Journal of Inequalities in Pure and Applied Mathematics

MONOTONICITY AND CONVEXITY OF FOUR SEQUENCES ORIGINATING FROM NANSON'S INEQUALITY

LIANG-CHENG WANG

School of Mathematical Science
Chongqing Institute of Technology
No. 4 of Xingsheng Lu
Yangjiaping, Chongqing City
400050, CHINA

EMail: wlc@cqit.edu.cn

ISSN (electronic): 1443-5756
271-05

volume 7, issue 4, article 150, 2006.

Received 12 September, 2005; accepted 31 October, 2006.

Communicated by: F. Qi

Abstract

In the short note, four sequences originating from Nanson's inequality are introduced, their monotonicities and convexities are obtained, and Nanson's inequality is refined.

2000 Mathematics Subject Classification: 26D15.
Key words: Monotonicity, Convexity, Sequence, Nanson's inequality, Refinement.

The author was supported in part by the Key Research Foundation of Chongqing Institute of Technology under Grant 2004ZD94.
The author appreciates heartily Professor Feng Qi for his valuably revising this paper word by word.

Contents

1. Introduction

A real sequence $\{a_i\}_{i=1}^{k}$ for $k > 2$ is called convex if

$$a_i + a_{i+2} \geq 2a_{i+1} \tag{1.1}$$

for $i \in \mathbb{N}$ with $i + 2 \leq k$.

The Nanson's inequality (see [3, p. 465] and [1, 2, 4]) reads that if $\{a_i\}_{i=1}^{2n+1}$ is a convex sequence, then

$$\frac{1}{n}\sum_{k=1}^{n} a_{2k} \leq \frac{1}{n+1}\sum_{k=0}^{n} a_{2k+1}. \tag{1.2}$$

The equality in (1.2) holds only if $\{a_i\}_{i=1}^{2n+1}$ is an arithmetic sequence.

It is clear that inequality (1.2) can be rewritten as

$$H(n) \triangleq n\sum_{k=0}^{n} a_{2k+1} - (n+1)\sum_{k=1}^{n} a_{2k} \geq 0. \tag{1.3}$$

Similar to $H(n)$, it can be introduced for given $n \in \mathbb{N}$ that

$$\begin{aligned} h(m) = (n-m+1) &\sum_{k=m-1}^{n} a_{2k+1} \\ &- (n-m+2)\sum_{k=m}^{n} a_{2k} \quad \text{for } 1 \leq m \leq n+1, \end{aligned} \tag{1.4}$$

$$
C(m) = \frac{1}{n(n+1)} \times \left[m \sum_{i=0}^{m} a_{2i+1} + (n-m) \sum_{i=1}^{m} a_{2i} + (n+1) \sum_{i=m+1}^{n} a_{2i} \right] \tag{1.5}
$$

for $0 \le m \le n$, and

$$
c(m) = \frac{1}{n(n+1)} \left[(n-m+1) \sum_{i=m-1}^{n} a_{2i+1} + (n+1) \sum_{i=1}^{m-1} a_{2i} + (m-1) \sum_{i=m}^{n} a_{2i} \right] \tag{1.6}
$$

for $1 \le m \le n+1$, where $\sum_{i=q+1}^{q} b_i = 0$ is assumed for any $b_i \in \mathbb{R}$ and $q \in \mathbb{N}$.

The aim of this paper is to study monotonicity and convexity of H, h, C and c. From this, some new inequalities and refinements of (1.2) are deduced.

Our main results are the following two theorems.

Theorem 1.1. *Let $\{a_i\}_{i=1}^{2n+1}$ for $n \ge 1$ be a convex sequence. Then*

1. *the sequence $\{H(j)\}_{j=1}^{n}$ is increasing and convex,*
2. *the sequence $\{C(j)\}_{j=0}^{n}$ satisfies*

$$
\frac{1}{n} \sum_{i=1}^{n} a_{2i} = C(0) \le C(1) \le \cdots \le C(n-1) \le C(n) = \frac{1}{n+1} \sum_{i=0}^{n} a_{2i+1}. \tag{1.7}
$$

Theorem 1.2. *Let* $\{a_i\}_{i=1}^{2n+1}$ *for* $n \geq 1$ *be a convex sequence. Then*

1. *the sequence* $\{h(j)\}_{j=1}^{n+1}$ *is decreasing and convex,*
2. *the sequence* $\{c(j)\}_{j=1}^{n+1}$ *satisfies*

$$\frac{1}{n}\sum_{i=1}^{n} a_{2i} = c(n+1) \leq c(n) \leq \cdots \leq c(2) \leq c(1) = \frac{1}{n+1}\sum_{i=0}^{n} a_{2i+1}, \tag{1.8}$$

3. *and*

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} a_{2i} &= \frac{C(0)+c(n+1)}{2} \\ &\leq \frac{C(1)+c(n)}{2} \leq \cdots \\ &\leq \frac{C(n-1)+c(2)}{2} \\ &\leq \frac{C(n)+c(1)}{2} = \frac{1}{n+1}\sum_{i=0}^{n} a_{2i+1}. \end{aligned} \tag{1.9}$$

Remark 1. *Inequalities* (1.7), (1.8) *and* (1.9) *are refinements of* (1.2).

2. Proofs of the Theorems

Proof of Theorem 1.1. If $\{a_i\}_{i=1}^n$ is convex, then it is easy to see that

$$\begin{aligned}(2.1)\quad a_i - a_{i+1} - a_{n-1} + a_n &= (a_i - 2a_{i+1} + a_{i+2}) + (a_{i+1} - 2a_{i+2} + a_{i+3}) + \cdots \\ &\quad + (a_{n-4} - 2a_{n-3} + a_{n-2}) + (a_{n-3} - 2a_{n-2} + a_{n-1}) \\ &\quad + (a_{n-2} - 2a_{n-1} + a_n) \geq 0.\end{aligned}$$

From (1.1) and (2.1), it follows that

$$\begin{aligned}&H(j) - H(j-1) \\ &= j\sum_{i=0}^{j} a_{2i+1} - (j+1)\sum_{i=1}^{j} a_{2i} - (j-1)\sum_{i=0}^{j-1} a_{2i+1} + j\sum_{i=1}^{j-1} a_{2i} \\ &= \left(j\sum_{i=0}^{j} a_{2i+1} - (j-1)\sum_{i=0}^{j-1} a_{2i+1}\right) + \left(j\sum_{i=1}^{j-1} a_{2i} - (j+1)\sum_{i=1}^{j} a_{2i}\right) \\ &= \left(ja_{2j+1} + \sum_{i=0}^{j-1} a_{2i+1}\right) - \left(ja_{2j} + \sum_{i=1}^{j} a_{2i}\right) \\ &= \sum_{i=1}^{j} (a_{2i-1} - a_{2i} - a_{2j} + a_{2j+1}) \\ &\geq 0,\end{aligned}$$

which implies the increasing monotonicity of $H(j)$ for $1 \leq j \leq n$.

It is obvious that

$$C(k) = \frac{1}{n(n+1)}\left[H(k) + (n+1)\sum_{i=1}^{n} a_{2i}\right] = \frac{H(k)}{n(n+1)} + \frac{1}{n}\sum_{i=1}^{n} a_{2i}. \tag{2.2}$$

From the increasingly monotonic property of $H(j)$ for $1 \le j \le n$, inequalities in (1.7) are concluded.

For $j = 1, 2, \dots, n-2$, direct calculation gives

$$\begin{aligned}
&H(j) - 2H(j+1) + H(j+2) \\
&= \left(j\sum_{i=0}^{j} a_{2i+1} - (j+1)\sum_{i=1}^{j} a_{2i}\right) - 2\left((j+1)\sum_{i=0}^{j+1} a_{2i+1} - (j+2)\sum_{i=1}^{j+1} a_{2i}\right) \\
&\quad + \left((j+2)\sum_{i=0}^{j+2} a_{2i+1} - (j+3)\sum_{i=1}^{j+2} a_{2i}\right) \\
&= \left(j\sum_{i=0}^{j} a_{2i+1} - (j+1)\sum_{i=0}^{j+1} a_{2i+1}\right) + \left((j+2)\sum_{i=0}^{j+2} a_{2i+1} - (j+1)\sum_{i=0}^{j+1} a_{2i+1}\right) \\
&\quad + \left((j+2)\sum_{i=1}^{j+1} a_{2i} - (j+1)\sum_{i=1}^{j} a_{2i}\right) + \left((j+2)\sum_{i=1}^{j+1} a_{2i} - (j+3)\sum_{i=1}^{j+2} a_{2i}\right) \\
&= \left(-ja_{2j+3} - \sum_{i=0}^{j+1} a_{2i+1}\right) + \left((j+1)a_{2j+5} + \sum_{i=0}^{j+2} a_{2i+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left((j+1)a_{2j+2}+\sum_{i=1}^{j+1}a_{2i}\right)+\left(-(j+2)a_{2j+4}-\sum_{i=1}^{j+2}a_{2i}\right)\\
&=(j+1)a_{2j+2}-ja_{2j+3}-(j+2)a_{2j+4}+(j+1)a_{2j+5}\\
&\quad+\left(\sum_{i=1}^{j+1}a_{2i}-\sum_{i=1}^{j+2}a_{2i}\right)+\left(\sum_{i=0}^{j+2}a_{2i+1}-\sum_{i=0}^{j+1}a_{2i+1}\right)\\
&=(j+1)a_{2j+2}-ja_{2j+3}-(j+3)a_{2j+4}+(j+2)a_{2j+5}\\
&=(j+1)(a_{2j+2}-2a_{2j+3}+a_{2j+4})+(j+2)(a_{2j+3}-2a_{2j+4}+a_{2j+5})\\
&\geq 0
\end{aligned}
$$

which implies that the sequence $\{H(j)\}_{j=1}^{n}$ is convex. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. By the same arguments as in Theorem 1.1, the decreasing and convex properties of the sequences $\{h(j)\}_{j=1}^{n+1}$ and $\{c(j)\}_{j=1}^{n+1}$ are immediately obtained.

Adding (1.7) and (1.8) yields (1.9). The proof of Theorem 1.2 is complete. □

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