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APPROXIMATION OF B-CONTINUOUS AND B-DIFFERENTIABLE FUNCTIONS BY GBS OPERATORS OF BERNSTEIN BIVARIATE POLYNOMIALS

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Abstract

In this paper we give an approximation of B-continuous and B-differentiable functions by GBS operators of Bernstein bivariate polynomials.

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Key words: Linear positive operators, Bernstein bivariate polynomials, GBS operators, B-differentiable functions, approximation of B-differentiable functions by GBS operators, mixed modulus of smoothness.

Contents

1. Preliminaries

In this section, we recall some results which we will use in this article.

In the following, let X and Y be compact real intervals. A function $f : X \times Y \to \mathbb{R}$ is called a B-continuous (Bögel-continuous) function in $(x_0, y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f\left((x,y),(x_0,y_0)\right) = 0.$$

Here

$$\Delta f\left((x,y),(x_0,y_0)\right) = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$$

denotes a so-called mixed difference of f.

A function $f : X \times Y \to \mathbb{R}$ is called a B-differentiable (Bögel-differentiable) function in $(x_0, y_0) \in X \times Y$ if it exists and if the limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f\left((x,y),(x_0,y_0)\right)}{(x-x_0)(y-y_0)}.$$

The limit is named the B-differential of f in the point (x_0, y_0) and is denoted by $D_B f(x_0, y_0)$.

The definitions of B-continuity and B-differentiability were introduced by K. Bögel in the papers [5] and [6].

The function $f : X \times Y \to \mathbb{R}$ is B-bounded on $X \times Y$ if there exists $K > 0$ such that

$$\left|\Delta f\left((x,y),(s,t)\right)\right| \le K$$

for any $(x,y),(s,t) \in X \times Y$.

We shall use the function sets $B(X\times Y)=\{f:X\times Y\to\mathbb{R}|f$ bounded on $X\times Y\}$ with the usual sup-norm $\|\cdot\|_\infty$, $B_b(X\times Y)=\{f:X\times Y\to\mathbb{R}|f$ B-bounded on $X\times Y\}$ and we set $\|f\|_B=\sup_{(x,y),(s,t)\in X\times Y}|\Delta f((x,y),(s,t))|$, where

$$\begin{aligned} &f\in B_b(X\times Y),\\ &C_b(X\times Y)=\{f:X\times Y\to\mathbb{R}|f\ B-\text{continuous on } X\times Y\},\\ \text{and}\quad &D_b(X\times Y)=\{f:X\times Y\to\mathbb{R}|f\ B-\text{differentiable on } X\times Y\}.\end{aligned}$$

Let $f\in B_b(X\times Y)$. The function $\omega_{\text{mixed}}(f;\cdot,\cdot):[0,\infty)\times[0,\infty)\to\mathbb{R}$, defined by

$$\omega_{\text{mixed}}(f;\delta_1,\delta_2)=\sup\{|\Delta f((x,y),(s,t))|:|x-s|\le\delta_1,|y-t|\le\delta_2\} \tag{1.1}$$

for any $(\delta_1,\delta_2)\in[0,\infty)\times[0,\infty)$ is called the mixed modulus of smoothness.

For related topics, see [1], [2], [3] and [10].

Let $L:C_b(X\times Y)\to B(X\times Y)$ be a linear positive operator. The operator $UL:C_b(X\times Y)\to B(X\times Y)$ defined for any function $f\in C_b(X\times Y)$ and any $(x,y)\in X\times Y$ by

$$(ULf)(x,y)=(L(f(\cdot,y)+f(x,*)-f(\cdot,*)))(x,y) \tag{1.2}$$

is called the GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and second variable.

Let the functions $e_{ij}:X\times Y\to\mathbb{R}$, $(e_{ij})(x,y)=x^iy^j$ for any $(x,y)\in X\times Y$, where $i,j\in\mathbb{N}$. The following theorem is proved in [1].

Theorem 1.1. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in C_b(X \times Y)$, any $(x, y) \in (X \times Y)$ and any $\delta_1, \delta_2 > 0$, we have*

$$|f(x,y) - (ULf)(x,y)| \leq |f(x,y)|\,|1 - (Le_{00})(x,y)| + \Big[(Le_{00})(x,y) + \delta_1^{-1}\sqrt{(L(\cdot - x)^2)\,(x,y)} + \delta_2^{-1}\sqrt{(L(* - y)^2)\,(x,y)} + \delta_1^{-1}\delta_2^{-1}\sqrt{(L(\cdot - x)^2(* - y)^2)\,(x,y)}\Big]\omega_{mixed}(f;\delta_1,\delta_2). \tag{1.3}$$

In the following, we need the following theorem for estimating the rate of the convergence of the B-differentiable functions (see [11]).

Theorem 1.2. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in D_b(X \times Y)$ with $D_B f \in B(X \times Y)$, any $(x, y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, we have*

$$\begin{aligned} &|f(x,y) - (ULf)(x,y)| \\ &\leq |f(x,y)||1-(Le_{00})(x,y)|+3\|D_Bf\|_\infty\sqrt{(L(\cdot - x)^2(* - y)^2)\,(x,y)} \\ &\quad + \Big[\sqrt{(L(\cdot - x)^2(* - y)^2)(x,y)} + \delta_1^{-1}\sqrt{(L(\cdot - x)^4(* - y)^2)(x,y)} \\ &\quad + \delta_2^{-1}\sqrt{(L(\cdot - x)^2(* - y)^4)(x,y)} \\ &\quad + \delta_1^{-1}\delta_2^{-1}\left(L(\cdot - x)^2(* - y)^2\right)(x,y)\Big]\omega_{mixed}(D_Bf;\delta_1,\delta_2). \end{aligned} \tag{1.4}$$

2. Main Results

Let the sets $\Delta_2 = \{(x,y) \in \mathbb{R} \times \mathbb{R} | x, y \geq 0, x + y \leq 1\}$ and $\mathcal{F}(\Delta_2) = \{f | f : \Delta_2 \to \mathbb{R}\}$. For m a non zero natural number, let the operators $B_m : \mathcal{F}(\Delta_2) \to \mathcal{F}(\Delta_2)$, defined for any function $f \in \mathcal{F}(\Delta_2)$ by

$$(B_m f)(x,y) = \sum_{\substack{k,j=0 \\ k+j \leq m}} p_{m,k,j}(x,y) f\left(\frac{k}{m}, \frac{j}{m}\right) \tag{2.1}$$

for any $(x,y) \in \Delta_2$, where

$$p_{m,k,j}(x,y) = \frac{m!}{k!j!(m-k-j)!} x^k y^j (1-x-y)^{m-k-j}. \tag{2.2}$$

The operators are named Bernstein bivariate polynomials (see [8]).

Lemma 2.1. *The operators $(B_m)_{m \geq 1}$ are linear and positive on $\mathcal{F}(\Delta_2)$.*

Proof. The proof follows immediately. □

For m a non zero natural number, let the GBS operator of Bernstein bivariate polynomials UB_m (see [1]), $UB_m : C_b(\Delta_2) \to B(\Delta_2)$ defined for any function $f \in C_b(\Delta_2)$ and any $(x,y) \in \Delta_2$ by

$$\begin{aligned}(UB_m f)(x,y) &\\ &= \left(B_m(f(x,*) + f(\cdot,y) - f(\cdot,*))\right)(x,y) \\ &= \sum_{\substack{k,j=0 \\ k+j \leq m}} p_{m,k,j}(x,y) \left[f\left(x, \frac{j}{m}\right) + f\left(\frac{k}{m}, y\right) - f\left(\frac{k}{m}, \frac{j}{m}\right)\right].\end{aligned} \tag{2.3}$$

Lemma 2.2. *The operators* $(B_m)_{m\geq 1}$ *verify for any* $(x,y)\in\Delta_2$ *the following:*

$$(B_m e_{00})(x,y)=1; \tag{2.4}$$

$$\left(B_m(\cdot-x)^2\right)(x,y)=\frac{x(1-x)}{m}; \tag{2.5}$$

$$\left(B_m(*-y)^2\right)(x,y)=\frac{y(1-y)}{m}; \tag{2.6}$$

$$\begin{aligned}\left(B_m(\cdot-x)^2(*-y)^2\right)(x,y)\\ =\frac{3(m-2)}{m^3}x^2y^2-\frac{m-2}{m^3}(x^2y+xy^2)+\frac{m-1}{m^3}xy;\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\left(B_m(\cdot-x)^4(*-y)^2\right)(x,y)\\ &=-\frac{5(3m^2-26m+24)}{m^5}x^4y^2+\frac{6(3m^2-26m+24)}{m^5}x^3y^2\\ &\quad-\frac{6(m^2-7m+6)}{m^5}x^3y-\frac{3m^2-41m+42}{m^5}x^2y^2\\ &\quad+\frac{3m^2-26m+24}{m^5}x^4y+\frac{3m^2-17m+14}{m^5}x^2y\\ &\quad-\frac{m-2}{m^5}xy^2+\frac{m-1}{m^5}xy\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\left(B_m(\cdot-x)^2(*-y)^4\right)(x,y)\\ &=-\frac{5(m^2-26m+24)}{m^5}x^2y^4+\frac{6(3m^2-26m+24)}{m^5}x^2y^3\end{aligned} \tag{2.9}$$

$$-\frac{6(m^2-7m+6)}{m^5}xy^3-\frac{3m^2-41m+42}{m^5}x^2y^2$$
$$+\frac{3m^2-26m+24}{m^5}xy^4+\frac{3m^2-17m+14}{m^5}xy^2$$
$$-\frac{m-2}{m^5}x^2y+\frac{m-1}{m^5}xy$$

for any non zero natural number m.

Proof. Let $(x,y)\in\Delta_2$ and m be a non zero natural number. We have

$$\begin{aligned}(B_m e_{00})(x,y) &= \sum_{\substack{k,j=0\\k+j\le m}} \frac{m!}{k!j!(m-k-j)!}x^k y^j(1-x-y)^{m-k-j}\\ &= (x+y+1-x-y)^m = 1,\end{aligned}$$

so (2.4) holds,

$$\begin{aligned}(B_m e_{10})(x,y) &= \sum_{\substack{k,j=0\\k+j\le m}} \frac{m!}{k!j!(m-k-j)!}x^k y^j(1-x-y)^{m-k-j}\frac{k}{m}\\ &= x\sum_{\substack{k=1,j=0\\k+j\le m}} \frac{(m-1)!}{(k-1)!j!(m-k-j)!}x^{k-1}y^j(1-x-y)^{m-k-j}\\ &= x,\end{aligned}$$

it results that

$$(B_m e_{10})(x,y) = x \tag{2.10}$$

and similarly

$$(B_m e_{01})(x,y) = y. \tag{2.11}$$

In the same way, using the formulas

$$k^2 = k(k-1) + k,$$
$$k^3 = k(k-1)(k-2) + 3k(k-1) + k,$$
$$k^4 = k(k-1)(k-2)(k-3) + 6k(k-1)(k-2) + 7k(k-1) + k,$$

we obtain

$$(B_m e_{20})(x,y) = \frac{m-1}{m}x^2 + \frac{1}{m}x, \tag{2.12}$$

$$(B_m e_{30})(x,y) = \frac{(m-1)(m-2)}{m^2}x^3 + \frac{3(m-1)}{m^2}x^2 + \frac{1}{m^2}x, \tag{2.13}$$

$$(B_m e_{40})(x,y) = \frac{(m-1)(m-2)(m-3)}{m^3}x^4 + \frac{6(m-1)(m-2)}{m^3}x^3 + \frac{7(m-1)}{m^3}x^2 + \frac{1}{m^3}x \tag{2.14}$$

and similarly the relations $(B_m e_{02})(x,y)$, $(B_m e_{03})(x,y)$, $(B_m e_{04})(x,y)$.

We have

$$\begin{aligned}(B_m e_{11})(x,y) &= \frac{m-1}{m} y \sum_{\substack{k=0,j=1\\k+j\le m}} \frac{(m-1)!}{k!(j-1)!(m-k-j)!} x^k y^{j-1}(1-x-y)^{m-k-j} \frac{k}{m-1} \\ &= \frac{m-1}{m} y (B_{m-1} e_{10})(x,y),\end{aligned}$$

$$
\begin{aligned}
&(B_m e_{21})(x,y) \\
&= \left(\frac{m-1}{m}\right)^2 y \sum_{\substack{k=0,j=1\\ k+j\le m}} \frac{(m-1)!}{k!(j-1)!(m-k-j)!} x^k y^{j-1}(1-x-y)^{m-k-j}\left(\frac{k}{m-1}\right)^2 \\
&= \left(\frac{m-1}{m}\right)^2 y(B_{m-1}e_{20})(x,y),
\end{aligned}
$$

and in the same way, we write $(B_m e_{31})(x,y)$, $(B_m e_{41})(x,y)$, $(B_m e_{32})(x,y)$, $(B_m e_{42})(x,y)$. Taking (2.12) - (2.14) into account, we obtain

$$(B_m e_{11})(x,y) = \frac{m-1}{m} xy, \tag{2.15}$$

$$(B_m e_{21})(x,y) = \frac{(m-1)(m-2)}{m^2} x^2 y + \frac{m-1}{m^2} xy, \tag{2.16}$$

$$
\begin{aligned}
(B_m e_{31})(x,y) &= \frac{(m-1)(m-2)(m-3)}{m^3} x^3 y \\
&\quad + \frac{3(m-1)(m-2)}{m^3} x^2 y + \frac{m-1}{m^3} xy,
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
(B_m e_{41})(x,y) &= \frac{(m-1)(m-2)(m-3)(m-4)}{m^4} x^4 y \\
&\quad + \frac{6(m-1)(m-2)(m-3)}{m^4} x^3 y \\
&\quad + \frac{7(m-1)(m-2)}{m^4} x^2 y + \frac{m-1}{m^4} xy,
\end{aligned}
\tag{2.18}
$$

$$(2.19)\quad (B_m e_{22})(x,y) = \frac{(m-1)(m-2)(m-3)}{m^3}x^2y^2 + \frac{(m-1)(m-2)}{m^3}(x^2y + xy^2) + \frac{m-1}{m^3}xy,$$

$$(2.20)\quad (B_m e_{32})(x,y) = \frac{(m-1)(m-2)(m-3)(m-4)}{m^4}x^3y^2 + \frac{(m-1)(m-2)(m-3)}{m^4}x^3y + \frac{3(m-1)(m-2)(m-3)}{m^4}x^2y^2 + \frac{3(m-1)(m-2)}{m^4}x^2y + \frac{(m-1)(m-2)}{m^4}xy^2 + \frac{m-1}{m^4}xy,$$

$$(2.21)\quad (B_m e_{42})(x,y) = \frac{(m-1)(m-2)(m-3)(m-4)(m-5)}{m^5}x^4y^2 + \frac{(m-1)(m-2)(m-3)(m-4)}{m^5}x^4y + \frac{6(m-1)(m-2)(m-3)(m-4)}{m^5}x^3y^2 + \frac{6(m-1)(m-2)(m-3)}{m^5}x^3y + \frac{7(m-1)(m-2)(m-3)}{m^5}x^2y^2 + \frac{7(m-1)(m-2)}{m^5}x^2y + \frac{(m-1)(m-2)}{m^5}xy^2 + \frac{m-1}{m^5}xy$$

and similarly the relations $(B_m e_{12})(x,y)$, $(B_m e_{13})(x,y)$, $(B_m e_{14})(x,y)$, $(B_m e_{23})(x,y)$, $(B_m e_{24})(x,y)$.

Now, we have

$$(B_m(\cdot - x)^2)(x,y) = (B_m e_{20})(x,y) - 2x(B_m e_{10})(x,y) + x^2(B_m e_{02})(x,y),$$

$$\begin{aligned}(B_m(\cdot - x)^2(* - y)^2)(x,y) &= (B_m e_{22})(x,y) - 2y(B_m e_{21})(x,y) + y^2(B_m e_{20})(x,y)\\ &\quad - 2x(B_m e_{12})(x,y) + 4xy(B_m e_{11})(x,y) - 2xy^2(B_m e_{10})(x,y)\\ &\quad + x^2(B_m e_{02})(x,y) - 2x^2y(B_m e_{01})(x,y) + x^2y^2(B_m e_{00})(x,y),\end{aligned}$$

$$\begin{aligned}&(B_m(\cdot - x)^4(* - y)^2)(x,y)\\ &= (B_m e_{40})(x,y) - 2y(B_m e_{41})(x,y) + y^2(B_m e_{40})(x,y)\\ &\quad - 4x(B_m e_{32})(x,y) + 8xy(B_m e_{31})(x,y) - 4xy^2(B_m e_{30})(x,y)\\ &\quad + 6x^2(B_m e_{22})(x,y) - 12x^2y(B_m e_{21})(x,y) + 6x^2y^2(B_m e_{20})(x,y)\\ &\quad - 4x^3(B_m e_{12})(x,y) + 8x^3y(B_m e_{11})(x,y) - 4x^3y^2(B_m e_{10})(x,y)\\ &\quad + x^4(B_m e_{02})(x,y) - 2x^4y(B_m e_{01})(x,y) + x^4y^2(B_m e_{00})(x,y)\end{aligned}$$

and taking (2.9) – (2.21) into account, we obtain (2.5), (2.7) and (2.8). Similarly we obtain (2.9). □

Lemma 2.3. *The operators $(B_m)_{m\geq 1}$ verify for any $(x,y)\in\Delta_2$ the following inequalities:*

$$(B_m(\cdot - x)^2)(x,y) \leq \frac{1}{4m}, \tag{2.22}$$

$$(B_m(* - y)^2)(x,y) \leq \frac{1}{4m}, \tag{2.23}$$

for any non zero natural number m,

$$\left(B_m(\cdot - x)^2(* - y)^2\right)(x, y) \le \frac{9}{4m^2}, \tag{2.24}$$

for any natural number m, $m \ge 2$,

$$\left(B_m(\cdot - x)^4(* - y)^2\right)(x, y) \le \frac{9}{m^3}, \tag{2.25}$$

$$\left(B_m(\cdot - x)^2(* - y)^4\right)(x, y) \le \frac{9}{m^3}, \tag{2.26}$$

for any natural number m, $m \ge 8$.

Proof. Because $x(1-x) \le \frac{1}{4}$ for any $x \in [0, 1]$, (2.22) and (2.23) results.

From (2.7), we have

$$\begin{aligned}
&\left(B_m(\cdot - x)^2(* - y)^2\right)(x, y) \\
&= \frac{2(m-2)}{m^3} x^2 y^2 + \frac{m-2}{m^3} x(1-x)y(1-y) + \frac{1}{m^3} xy \\
&\le \frac{2(m-2)}{m^3} + \frac{m-2}{16m^3} + \frac{1}{m^3} \\
&= \frac{33m-50}{16m^3},
\end{aligned}$$

from where (2.24) results.

From (2.8), we have

$$\left(B_m(\cdot - x)^4(* - y)^2\right)(x,y)$$
$$= \frac{6(3m^2-26m+24)}{m^5}x^3y^2(1-x) + \frac{3m^2-26m+24}{m^5}x^4y(y+1)$$
$$- \frac{6(m^2-7m+6)}{m^5}x^3y + \frac{3m^2-17m+14}{m^5}x^2y(1-y)$$
$$+ \frac{24m-28}{m^5}x^2y^2 + \frac{m-2}{m^5}xy(1-y) + xy.$$

But

$$\frac{3m^2-26m+24}{m^5}x^4y(y+1) \le 2\frac{3m^2-26m+24}{m^5}x^2y$$
$$= \frac{6m^2-42m+36}{m^5}x^2y - \frac{10m-12}{m^5}x^2y$$
$$\le \frac{6m^2-42m+36}{m^5}x^2y - \frac{10m-12}{m^5}x^3y^2$$

and then, from the inequalities above, we obtain

$$\left(B_m(\cdot - x)^4(* - y)^2\right)(x,y) \tag{2.27}$$
$$\le \frac{6(3m^2-26m+24)}{m^5}x^3y^2(1-x) + \frac{6m^2-42m+36}{m^5}x^2y(1-y)$$
$$+ \frac{3m^2-17m+14}{m^5}x^2y(1-y) + \frac{10m-12}{m^5}x^2y^2(1-y)$$
$$+ \frac{14m-16}{m^5}x^2y^2 + \frac{m-2}{m^5}xy(1-y) + xy.$$

Because $x(1-x) \le \frac{1}{4}$, $y(1-y) \le \frac{1}{4}$, $xy \le 1$ for any $x, y \in [0,1]$, from (2.27) we have

$$\begin{aligned}\left(B_m(\cdot - x)^4(* - y)^2\right)(x,y) &\le \frac{6(3m^2-26m+24)}{4m^5} + \frac{6m^2-42m+36}{4m^5} \\ &\quad + \frac{3m^2-17m+14}{4m^5} + \frac{10m-12}{4m^5} + \frac{14m-16}{m^5} + \frac{m-2}{4m^5} + 1 \\ &= \frac{27m^2-148m+170}{m^5},\end{aligned}$$

from where (2.25) results. □

Theorem 2.4. *Let the function $f \in C_b(\Delta_2)$. Then, for any $(x,y) \in \Delta_2$, any natural number m, $m \ge 2$, we have*

$$\begin{aligned}(2.28)\quad &|f(x,y) - (UB_m f)(x,y)| \\ &\le \left(1 + \delta_1^{-1}\frac{1}{2\sqrt{m}} + \delta_2^{-1}\frac{1}{2\sqrt{m}} + \delta_1^{-1}\delta_2^{-1}\frac{3}{2m}\right)\omega_{mixed}(f;\delta_1,\delta_2)\end{aligned}$$

for any $\delta_1, \delta_2 > 0$ and

$$(2.29)\qquad |f(x,y) - (UB_m f)(x,y)| \le \frac{7}{2}\omega_{mixed}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right).$$

Proof. For the first inequality we apply Theorem 1.1 and Lemma 2.3. The inequality (2.29) is obtained from (2.28) by choosing $\delta_1 = \delta_2 = \frac{1}{\sqrt{m}}$. □

Corollary 2.5. *If $f \in C_b(\Delta_2)$, then*

$$\lim_{m\to\infty} (UB_m f)(x,y) = f(x,y) \tag{2.30}$$

uniformly on Δ_2.

Proof. Because $f \in C_b(\Delta_2)$, there results that f is uniform B-continuous on Δ_2 and then $\lim_{m\to\infty} \omega_{\text{mixed}}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right) = 0$ (see [2] or [3]). From (2.29), there results the conclusion. □

Theorem 2.6. *Let the function $f \in D_b(\Delta_2)$ with $D_B f \in B(\Delta_2)$. Then for any $(x,y) \in \Delta_2$, any natural number m, $m \geq 8$, we have*

$$|f(x,y) - (UB_m f)(x,y)| \leq \frac{9}{2m} \|D_b f\|_\infty + \left(\frac{3}{2m} + \delta_1^{-1} \frac{3}{m\sqrt{m}} + \delta_2^{-1} \frac{3}{m\sqrt{m}} + \delta_1^{-1}\delta_2^{-1} \frac{9}{4m^2} \right) \omega_{mixed}(D_B f; \delta_1, \delta_2) \tag{2.31}$$

for any $\delta_1, \delta_2 > 0$ and

$$|f(x,y) - (UB_m f)(x,y)| \leq \frac{3}{4m} \left(6\|D_B f\|_\infty + 13\omega_{mixed}\left(D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}} \right) \right). \tag{2.32}$$

Proof. It results from Theorem 1.2 and Lemma 2.3. □

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