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Abstract

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## Abstract

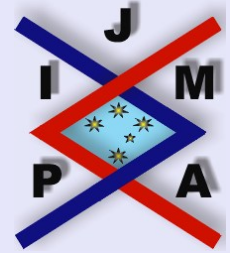
In this paper we introduce the class  $\mathcal{B}(p, n, \mu, \alpha)$  of analytic and  $p$ -valent functions to obtain some sufficient conditions and some angular properties for functions belonging to this class.

*2000 Mathematics Subject Classification:* 30C45.

*Key words:* Analytic and  $p$ -valent functions,  $p$ -valent starlike functions and  $p$ -valent convex functions, Strongly starlike and strongly convex functions.

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# 1. Introduction and Definitions

Let  $\mathcal{A}(p, n)$  denote the class of functions  $f(z)$  of the form

$$(1.1) \quad f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}),$$

which are analytic and  $p$ -valent in the open unit disk  $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ . In particular, we set  $\mathcal{A}(1, 1) =: \mathcal{A}$ . A function  $f(z) \in \mathcal{A}(p, n)$  is said to be in the class  $\mathcal{S}^*(p, n, \alpha)$  of  $p$ -valently starlike of order  $\alpha$  in  $\mathcal{U}$  if and only if it satisfies the inequality

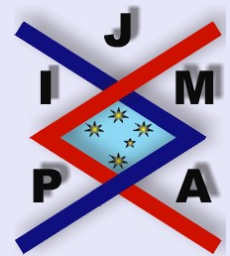
$$(1.2) \quad \operatorname{Re} \left( \frac{z f'(z)}{f(z)} \right) > \alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < p).$$

On the other hand, a function  $f(z) \in \mathcal{A}(p, n)$  is said to be in the class  $\mathcal{K}(p, n, \alpha)$  of  $p$ -valently convex of order  $\alpha$  in  $\mathcal{U}$  if and only if it satisfies the inequality

$$(1.3) \quad \operatorname{Re} \left( 1 + \frac{z f''(z)}{f'(z)} \right) > \alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < p).$$

Furthermore, a function  $f(z) \in \mathcal{A}(p, n)$  is said to be in the class  $\mathcal{C}(p, n, \alpha)$  of  $p$ -valently close-to-convex of order  $\alpha$  in  $\mathcal{U}$  if and only if it satisfies the inequality

$$(1.4) \quad \operatorname{Re} \left( \frac{f'(z)}{z^{p-1}} \right) > \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < p).$$



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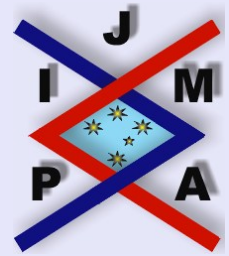


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In particular, we write  $\mathcal{S}^*(1, 1, 0) =: \mathcal{S}^*$ ,  $\mathcal{K}(1, 1, 0) =: \mathcal{K}$  and  $\mathcal{C}(1, 1, 0) =: \mathcal{C}$ , where  $\mathcal{S}^*$ ,  $\mathcal{K}$  and  $\mathcal{C}$  are the usual subclasses of  $\mathcal{A}$  consisting of functions which are starlike, convex and close-to-convex, respectively.

Let  $\overline{\mathcal{S}}^*(p, n, \alpha_1, \alpha_2)$  be the subclass of  $\mathcal{A}(p, n)$  which satisfies

$$(1.5) \quad -\frac{\pi\alpha_1}{2} < \arg \frac{zf'(z)}{f(z)} < \frac{\pi\alpha_2}{2}, \quad (z \in \mathcal{U}; 0 < \alpha_1, \alpha_2 \leq p),$$

and let  $\overline{\mathcal{K}}(p, n, \alpha_1, \alpha_2)$  be the subclass of  $\mathcal{A}(p, n)$  which satisfies

$$(1.6) \quad -\frac{\pi\alpha_1}{2} < \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{\pi\alpha_2}{2}, \quad (z \in \mathcal{U}; 0 < \alpha_1, \alpha_2 \leq p).$$

We note that  $\overline{\mathcal{S}}^*(1, 1, \alpha_1, \alpha_2) =: \mathcal{S}^*(\alpha_1, \alpha_2)$ ,  $\overline{\mathcal{K}}(1, 1, \alpha_1, \alpha_2) =: \mathcal{K}(\alpha_1, \alpha_2)$ , where  $\mathcal{S}^*(\alpha_1, \alpha_2)$  and  $\mathcal{K}(\alpha_1, \alpha_2)$  are the subclasses of  $\mathcal{A}$  introduced and studied by Takahashi and Nunokawa [7]. Also, we note that  $\overline{\mathcal{S}}^*(1, 1, \alpha, \alpha) =: \mathcal{S}_{st}^*(\alpha)$  and  $\overline{\mathcal{K}}(1, 1, \alpha, \alpha) =: \mathcal{K}_{st}(\alpha)$  where  $\mathcal{S}_{st}^*(\alpha)$  and  $\mathcal{K}_{st}(\alpha)$  are the familiar classes of strongly starlike functions of order  $\alpha$  and strongly convex functions of order  $\alpha$ , respectively.

The object of the present paper is to investigate various properties of the following classes of analytic and  $p$ -valent functions defined as follows.

**Definition 1.1.** A function  $f(z) \in \mathcal{A}(p, n)$  is said to be a member of the class  $\mathcal{B}(p, n, \mu, \alpha)$  if and only if

$$(1.7) \quad \left| \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) - p \right| < p - \alpha, \quad (p \in \mathbb{N}),$$

for some  $\alpha$  ( $0 \leq \alpha < p$ ),  $\mu \geq 0$  and for all  $z \in \mathcal{U}$ .

Note that condition (1.7) implies that

$$(1.8) \quad \operatorname{Re} \left( \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right) > \alpha.$$

We note that  $\mathcal{B}(p, n, 2, \alpha) \equiv \mathcal{S}^*(p, n, \alpha)$ ,  $\mathcal{B}(p, n, 1, \alpha) \equiv \mathcal{C}(p, n, \alpha)$ . The class  $\mathcal{B}(1, 1, 3, \alpha) \equiv \mathcal{B}(\alpha)$  is the class which has been introduced and studied by Frasin and Darus [3] (see also [1, 2]).

In order to derive our main results, we have to recall the following lemmas.

**Lemma 1.1 ([4]).** *Let  $w(z)$  be analytic in  $\mathcal{U}$  and such that  $w(0) = 0$ . Then if  $|w(z)|$  attains its maximum value on circle  $|z| = r < 1$  at a point  $z_o \in \mathcal{U}$ , we have*

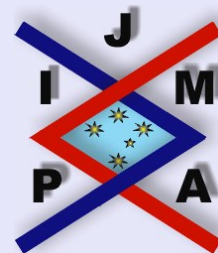
$$(1.9) \quad z_o w'(z) = k w(z_o),$$

where  $k \geq 1$  is a real number.

**Lemma 1.2 ([6]).** *Let  $\Omega$  be a set in the complex plane  $\mathbb{C}$  and suppose that  $\Phi(z)$  is a mapping from  $\mathbb{C}^2 \times \mathcal{U}$  to  $\mathbb{C}$  which satisfies  $\Phi(ix, y; z) \notin \Omega$  for  $z \in \mathcal{U}$ , and for all real  $x, y$  such that  $y \leq -n(1 + u_2^2)/2$ . If the function  $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \dots$  is analytic in  $\mathcal{U}$  such that  $\Phi(q(z), zq'(z); z) \in \Omega$  for all  $z \in \mathcal{U}$ , then  $\operatorname{Re} q(z) > 0$ .*

**Lemma 1.3 ([5]).** *Let  $q(z)$  be analytic in  $\mathcal{U}$  with  $q(0) = 1$  and  $q(z) \neq 0$  for all  $z \in \mathcal{U}$ . If there exist two points  $z_1, z_2 \in \mathcal{U}$  such that*

$$(1.10) \quad -\frac{\pi\alpha_1}{2} = \arg q(z_1) < \arg q(z) < \arg q(z_2) = \frac{\pi\alpha_2}{2}$$



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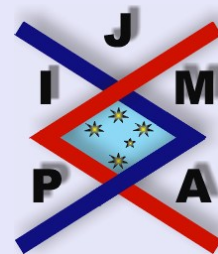
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for  $\alpha_1 > 0$ ,  $\alpha_2 > 0$ , and for  $|z| < |z_1| = |z_2|$ , then we have

$$(1.11) \quad \frac{z_1 q'(z_1)}{q(z_1)} = -i \left( \frac{\alpha_1 + \alpha_2}{2} \right) m \quad \text{and} \quad \frac{z_2 q'(z_2)}{q(z_2)} = i \left( \frac{\alpha_1 + \alpha_2}{2} \right) m$$

where

$$(1.12) \quad m \geq \frac{1 - |a|}{1 + |a|} \quad \text{and} \quad a = i \tan \frac{\pi}{4} \left( \frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2} \right).$$




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## 2. Sufficient Conditions for Starlikeness and Close-to-convexity

Making use of Lemma 1.1, we first prove

**Theorem 2.1.** *If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

$$(2.1) \quad \left| 1 + \frac{zf''(z)}{f'(z)} + (\mu - 1) \left( p - \frac{zf'(z)}{f(z)} \right) + \gamma \left( \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) - p \right) \right| < \frac{(p - \alpha)(\gamma(2p - \alpha) + 1)}{2p - \alpha}, \quad (z \in \mathcal{U}),$$

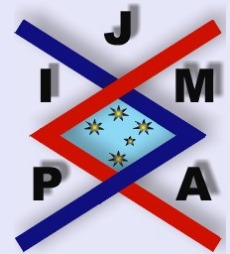
for some  $\alpha$  ( $0 \leq \alpha < p$ ) and  $\mu, \gamma \geq 0$ , then  $f(z) \in \mathcal{B}(p, n, \mu, \alpha)$ .

*Proof.* Define the function  $w(z)$  by

$$(2.2) \quad \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) = p + (p - \alpha)w(z).$$

Then  $w(z)$  is analytic in  $\mathcal{U}$  and  $w(0) = 0$ . It follows from (2.2) that

$$1 + \frac{zf''(z)}{f'(z)} - p + (\mu - 1) \left( p - \frac{zf'(z)}{f(z)} \right) + \gamma \left( \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) - p \right)$$



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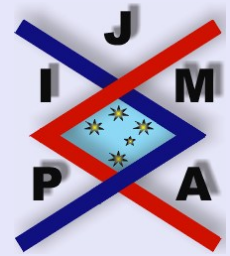


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$$= \gamma(p - \alpha)w(z) + \frac{(p - \alpha)zw'(z)}{p + (p - \alpha)w(z)}.$$

Suppose that there exists  $z_0 \in \mathcal{U}$  such that

$$(2.3) \quad \max_{|z| < z_0} |w(z)| = |w(z_0)| = 1.$$

Then from Lemma 1.1, we have (1.9). Therefore, letting  $w(z_0) = e^{i\theta}$ , with  $k \geq 1$ , we obtain that

$$\begin{aligned} & \left| 1 + \frac{z_0 f''(z_0)}{f'(z_0)} + (\mu - 1) \left( p - \frac{z_0 f'(z_0)}{f(z_0)} \right) + \gamma \left( \left( \frac{z_0^p}{f(z_0)} \right)^{\mu-1} z_0^{1-p} f'(z_0) - p \right) \right| \\ &= \left| \gamma(p - \alpha)w(z_0) + \frac{(p - \alpha)zw'(z_0)}{p + (p - \alpha)w(z_0)} \right| \\ &\geq \operatorname{Re} \left\{ \gamma(p - \alpha) + \frac{(p - \alpha)k}{p + (p - \alpha)w(z_0)} \right\} \\ &> \gamma(p - \alpha) + \frac{p - \alpha}{2p - \alpha} \\ &= \frac{(p - \alpha)(\gamma(2p - \alpha) + 1)}{2p - \alpha}, \end{aligned}$$

which contradicts our assumption (2.1). Therefore we have  $|w(z)| < 1$  in  $\mathcal{U}$ .

Finally, we have

$$(2.4) \quad \left| \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) - p \right| = (p - \alpha)|w(z)| < p - \alpha \quad (z \in \mathcal{U}),$$

that is,  $f(z) \in \mathcal{B}(p, n, \mu, \alpha)$ . □



Letting  $\mu = 1$  in Theorem 2.1, we obtain

**Corollary 2.2.** *If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

$$(2.5) \quad \left| 1 + \frac{zf''(z)}{f'(z)} + \gamma(z^{1-p}f'(z) - p) \right| < \frac{(p - \alpha)(\gamma(2p - \alpha) + 1)}{2p - \alpha}, \quad (z \in \mathcal{U}),$$

for some  $\alpha$  ( $0 \leq \alpha < p$ ) and  $\gamma \geq 0$ , then  $f(z) \in \mathcal{C}(p, n, \alpha)$ .

Letting  $p = n = 1$  and  $\gamma = \alpha = 0$  in Corollary 2.2, we easily obtain

**Corollary 2.3.** *If  $f(z) \in \mathcal{A}$  satisfies*

$$(2.6) \quad \left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{1}{2}, \quad (z \in \mathcal{U}),$$

then  $f(z) \in \mathcal{C}$ .

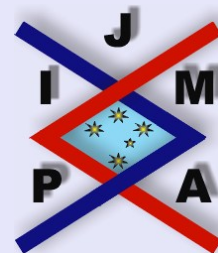
Letting  $\mu = 2$  and  $\gamma = 1$  in Theorem 2.1, we obtain

**Corollary 2.4.** *If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

$$(2.7) \quad \left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{(p - \alpha)(2p - \alpha + 1)}{2p - \alpha} \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

then  $f(z) \in \mathcal{S}^*(p, n, \alpha)$ .

Letting  $p = n = 1$  and  $\alpha = 0$  in Corollary 2.4, we easily obtain



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**Corollary 2.5.** *If  $f(z) \in \mathcal{A}$  satisfies*

$$(2.8) \quad \left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{3}{2} \quad (z \in \mathcal{U}),$$

*then  $f(z) \in \mathcal{S}^*$ .*

Next we prove

**Theorem 2.6.** *If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

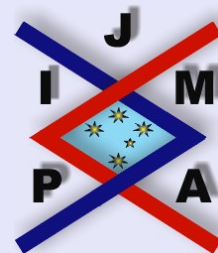
$$(2.9) \quad \operatorname{Re} \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right] \\ \times \left\{ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) + 1 + \frac{zf''(z)}{f'(z)} - (\mu-1) \frac{zf'(z)}{f(z)} \right\} \\ > \delta \left( \delta + \frac{n}{2} \right) + p \left( \delta(2-\mu) - \frac{n}{2} \right),$$

*then  $f(z) \in \mathcal{B}(p, n, \mu, \delta)$ , where  $0 \leq \delta < p$ .*

*Proof.* Define the function  $q(z)$  by

$$(2.10) \quad \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) = \delta + (p-\delta)q(z).$$

Then, we see that  $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \dots$  is analytic in  $\mathcal{U}$ . A



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computation shows that

$$\begin{aligned} & \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right]^2 \\ & + \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \left( 1 + \frac{z f''(z)}{f'(z)} - (\mu-1) \frac{z f'(z)}{f(z)} \right) \\ & = (p-\delta) z q'(z) + (p-\delta)^2 q^2(z) \\ & \quad + (p-\delta)[p(2-\mu) + 2\delta] q(z) + \delta p(2-\mu) + \delta^2 \\ & = \Phi(q(z), z q'(z); z), \end{aligned}$$

where

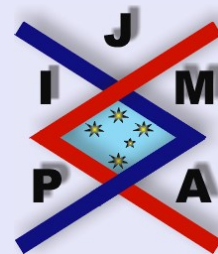
$$\Phi(r, s; t) = (p-\delta)s + (p-\delta)^2 r^2 + (p-\delta)[p(2-\mu) + 2\delta]r + \delta p(2-\mu) + \delta^2.$$

For all real  $x, y$  satisfying  $y \leq -n(1+x^2)/2$ , we have

$$\begin{aligned} \operatorname{Re} \Phi(ix, y; z) & = (p-\delta)y - (p-\delta)^2 x^2 + \delta p(2-\mu) + \delta^2 \\ & \leq -\frac{n}{2}(p-\delta) - (p-\delta) \left[ \frac{n}{2} + p-\delta \right] x^2 + \delta p(2-\mu) + \delta^2 \\ & \leq \delta p(2-\mu) + \delta^2 - \frac{n}{2}(p-\delta) \\ & = \delta \left( \delta + \frac{n}{2} \right) + p \left( \delta(2-\mu) - \frac{n}{2} \right). \end{aligned}$$

Let

$$\Omega = \left\{ w : \operatorname{Re} w > \delta \left( \delta + \frac{n}{2} \right) + p \left( \delta(2-\mu) - \frac{n}{2} \right) \right\}.$$



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Then  $\Phi(q(z), zq'(z); z) \in \Omega$  and  $\Phi(ix, y; z) \notin \Omega$  for all real  $x$  and  $y \leq -n(1 + x_2^2)/2$ ,  $z \in \mathcal{U}$ . By using Lemma 1.2, we have  $\operatorname{Re} q(z) > 0$ , that is,  $f(z) \in \mathcal{B}(p, n, \mu, \delta)$ .  $\square$

Letting  $\mu = 1$  in Theorem 2.6, we have the following:

**Corollary 2.7.** *If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

$$(2.11) \quad \operatorname{Re} \left\{ (z^{1-p} f'(z))^2 + z^{1-p} f'(z) + z^{2-p} f''(z) \right\} > \delta \left( \delta + \frac{n}{2} \right) + p \left( \delta - \frac{n}{2} \right),$$

then  $f(z) \in \mathcal{C}(p, n, \delta)$ , where  $0 \leq \delta < p$ .

Letting  $p = n = 1$  and  $\delta = 0$  in Corollary 2.7, we easily get

**Corollary 2.8.** *If  $f(z) \in \mathcal{A}$  satisfies*

$$(2.12) \quad \operatorname{Re} \left\{ (f'(z))^2 + f'(z) + z f''(z) \right\} > -\frac{1}{2},$$

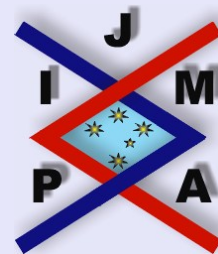
then  $f(z) \in \mathcal{C}$ .

Letting  $\mu = 2$  in Theorem 2.6, we have

**Corollary 2.9.** *If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

$$\operatorname{Re} \left( \frac{z f'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} \right) > \delta \left( \delta + \frac{n}{2} \right) - \frac{n}{2} p.$$

then  $f(z) \in \mathcal{S}^*(p, n, \delta)$ , where  $0 \leq \delta < p$ .



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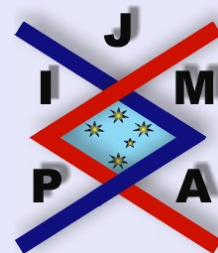
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Letting  $p = n = 1$  and  $\delta = 0$  in Corollary 2.9, we easily get

**Corollary 2.10.** *If  $f(z) \in \mathcal{A}$  satisfies*

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} \right) > -\frac{1}{2}.$$

*then  $f(z) \in \mathcal{S}^*$ .*



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### 3. Argument Properties

**Theorem 3.1.** Suppose that  $\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z) \neq \delta$  for  $z \in \mathcal{U}$  and  $0 \leq \delta < p$ . If  $f(z) \in \mathcal{A}(p, n)$  satisfies

$$\begin{aligned}
 & -\frac{\pi}{2}\alpha_1 - \tan^{-1} \left( \frac{1 - |a| (\alpha_1 + \alpha_2)(p - \delta)}{1 + |a| 2\gamma} \right) \\
 & < \arg \left\{ \left( \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right) \right. \\
 & \quad \times \left. \left( 1 + \frac{zf''(z)}{f'(z)} - p + (\mu - 1) \left( p - \frac{zf'(z)}{f(z)} \right) + \frac{\gamma}{p - \delta} \right) - \frac{\gamma\delta}{p - \delta} \right\} \\
 (3.1) \quad & < \frac{\pi}{2}\alpha_2 + \tan^{-1} \left( \frac{1 - |a| (\alpha_1 + \alpha_2)(p - \delta)}{1 + |a| 2\gamma} \right)
 \end{aligned}$$

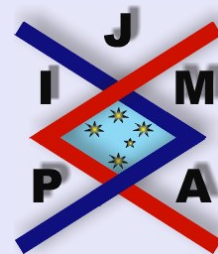
for  $\alpha_1, \alpha_2, \gamma > 0$ , then

$$(3.2) \quad -\frac{\pi}{2}\alpha_1 < \arg \left( \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) - \delta \right) < \frac{\pi}{2}\alpha_2.$$

*Proof.* Define the function  $q(z)$  by

$$(3.3) \quad q(z) = \frac{1}{p - \delta} \left( \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) - \delta \right).$$

Then we see that  $q(z)$  is analytic in  $\mathcal{U}$ ,  $q(0) = 1$ , and  $q(z) \neq 0$  for all  $z \in \mathcal{U}$ . It



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follows from (3.3) that

$$\begin{aligned} & \left( \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right) \\ & \times \left( 1 + \frac{z f''(z)}{f'(z)} - p + (\mu - 1) \left( p - \frac{z f'(z)}{f(z)} \right) + \frac{\gamma}{p - \delta} \right) - \frac{\gamma \delta}{p - \delta} \\ & = (p - \delta) z q'(z) + \gamma q(z). \end{aligned}$$

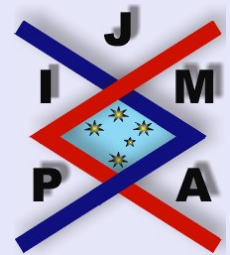
Suppose that there exists two points  $z_1, z_2 \in \mathcal{U}$  such that the condition (1.10) is satisfied, then by Lemma 1.3, we obtain (1.11) under the constraint (1.12). Therefore, we have

$$\begin{aligned} \arg(\gamma q(z_1) + (p - \delta) z q'(z_1)) &= \arg q(z_1) + \arg \left( \gamma + (p - \delta) \frac{z_1 q'(z_1)}{q(z_1)} \right) \\ &= -\frac{\pi}{2} \alpha_1 + \arg \left( \gamma - i \frac{(\alpha_1 + \alpha_2)(p - \delta)}{2} m \right) \\ &= -\frac{\pi}{2} \alpha_1 - \tan^{-1} \left( \frac{(\alpha_1 + \alpha_2)(p - \delta)}{2\gamma} m \right) \\ &\leq \frac{\pi}{2} \alpha_1 - \tan^{-1} \left( \frac{1 - |a|}{1 + |a|} \frac{(\alpha_1 + \alpha_2)(p - \delta)}{2\gamma} \right) \end{aligned}$$

and

$$\arg(\gamma q(z_2) + (p - \delta) z q'(z_2)) \geq \frac{\pi}{2} \alpha_2 + \tan^{-1} \left( \frac{1 - |a|}{1 + |a|} \frac{(\alpha_1 + \alpha_2)(p - \delta)}{2\gamma} \right),$$

which contradict the assumptions of the theorem. This completes the proof.  $\square$



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Letting  $\mu = 1$  in Theorem 3.1, we have

**Corollary 3.2.** *Suppose that  $z^{1-p}f'(z) \neq \delta$  for  $z \in \mathcal{U}$  and  $0 \leq \delta < p$ . If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

$$(3.4) \quad \begin{aligned} & -\frac{\pi}{2}\alpha_1 - \tan^{-1} \left( \frac{1 - |a| (\alpha_1 + \alpha_2)(p - \delta)}{1 + |a| 2\gamma} \right) \\ & < \arg \left\{ z^{1-p}f'(z) \left( 1 + \frac{zf''(z)}{f'(z)} - p + \frac{\gamma}{p - \delta} \right) - \frac{\gamma\delta}{p - \delta} \right\} \\ & < \frac{\pi}{2}\alpha_2 + \tan^{-1} \left( \frac{1 - |a| (\alpha_1 + \alpha_2)(p - \delta)}{1 + |a| 2\gamma} \right) \end{aligned}$$

for  $\alpha_1, \alpha_2, \gamma > 0$ , then

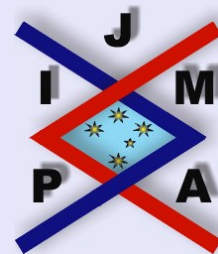
$$(3.5) \quad -\frac{\pi}{2}\alpha_1 < \arg(z^{1-p}f'(z) - \delta) < \frac{\pi}{2}\alpha_2.$$

Letting  $\alpha_1 = \alpha_2 = 1$  in Corollary 3.2, we have

**Corollary 3.3.** *Suppose that  $z^{1-p}f'(z) \neq \delta$  for  $z \in \mathcal{U}$  and  $0 \leq \delta < p$ . If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

$$(3.6) \quad \left| \arg \left\{ z^{1-p}f'(z) \left( 1 + \frac{zf''(z)}{f'(z)} - p + \frac{\gamma}{p - \delta} \right) - \frac{\gamma\delta}{p - \delta} \right\} \right| < \frac{\pi}{2} + \tan^{-1} \left( \frac{p - \delta}{\gamma} \right)$$

for  $\gamma > 0$ , then  $f(z) \in \mathcal{C}(p, n, \delta)$ .



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Letting  $\mu = 2$  in Theorem 3.1, we have

**Corollary 3.4.** *Suppose that  $zf'(z)/f(z) \neq \delta$  for  $z \in \mathcal{U}$  and  $0 \leq \delta < p$ . If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

$$\begin{aligned}
 & -\frac{\pi}{2}\alpha_1 - \tan^{-1} \left( \frac{1 - |a| (\alpha_1 + \alpha_2)(p - \delta)}{1 + |a| 2\gamma} \right) \\
 & < \arg \left\{ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{\gamma}{p - \delta} \right) - \frac{\gamma\delta}{p - \delta} \right\} \\
 (3.7) \quad & < \frac{\pi}{2}\alpha_2 + \tan^{-1} \left( \frac{1 - |a| (\alpha_1 + \alpha_2)(p - \delta)}{1 + |a| 2\gamma} \right)
 \end{aligned}$$

for  $\alpha_1, \alpha_2, \gamma > 0$ , then

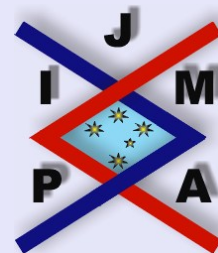
$$(3.8) \quad -\frac{\pi}{2}\alpha_1 < \arg \left( \frac{zf'(z)}{f(z)} - \delta \right) < \frac{\pi}{2}\alpha_2$$

Letting  $\alpha_1 = \alpha_2 = 1$  in Corollary 3.4, we have

**Corollary 3.5.** *Suppose that  $zf'(z)/f(z) \neq \delta$  for  $z \in \mathcal{U}$  and  $0 \leq \delta < p$ . If  $f(z) \in \mathcal{A}(p, n)$  satisfies*

$$\begin{aligned}
 (3.9) \quad & \left| \arg \left\{ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{\gamma}{p - \delta} \right) - \frac{\gamma\delta}{p - \delta} \right\} \right| \\
 & < \frac{\pi}{2} + \tan^{-1} \left( \frac{p - \delta}{\gamma} \right)
 \end{aligned}$$

for  $\gamma > 0$ , then  $f(z) \in \mathcal{S}^*(p, n, \delta)$ .



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Letting  $\alpha_1 = \alpha_2$ ,  $\mu = p = n = 1$  and  $\delta = 0$  in Theorem 3.1, we have

**Corollary 3.6.** *If  $f(z) \in \mathcal{A}$  satisfies*

$$(3.10) \quad \left| \arg \left( z f''(z) + f'(z)(\gamma + 1) - \frac{z(f'(z))^2}{f(z)} \right) \right| < \frac{\pi}{2} \alpha + \tan^{-1} \frac{\alpha}{\gamma}$$

for  $\gamma > 0$ , then

$$(3.11) \quad |\arg f'(z)| < \frac{\pi}{2} \alpha, \quad (0 < \alpha \leq 1).$$

Taking  $\alpha_1 = \alpha_2$ ,  $p = n = 1$ ,  $\mu = 2$  and  $\delta = 0$  in Theorem 3.1, we obtain

**Corollary 3.7.** *If  $f(z) \in \mathcal{A}$  satisfies*

$$(3.12) \quad \left| \arg \left( \frac{z^2 f''(z)}{f(z)} - \left( \frac{z f'(z)}{f(z)} \right)^2 + \frac{z f'(z)}{f(z)} (\gamma + 1) \right) \right| < \frac{\pi}{2} \alpha + \tan^{-1} \frac{\alpha}{\gamma}$$

for  $\gamma > 0$ , then  $f(z) \in \mathcal{S}_{st}^*(\alpha)$ .

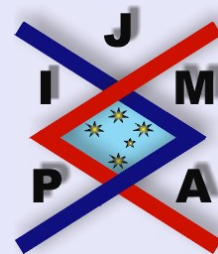
Finally, we prove

**Theorem 3.8.** *Let  $q(z)$  analytic in  $\mathcal{U}$  with  $q(0) = 1$ , and  $q(z) \neq 0$ . If*

$$(3.13) \quad -\frac{\pi}{2} \eta_1 < \arg \left\{ q(z) + \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right]^{-1} z q'(z) \right\} < \frac{\pi}{2} \eta_2$$

for some  $f(z) \in \mathcal{B}(p, n, \mu, \alpha)$  and  $(0 < \eta_1, \eta_2 \leq 1)$  then

$$(3.14) \quad -\frac{\pi}{2} \alpha_1 < \arg q(z) < \frac{\pi}{2} \alpha_2$$



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where  $\alpha_1$  and  $\alpha_2$  ( $0 < \alpha_1, \alpha_2 \leq 1$ ) are the solutions of the following equations:

$$(3.15) \quad \eta_1 = \alpha_1 + \frac{2}{\pi} \tan^{-1} \left( \frac{(\alpha_1 + \alpha_2)m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]}{2(2p - \alpha) + (\alpha_1 + \alpha_2)m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]} \right)$$

and

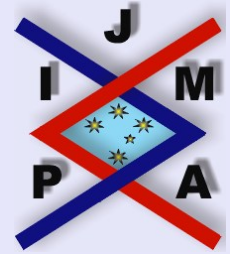
$$(3.16) \quad \eta_2 = \alpha_2 + \frac{2}{\pi} \tan^{-1} \left( \frac{(\alpha_1 + \alpha_2)m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]}{2(2p - \alpha) + (\alpha_1 + \alpha_2)m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]} \right)$$

*Proof.* Suppose that there exists two points  $z_1, z_2 \in \mathcal{U}$  such that the condition (1.10) is satisfied, then by Lemma 1.3, we obtain (1.11) under the constraint (1.12). Since  $f \in \mathcal{B}(p, n, \mu, \alpha)$ , we have

$$(3.17) \quad \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) = \rho \exp \left( \frac{i\pi\phi}{2} \right),$$

where

$$(3.18) \quad \left\{ \begin{array}{l} \alpha < \rho < 2p - \alpha \\ -\frac{2}{\pi} \sin^{-1} \frac{p - \alpha}{p} < \phi < \frac{2}{\pi} \sin^{-1} \frac{p - \alpha}{p}. \end{array} \right.$$



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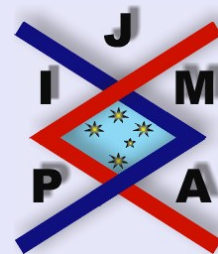
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Thus, we obtain

$$\begin{aligned}
 & \arg \left\{ q(z_1) + \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right]^{-1} z q'(z_1) \right\} \\
 &= \arg q(z_1) + \arg \left( 1 + \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right]^{-1} \frac{z_1 q'(z_1)}{q(z_1)} \right) \\
 &= -\frac{\pi}{2} \alpha_1 + \arg \left( 1 - i \left( \frac{\alpha_1 + \alpha_2}{2} \right) m \left[ \rho \exp \left( \frac{i\pi\phi}{2} \right) \right]^{-1} \right) \\
 &\leq -\frac{\pi}{2} \alpha_1 - \tan^{-1} \left( \frac{(\alpha_1 + \alpha_2) m \sin \left[ \frac{\pi}{2} (1 - \phi) \right]}{2\rho + (\alpha_1 + \alpha_2) m \sin \left[ \frac{\pi}{2} (1 - \phi) \right]} \right) \\
 &\leq -\frac{\pi}{2} \alpha_1 - \tan^{-1} \left( \frac{(\alpha_1 + \alpha_2) m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]}{2(2p - \alpha) + (\alpha_1 + \alpha_2) m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]} \right) \\
 &= -\frac{\pi}{2} \eta_1
 \end{aligned}$$

and

$$\arg(q(z_1) + \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right]^{-1} z q'(z_1))$$



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$$\geq \frac{\pi}{2}\alpha_2 + \tan^{-1} \left( \frac{(\alpha_1 + \alpha_2)m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]}{2(2p - \alpha) + (\alpha_1 + \alpha_2)m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]} \right)$$

$$= \frac{\pi}{2}\eta_2,$$

where  $\eta_1$  and  $\eta_2$  being given by (3.15) and (3.16), respectively, which contradicts the assumption (3.13). This completes the proof of Theorem 3.8.  $\square$

Letting  $q(z) = zf'(z)/f(z)$  in Theorem 3.8, we have

**Corollary 3.9.** *Let  $0 < \eta_1, \eta_2 \leq 1$ . If*

$$(3.19) \quad -\frac{\pi}{2}\eta_1 < \arg \left\{ q(z) + \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right]^{-1} zq'(z) \right\} < \frac{\pi}{2}\eta_2$$

for some  $f(z) \in \mathcal{B}(p, n, \mu, \alpha)$  then  $f(z) \in \overline{\mathcal{S}}^*(p, n, \alpha_1, \alpha_2)$ , where  $0 < \alpha_1, \alpha_2 \leq 1$ .

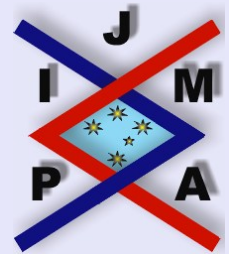
Letting  $\eta_1 = \eta_2$  in Corollary 3.9, we have

**Corollary 3.10.** *Let  $0 < \eta_1 \leq 1$ . If*

$$(3.20) \quad \left| \arg \left\{ \frac{zf'(z)}{f(z)} + \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right]^{-1} z \left( \frac{zf'(z)}{f(z)} \right)' \right\} \right| < \frac{\pi}{2}\eta_1$$

for some  $f(z) \in \mathcal{B}(p, n, \mu, \alpha)$ , then

$$(3.21) \quad \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2}\alpha_1 \quad (0 < \alpha_1 \leq 1),$$



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that is,  $f(z) \in \mathcal{S}_{st}^*(\alpha_1)$ , where  $\alpha_1$  is the solutions of the following equation:

$$(3.22) \quad \eta = \alpha_1 + \frac{2}{\pi} \tan^{-1} \left( \frac{2\alpha_1 m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]}{2(2p - \alpha) + 2\alpha_1 m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]} \right).$$

Letting  $q(z) = q(z) = 1 + (zf''(z)/f'(z))$  in Theorem 3.8, we have

**Corollary 3.11.** *Let  $0 < \eta_1, \eta_2 \leq 1$ . If*

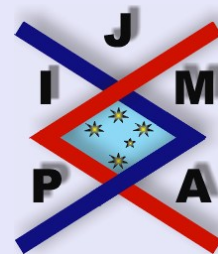
$$(3.23) \quad -\frac{\pi}{2}\eta_1 < \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} + \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right]^{-1} z \left( 1 + \frac{zf''(z)}{f'(z)} \right)' \right\} < \frac{\pi}{2}\eta_1$$

for some  $f(z) \in \mathcal{B}(p, n, \mu, \alpha)$  then  $f(z) \in \mathcal{K}(p, n, \alpha_1, \alpha_2)$ , where  $0 < \alpha_1, \alpha_2 \leq 1$ .

Letting  $\eta_1 = \eta_2$  in Corollary 3.11, we have

**Corollary 3.12.** *Let  $0 < \eta_1 \leq 1$ . If*

$$(3.24) \quad \left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} + \left[ \left( \frac{z^p}{f(z)} \right)^{\mu-1} z^{1-p} f'(z) \right]^{-1} z \left( 1 + \frac{zf''(z)}{f'(z)} \right)' \right\} \right| < \frac{\pi}{2}\eta_1$$



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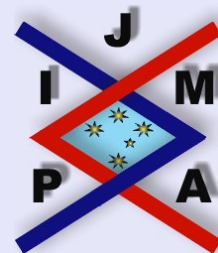
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for some  $f(z) \in \mathcal{B}(p, n, \mu, \alpha)$  then

$$(3.25) \quad \left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \alpha_1 \quad (0 < \alpha_1 \leq 1),$$

that is,  $f(z) \in \mathcal{K}_{st}(\alpha_1)$ , where  $\alpha_1$  is the solution of the following equation:

$$(3.26) \quad \eta_1 = \alpha_1 + \frac{2}{\pi} \tan^{-1} \left( \frac{2\alpha_1 m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]}{2(2p - \alpha) + 2\alpha_1 m \sin \left[ \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{p-\alpha}{p} \right) \right]} \right).$$




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