WHEN LAGRANGEAN AND QUASI-ARITHMETIC MEANS COINCIDE

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Abstract: We give a complete characterization of functions f generating the same Lagrangean mean L_f and quasi-arithmetic mean Q_f. We also solve the equation $L_f = Q_g$ imposing some additional conditions on f and g.

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1. Introduction

We consider the problem when the Lagrangean and quasi-arithmetic means coincide. The Lagrangean means are related to the basic mean value theorem. The family of quasi-arithmetic means naturally generalizes all the classical means. Thus these two types of means, coming from different roots, are not closely related. On the other hand they enjoy a common property, namely, each of them is generated by a single variable function. With this background, the question: *When do these two types of means coincide?* seems to be interesting.

To present the main results we recall some definitions.

Let $I \subset \mathbb{R}$ be a real interval and $f : I \to \mathbb{R}$ be a continuous and strictly monotonic function. The function $L_f : I^2 \to \mathbb{R}$, defined by

$$L_f(x, y) := \begin{cases} f^{-1}\left(\frac{1}{y-x}\int_x^y f(\xi)\,d\xi\right), & \text{if} \quad x \neq y, \\ x, & \text{if} \quad x = y, \end{cases}$$

is a strict symmetric mean, and it is called a *Lagrangean* one (cf. P.S. Bullen, D.S. Mitrinović, P.M. Vasić [3], Chap. VII, p. 343; L. R. Berrone, J. Moro [2], and the references therein). The function $Q_f : I^2 \to \mathbb{R}$, given by

$$Q_f(x, y) := f^{-1}\left(\frac{f(x) + f(y)}{2}\right),$$

is called a *quasi-arithmetic mean* (cf., for instance, J. Aczél [1], Chap. VI, p. 276; P. S. Bullen, D. S. Mitrinović, P. M. Vasić [3], Chap. IV, p. 215; M. Kuczma [4], Chap. VIII, p. 189). In both cases, we say that f is the *generator* of the mean.

In Section 3 we give a complete solution of the equation $L_f = Q_f$. We show that this happens if and only if both the means are simply the arithmetic mean A. The general problem *when* $L_f = Q_g$ turns out to be much more difficult. We solve it in Section 4, imposing some conditions on the generators f and g.

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2. Some Definitions and Auxiliary Results

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to \mathbb{R}$ is said to be a *mean on* I if

$$\min(x, y) \le M(x, y) \le \max(x, y), \qquad x, y \in I.$$

If, in addition, these inequalities are sharp whenever $x \neq y$, the mean M is called *strict*, and M is called *symmetric* if $M(x, y) = M(y, x)$ for all $x, y \in I$.

Note that if $M : I^2 \to \mathbb{R}$ is a mean, then for every interval $J \subset I$ we have $M(J^2) = J$; in particular, $M(I^2) = I$. Moreover, M is *reflexive*, that is $M(x, x) = x$ for all $x \in I$.

By A we denote the restriction of the *arithmetic* mean to the set I^2, i.e.

$$A(x, y) := \frac{x + y}{2}, \qquad x, y \in I.$$

We shall need the following basic result about the Jensen functional equation (cf. [4], Th. XIII.2.2).

Theorem 2.1. *Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is a continuous solution of the equation*

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2} \tag{2.1}$$

if and only if

$$f(x) = ax + b, \qquad x, y \in I,$$

with some $a, b \in \mathbb{R}$.

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3. The Case of Common Generators

It is well known that $Q_f = Q_g$, i.e., f and g are equivalent generators of the quasi-arithmetic mean if and only if $g = af + b$ for some $a, b \in \mathbb{R}$, $a \neq 0$ (cf. [1], Sec. 6.4.3, Th. 2; [3], Chap. VI, p. 344). Similarly, $L_f = L_g$ if and only if $g = cf + d$ for some $c, d \in \mathbb{R}$, $c \neq 0$ (cf. [2], Cor. 7; [3], Chap. VI, p. 344).

The main result of this section gives a complete characterization of functions f such that $L_f = Q_f$. Two different proofs are presented. The first is based on an elementary theory of differential equations; in the second one we apply Theorem 2.1.

Theorem 3.1. *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a continuous strictly monotonic function. Then the following conditions are pairwise equivalent:*

(i) $L_f = Q_f$;

(ii) there are $a, b \in \mathbb{R}$, $a \neq 0$, *such that*

$$f(x) = ax + b, \qquad x \in I;$$

(iii) $L_f = Q_f = A$.

First Proof. We only show the implication (i) $\Rightarrow$(ii), as the remaining are obvious. Assume that (i) holds true. From the definition of L_f and Q_f we have

$$f^{-1}\left(\frac{f(x)+f(y)}{2}\right) = f^{-1}\left(\frac{1}{y-x}\int_x^y f(\xi)\,d\xi\right), \qquad x, y \in I, \quad x \neq y,$$

or, equivalently,

$$\frac{f(x)+f(y)}{2} = \frac{1}{y-x}\int_x^y f(\xi)\,d\xi, \qquad x, y \in I, \quad x \neq y.$$

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Let $F : I \to \mathbb{R}$ be any primitive function of f. Then the condition above can be written in the form

$$\frac{f(x)+f(y)}{2} = \frac{F(y)-F(x)}{y-x}, \qquad x, y \in I, \quad x \neq y. \tag{3.1}$$

This implies that f is differentiable and, consequently, F is twice differentiable. Fix an arbitrary $y \in I$. Differentiating both sides of this equality with respect to x, we obtain

$$\frac{f'(x)}{2} = \frac{F'(x)(x-y) - F(x) + F(y)}{(x-y)^2}, \qquad x, y \in I, \quad x \neq y.$$

Hence, using the relation $f' = F''$, we get

$$F''(x)(x-y)^2 = 2F'(x)(x-y) - 2F(x) + 2F(y), \qquad x \in I.$$

Solving this differential equation of the second order on two disjoint intervals $(-\infty, y) \cap I$ and $(y, \infty) \cap I$, and then using the twice differentiability of F at the point y, we obtain

$$F(x) = \frac{a}{2}x^2 + bx + p, \qquad x \in I,$$

with some $a, b, p \in \mathbb{R}, a \neq 0$. Since F is a primitive function of f, we get

$$f(x) = F'(x) = ax + b, \qquad x \in I,$$

which completes the proof . $\square$

Second Proof. Again, let F be a primitive function of f. In the same way, as is in the previous proof, we show that (3.1) is satisfied. It follows that

$$2[F(y) - F(x)] = (y-x)[f(x) + f(y)], \qquad x, y \in I,$$

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and, consequently, since

$$2\left[F(y)-F(z)\right]+2\left[F(z)-F(x)\right]=2\left[F(y)-F(x)\right], \qquad x,\ y,\ z\in I,$$

we get

$$(y-z)\left[f(z)+f(y)\right]+(z-x)\left[f(x)+f(z)\right]=(y-x)\left[f(x)+f(y)\right]$$

for all $x,\ y,\ z\in I$. Setting here $z=\frac{x+y}{2}$, we have

$$\frac{y-x}{2}\left[f\left(\frac{x+y}{2}\right)+f(y)\right]+\frac{y-x}{2}\left[f\left(x\right)+f\left(\frac{x+y}{2}\right)\right]=(y-x)\left[f(x)+f(y)\right]$$

for all $x,\ y\in I$, i.e., f satisfies equation (2.1). In view of Theorem 2.1, the continuity of f implies that $f(x)=ax+b,\ x\in I$, for some $a,\ b\in\mathbb{R}$. Since f is strictly monotonic we infer that $a\neq 0$. $\square$

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4. Equality of Lagrangean and Quasi-arithmetic Means Under Some Convexity Assumptions

In this section we examine the equation $L_f = Q_g$, imposing some additional conditions on f and g.

Theorem 4.1. *Let $I \subset \mathbb{R}$ be an interval, and $f, g : I \to \mathbb{R}$ be continuous and strictly monotonic functions. Assume that $g \circ f^{-1}$ and g are of the same type of convexity. Then the following conditions are pairwise equivalent:*

(i) $L_f = Q_g$;

(ii) there are $a, b, c, d \in \mathbb{R}$, $a \neq 0, c \neq 0$, *such that*

$$f(x) = ax + b, \qquad g(x) = cx + d, \qquad x \in I;$$

(iii) $L_f = Q_g = A$.

Proof. Assume, for instance, that $g \circ f^{-1}$ and g are convex. Let $F : I \to \mathbb{R}$ be any primitive function of f. Then the condition $L_f = Q_g$ can be written in the form

$$f^{-1}\left(\frac{F(y) - F(x)}{y - x}\right) = g^{-1}\left(\frac{g(x) + g(y)}{2}\right), \qquad x, y \in I, \quad x \neq y, \tag{4.1}$$

or, equivalently,

$$F(y) - F(x) = f \circ g^{-1}\left(\frac{g(x) + g(y)}{2}\right)(y - x), \qquad x, y \in I.$$

Using the identity

$$F(y) - F(x) = [F(y) - F(z)] + [F(z) - F(x)],$$

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we get

$$f\circ g^{-1}\left(\frac{g(x)+g(y)}{2}\right)(y-x)=f\circ g^{-1}\left(\frac{g(z)+g(y)}{2}\right)(y-z)$$
$$+f\circ g^{-1}\left(\frac{g(x)+g(z)}{2}\right)(z-x)$$

for all $x,\ y,z\in I$. Putting here $\lambda=\frac{y-z}{y-x}$ and $z=\lambda x+(1-\lambda)\,y$, we have

$$\frac{g(x)+g(y)}{2}=\left(f\circ g^{-1}\right)^{-1}\left(\lambda\left(f\circ g^{-1}\right)\left(\frac{g(\lambda x+(1-\lambda)\,y)+g(y)}{2}\right)\right.$$
$$\left.+(1-\lambda)\left(f\circ g^{-1}\right)\left(\frac{g(x)+g(\lambda x+(1-\lambda)\,y)}{2}\right)\right)$$

for all $x,\ y\in I$ and $\lambda\in[0,1]$. Using the convexity of $\left(f\circ g^{-1}\right)^{-1}$, we obtain

$$\lambda g(x)+(1-\lambda)\,g(y)\le g(\lambda x+(1-\lambda)\,y)\,,\qquad x,\ y\in I,\ \ \lambda\in[0,1],$$

i.e, g is concave. On the other hand, by the assumption, g is convex. Hence we infer that there are $c,\ d\in\mathbb{R},\ c\neq 0,$ such that

$$g(x)=cx+d,\qquad x\in I.$$

Making use of (4.1), we obtain

$$f^{-1}\left(\frac{F(y)-F(x)}{y-x}\right)=\frac{x+y}{2},$$

whence

$$F(y)-F(x)=(y-x)\,f\left(\frac{x+y}{2}\right)$$

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for all $x, y \in I$. In particular, we deduce that f is differentiable. Differentiating both sides with respect to x and then with respect to y we get

$$-f(x) = -f\left(\frac{x+y}{2}\right) + \frac{y-x}{2} f'\left(\frac{x+y}{2}\right)$$

and

$$f(y) = f\left(\frac{x+y}{2}\right) + \frac{y-x}{2} f'\left(\frac{x+y}{2}\right)$$

for all $x, y \in I$, which means that f satisfies the Jensen equation. Now, using Theorem 2.1, we complete the proof. $\square$

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