$$\frac{\frac{zf'(z)}{f(z)} - \alpha}{1 - \alpha} \neq \frac{\zeta - 1}{\zeta + 1} \qquad (z \in \mathbb{U};\ \zeta \in \mathbb{C};\ |\zeta| = 1), \tag{1.5}$$

which readily yields

$$(\zeta + 1) z f'(z) + (1 - 2\alpha - \zeta) f(z) \neq 0$$

$$\big(f(z) \in \mathcal{S}^*(\alpha);\ z \in \mathbb{U};\ \zeta \in \mathbb{C};\ |\zeta| = 1\big).$$

Thus we find that

$$(\zeta + 1)z + (\zeta + 1)\left(\sum_{n=2}^{\infty} n a_n z^n\right) + (1 - 2\alpha - \zeta)\left(z + \sum_{n=2}^{\infty} a_n z^n\right) \neq 0$$
$$(z \in \mathbb{U};\ \zeta \in \mathbb{C};\ |\zeta| = 1),$$

that is, that

$$2(1-\alpha) z \left(1 + \sum_{n=2}^{\infty} \frac{n + 1 - 2\alpha + (n-1)\zeta}{2(1-\alpha)}\ a_n z^{n-1}\right) \neq 0 \tag{1.6}$$
$$(z \in \mathbb{U};\ \zeta \in \mathbb{C};\ |\zeta| = 1).$$

Now, dividing both sides of (1.6) by $2(1-\alpha)z\ \ (z \neq 0)$, we obtain

$$1 + \sum_{n=2}^{\infty} \frac{n + 1 - 2\alpha + (n-1)\zeta}{2(1-\alpha)}\ a_n z^{n-1} \neq 0$$
$$(z \in \mathbb{U};\ \zeta \in \mathbb{C};\ |\zeta| = 1),$$

which completes the proof of Lemma 2 (see also Remark 2 below). □

Remark 1. It follows from the normalization conditions:

$$a_0 = 0 \quad \text{and} \quad a_1 = 1$$

that

$$A_0 = \frac{1 - 2\alpha - x}{2 - 2\alpha}\ a_0 = 0 \quad \text{and} \quad A_1 = \frac{2 - 2\alpha}{2 - 2\alpha}\ a_1 = 1.$$

Remark 2. The assertion (1.4) of Lemma 2 is equivalent to

$$\frac{1}{z}\left(f(z) * \frac{z + \frac{\zeta+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2}\right) \neq 0 \qquad (z \in \mathbb{U}),$$

which was given earlier by Silverman *et al.* $[2]$. Furthermore, in its special case when $\alpha = 0$, Lemma 2 yields a recent result of Nezhmetdinov and Ponnusamy $[1]$ for the sufficient conditions involving the coefficients of $f(z)$ to be in the class $\mathcal{S}^*$.

The object of the present paper is to give some generalizations of the aforementioned result due to Nezhmetdinov and Ponnusamy [1]. We also briefly discuss several interesting corollaries and consequences of our main results.

2. COEFFICIENT CONDITIONS FOR FUNCTIONS IN THE CLASS $\mathcal{S}^*(\alpha)$

Our first result for functions $f(z)$ to be in the class $\mathcal{S}^*(\alpha)$ is contained in Theorem 1 below.

Theorem 1. *If $f(z) \in \mathcal{A}$ satisfies the following condition*:

$$\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}\,(j+1-2\alpha)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right| + \left|\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}(-1)^{k-j}\,(j-1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right) \leqq 2(1-\alpha) \tag{2.1}$$

$$(0 \leqq \alpha < 1;\ \beta \in \mathbb{R};\ \gamma \in \mathbb{R}),$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Proof. First of all, we note that

$$(1-z)^\beta \neq 0 \quad \text{and} \quad (1+z)^\gamma \neq 0 \qquad (z \in \mathbb{U};\ \beta \in \mathbb{R};\ \gamma \in \mathbb{R}).$$

Hence, if the following inequality:

$$\left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right)(1-z)^\beta(1+z)^\gamma \neq 0 \qquad (z \in \mathbb{U};\ \beta \in \mathbb{R};\ \gamma \in \mathbb{R}) \tag{2.2}$$

holds true, then we have

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

which is the relation (1.4) of Lemma 2. It is easily seen that (2.1) is equivalent to

$$\left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right)\left(\sum_{n=0}^{\infty}(-1)^n\, b_n z^n\right)\left(\sum_{n=0}^{\infty} c_n z^n\right) \neq 0, \tag{2.3}$$

where, for convenience,

$$b_n := \binom{\beta}{n} \quad \text{and} \quad c_n := \binom{\gamma}{n}.$$

Considering the Cauchy product of the first two factors, (2.3) can be rewritten as follows:

$$\left(1 + \sum_{n=2}^{\infty} B_n z^{n-1}\right)\left(\sum_{n=0}^{\infty} c_n z^n\right) \neq 0, \tag{2.4}$$

where

$$B_n := \sum_{j=1}^{n} (-1)^{n-j}\ A_j b_{n-j}.$$

Furthermore, by applying the same method for the Cauchy product in (2.4), we find that

$$1 + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n} B_k c_{n-k} \right) z^{n-1} \neq 0 \qquad (z \in \mathbb{U})$$

or, equivalently, that

$$1 + \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n} \left(\sum_{j=1}^{k} (-1)^{k-j} A_j b_{k-j} \right) c_{n-k} \right] z^{n-1} \neq 0 \qquad (z \in \mathbb{U}).$$

Thus, if $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left(\sum_{j=1}^{k} (-1)^{k-j} A_j b_{k-j} \right) c_{n-k} \right| \leqq 1,$$

that is, if

$$\begin{aligned} &\frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left(\sum_{j=1}^{k} (-1)^{k-j} [(j+1-2\alpha) + (j-1)\zeta] a_j b_{k-j} \right) c_{n-k} \right| \\ &\leqq \frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^{n} \left[\sum_{j=1}^{k} (-1)^{k-j}\ (j+1-2\alpha) a_j b_{k-j} \right] c_{n-k} \right| \right. \\ &\qquad\qquad \left. + |\zeta| \left| \sum_{k=1}^{n} \left[\sum_{j=1}^{k} (-1)^{k-j}\ (j-1) b_{k-j} a_j \right] c_{n-k} \right| \right) \\ &\leqq 1 \qquad (0 \leqq \alpha < 1;\ \zeta \in \mathbb{C};\ |\zeta| = 1), \end{aligned}$$

then $f(z) \in \mathcal{S}^*(\alpha)$. This completes the proof of Theorem 1. □

Setting $\alpha = 0$ in Theorem 1, we deduce the following corollary.

Corollary 1. *If $f(z) \in \mathcal{A}$ satisfies the following condition*:

$$\begin{aligned} \sum_{n=2}^{\infty} &\left(\left| \sum_{k=1}^{n} \left[\sum_{j=1}^{k} (-1)^{k-j}\ (j+1) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right. \\ &\left. + \left| \sum_{k=1}^{\infty} \left[\sum_{j=1}^{k} (-1)^{k-j}\ (j-1) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right) \leqq 2 \\ &\qquad (\beta \in \mathbb{R};\ \gamma \in \mathbb{R}), \end{aligned} \tag{2.5}$$

then $f(z) \in \mathcal{S}^*$.

Remark 3. If, in the hypothesis (2.5) of Corollary 1, we set

$$\beta - 1 = \gamma = 0 \quad \text{or} \quad \beta = \gamma = 1 \quad \text{or} \quad \beta - 2 = \gamma = 0,$$

we arrive at the result given by Nezhmetdinov and Ponnusamy $[1]$. Moreover, for $\beta = \gamma = 0$ in Theorem 1, we obtain Corollary 2 below.

Corollary 2. *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality*:

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq 1-\alpha \qquad (0 \leqq \alpha < 1), \tag{2.6}$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

In particular, by putting $\alpha = 0$ in (2.6), we get the following well-known coefficient condition for the familiar class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$.

Corollary 3. *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality*:

$$\sum_{n=2}^{\infty} n|a_n| \leqq 1, \tag{2.7}$$

then $f(z) \in \mathcal{S}^*$.

We next derive the coefficient condition for functions $f(z)$ to be in the class $\mathcal{K}(\alpha)$.

Theorem 2. *If $f(z) \in \mathcal{A}$ satisfies the following condition*:

$$\begin{aligned}\sum_{n=2}^{\infty}\Bigg(&\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}\, j(j+1-2\alpha)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right| \\ &+\left|\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}(-1)^{k-j}\, j(j-1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\Bigg) \leqq 2(1-\alpha) \\ &(0 \leqq \alpha < 1;\ \beta \in \mathbb{R};\ \gamma \in \mathbb{R}),\end{aligned} \tag{2.8}$$

then $f(z) \in \mathcal{K}(\alpha)$.

Proof. Since $zf'(z)$ belongs to the class $\mathcal{S}^*(\alpha)$ if and only if $f(z)$ is in the class $\mathcal{K}(\alpha)$, and since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2.9}$$

and

$$zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n, \tag{2.10}$$

upon replacing a_j in Theorem 1 by ja_j, we readily prove Theorem 2. □

By considering some special values for the parameters $\alpha,\ \beta$ and γ, we can deduce the following corollaries.

Corollary 4. *If $f(z) \in \mathcal{A}$ satisfies the following condition*:

$$\begin{aligned}\sum_{n=2}^{\infty}\Bigg(&\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}\, j(j+1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right| \\ &+\left|\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}(-1)^{k-j}\, j(j-1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\Bigg) \leqq 2 \quad (\beta \in \mathbb{R};\ \gamma \in \mathbb{R}),\end{aligned} \tag{2.11}$$

then $f(z) \in \mathcal{K}$.

Corollary 5. *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality*:

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqq 1-\alpha \qquad (0 \leqq \alpha < 1), \tag{2.12}$$

then $f(z) \in \mathcal{K}(\alpha)$.

Corollary 6. *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality*:

$$\sum_{n=2}^{\infty} n^2|a_n| \leqq 1, \tag{2.13}$$

then $f(z) \in \mathcal{K}$.

3. COEFFICIENT CONDITIONS FOR FUNCTIONS IN THE CLASS $\mathcal{SP}(\lambda, \alpha)$

In this section, we consider the subclass $\mathcal{SP}(\lambda, \alpha)$ of $\mathcal{A}$, which consists of functions $f(z) \in \mathcal{A}$ if and only if the following inequality holds true:

$$\Re\left[e^{i\lambda}\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right] > 0 \qquad \left(z \in \mathbb{U};\ 0 \leqq \alpha < 1;\ -\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right). \tag{3.1}$$

For $f(z) \in \mathcal{SP}(\lambda, \alpha)$, we first derive Lemma 3 below.

Lemma 3. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{SP}(\lambda, \alpha)$ if and only if*

$$1 + \sum_{n=2}^{\infty} C_n z^{n-1} \neq 0, \tag{3.2}$$

where

$$C_n := \frac{n-1+2(1-\alpha)e^{-i\lambda}\cos\lambda + (n-1)\zeta}{2(1-\alpha)e^{-i\lambda}\cos\lambda}\, a_n.$$

Proof. Letting

$$p(z) = \frac{e^{i\lambda}\left(\frac{zf'(z)}{f(z)} - \alpha\right) - i(1-\alpha)\sin\lambda}{(1-\alpha)\cos\lambda},$$

we see that

$$p(z) \in \mathcal{B} \quad \text{and} \quad \Re[p(z)] > 0 \qquad (z \in \mathbb{U}).$$

It follows from Lemma 1 that

$$\frac{e^{i\lambda}\left(\frac{zf'(z)}{f(z)} - \alpha\right) - i(1-\alpha)\sin\lambda}{(1-\alpha)\cos\lambda} \neq \frac{\zeta-1}{\zeta+1} \qquad (z \in \mathbb{U};\ \zeta \in \mathbb{C};\ |\zeta| = 1). \tag{3.3}$$

We need not consider Lemma 1 for the case when $z = 0$, because (3.3) implies that

$$p(0) \neq \frac{\zeta-1}{\zeta+1} \qquad (\zeta \in \mathbb{C};\ |\zeta| = 1).$$

It also follows from (3.3) that

$$\frac{e^{i\lambda}\left[zf'(z) - \alpha f(z)\right] - i(1-\alpha)f(z)\sin\lambda}{(1-\alpha)\cos\lambda} \neq \left(\frac{\zeta-1}{\zeta+1}\right) f(z)$$
$$(z \in \mathbb{U};\ \zeta \in \mathbb{C};\ |\zeta| = 1),$$

which readily yields

$$(\zeta+1)\left\{e^{i\lambda}[zf'(z)-\alpha f(z)]-i(1-\alpha)f(z)\sin\lambda\right\}\neq(\zeta-1)(1-\alpha)f(z)\cos\lambda$$
$$(z\in\mathbb{U};\ \zeta\in\mathbb{C};\ |\zeta|=1)$$

or, equivalently,

$$(3.4)\quad (\zeta+1)e^{i\lambda}zf'(z)-\alpha e^{i\lambda}f(z)-\zeta\alpha e^{i\lambda}f(z)-i(1-\alpha)f(z)\sin\lambda-i\zeta(1-\alpha)f(z)\sin\lambda$$
$$\neq\zeta(1-\alpha)f(z)\cos\lambda-(1-\alpha)f(z)\cos\lambda$$
$$(z\in\mathbb{U};\ \zeta\in\mathbb{C};\ |\zeta|=1).$$

We find from (3.4) that

$$(\zeta+1)e^{i\lambda}zf'(z)-\alpha e^{i\lambda}f(z)-\zeta\alpha e^{i\lambda}f(z)-\zeta(1-\alpha)e^{i\lambda}f(z)+(1-\alpha)e^{-i\lambda}f(z)\neq 0$$
$$(z\in\mathbb{U};\ \zeta\in\mathbb{C};\ |\zeta|=1),$$

that is, that

$$(1+\zeta)e^{i\lambda}zf'(z)+(e^{-i\lambda}-2\alpha\cos\lambda-\zeta e^{i\lambda})f(z)\neq 0$$
$$(z\in\mathbb{U};\ \zeta\in\mathbb{C};\ |\zeta|=1),$$

which, in light of (1.1) with $a_0=a_1-1=0$, assumes the following form:

$$(\zeta+1)e^{i\lambda}\left(z+\sum_{n=2}^{\infty}na_nz^n\right)+(e^{-i\lambda}-\zeta e^{i\lambda}-2\alpha\cos\lambda)\left(z+\sum_{n=2}^{\infty}a_nz^n\right)\neq 0$$
$$(z\in\mathbb{U};\ \zeta\in\mathbb{C};\ |\zeta|=1)$$

or, equivalently,

$$(3.5)\qquad 2(1-\alpha)z\cos\lambda\left(1+\sum_{n=2}^{\infty}\frac{n+e^{-2i\lambda}-2\alpha e^{-i\lambda}\cos\lambda+(n-1)\zeta}{2(1-\alpha)e^{-i\lambda}\cos\lambda}\,a_nz^{n-1}\right)\neq 0$$
$$(z\in\mathbb{U};\ \zeta\in\mathbb{C};\ |\zeta|=1).$$

Finally, upon dividing both sides of (3.5) by

$$2(1-\alpha)z\cos\lambda\neq 0$$

and noting that

$$e^{-2i\lambda}=-1+2e^{-i\lambda}\cos\lambda,$$

we obtain

$$1+\sum_{n=2}^{\infty}\frac{n-1+2(1-\alpha)e^{-i\lambda}\cos\lambda+(n-1)\zeta}{2(1-\alpha)e^{-i\lambda}\cos\lambda}\,a_n\neq 0$$
$$\left(0\leqq\alpha<1;\ -\frac{\pi}{2}<\lambda<\frac{\pi}{2};\ \zeta\in\mathbb{C};\ |\zeta|=1\right),$$

which completes the proof of Lemma 3 (see also the proof of a known result [1, Theorem 3.1]). □

By applying Lemma 3, we now prove Theorem 3 below.

Theorem 3. *If $f(z) \in \mathcal{A}$ satisfies the following condition*:

$$\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}\ [j-\alpha+(1-\alpha)e^{-2i\lambda}]\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right.$$

$$\left.+\left|\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}(-1)^{k-j}\ (j-1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right) \leqq 2(1-\alpha)\cos\lambda \tag{3.6}$$

$$\left(0 \leqq \alpha < 1;\ -\frac{\pi}{2} < \lambda < \frac{\pi}{2};\ \beta \in \mathbb{R};\ \gamma \in \mathbb{R}\right),$$

then $f(z) \in \mathcal{SP}(\lambda, \alpha)$.

Proof. Applying the same method as in the proof of Theorem 1, we see that $f(z)$ is in the class $\mathcal{SP}(\lambda, \alpha)$ if

$$\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}\ C_j b_{k-j}\right)c_{n-k}\right| \leqq 1 \tag{3.7}$$

where, as before,

$$b_n := \binom{\beta}{n} \qquad \text{and} \qquad c_n := \binom{\gamma}{n},$$

the coefficients C_n being given as in Lemma 3. It follows from the inequality (3.7) that

$$\frac{1}{|2(1-\alpha)e^{-i\lambda}\cos\lambda|}$$

$$\cdot\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}\Big((-1)^{k-j}(j-1+2(1-\alpha)e^{-i\lambda}\cos\lambda)+\zeta(j-1)\Big)a_j b_{k-j}\right]c_{n-k}\right|$$

$$\leqq \frac{1}{2(1-\alpha)\cos\lambda}$$

$$\cdot\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}\Big(j-\alpha+(1-\alpha)(-1+2e^{-i\lambda}\cos\lambda)\Big)b_{k-j}a_j\right]c_{n-k}\right|\right.$$

$$\left.+|\zeta|\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}\ (j-1)b_{k-j}a_j\right]c_{n-k}\right|\right)$$

$$\leqq 1 \qquad \left(0 \leqq \alpha < 1;\ -\frac{\pi}{2} < \lambda < \frac{\pi}{2};\ \zeta \in \mathbb{C};\ |\zeta| = 1\right), \tag{3.8}$$

which implies that, if $f(z)$ satisfies the hypothesis (3.6) of Theorem 3, then $f(z) \in \mathcal{SP}(\lambda, \alpha)$. This completes the proof of Theorem 3. □

In its special case when

$$\beta - 1 = \gamma = 0 \qquad \text{or} \qquad \beta = \gamma = 1 \qquad \text{or} \qquad \beta - 2 = \gamma = 0,$$

Theorem 3 would immediately yield the following corollary.

Corollary 7 (cf. [1])**.** *If $f(z) \in \mathcal{A}$ satisfies any one of the following conditions*:

$$\sum_{n=2}^{\infty}\Big(\big|[n-\alpha+(1-\alpha)e^{-2i\lambda}](a_n-a_{n-1})+a_{n-1}\big|+|(n-1)(a_n-a_{n-1})+a_{n-1}|\Big) \tag{3.9}$$

$$\leqq 2(1-\alpha)\cos\lambda \qquad \left(0 \leqq \alpha < 1;\ -\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right)$$

or

$$\sum_{n=2}^{\infty}\Big(\big|[n-\alpha+(1-\alpha)e^{-2i\lambda}](a_n-a_{n-2})+2a_{n-2}\big|+|(n-1)(a_n-a_{n-2})+2a_{n-2}|\Big) \leqq 2(1-\alpha)\cos\lambda \qquad \left(0\leqq\alpha<1;\ -\frac{\pi}{2}<\lambda<\frac{\pi}{2}\right) \tag{3.10}$$

or

$$\sum_{n=2}^{\infty}\Big(\big|[n-1-\alpha+(1-\alpha)e^{-2i\lambda}](a_n-2a_{n-1}+a_{n-2})+a_n-a_{n-2}\big| + |(n-2)(a_n-2a_{n-1}+a_{n-2})+a_n-a_{n-2}|\Big) \leqq 2(1-\alpha)\cos\lambda \qquad \left(0\leqq\alpha<1;\ -\frac{\pi}{2}<\lambda<\frac{\pi}{2}\right), \tag{3.11}$$

then $f(z)\in\mathcal{SP}(\lambda,\alpha)$.

Remark 4. For $\lambda=0$, Theorem 3 implies Theorem 1. Furthermore, by setting $\alpha=0$ in Theorem 3, we arrive at the following sufficient condition for functions $f(z)\in\mathcal{A}$ to be in the class $\mathcal{SP}(\lambda)$.

Corollary 8. *If* $f(z)\in\mathcal{A}$ *satisfies the following condition*:

$$\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}\,(j+e^{-2i\lambda})\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right| + \left|\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}(-1)^{k-j}\,(j-1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right) \leqq 2\cos\lambda \qquad \left(0\leqq\alpha<1;\ \beta\in\mathbb{R};\ \gamma\in\mathbb{R};\ -\frac{\pi}{2}<\lambda<\frac{\pi}{2}\right), \tag{3.12}$$

then

$$f(z)\in\mathcal{SP}(\lambda):=\mathcal{SP}(\lambda,0).$$

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