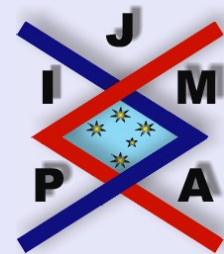


## APPROXIMATION OF $B$ -CONTINUOUS AND $B$ -DIFFERENTIABLE FUNCTIONS BY GBS OPERATORS OF BERNSTEIN BIVARIATE POLYNOMIALS

OVIDIU T. POP AND MIRCEA FARCAȘ

Vest University "Vasile Goldiș" of Arad Branch of Satu Mare  
26 Mihai Viteazu Street  
Satu Mare 440030, Romania.  
*EMail:* [ovidiutiberiu@yahoo.com](mailto:ovidiutiberiu@yahoo.com)

National College "Mihai Eminescu"  
5 Mihai Eminescu Street  
Satu Mare 440014, Romania.  
*EMail:* [mirceafarcas2005@yahoo.com](mailto:mirceafarcas2005@yahoo.com)



---

volume 7, issue 3, article 92,  
2006.

*Received 13 September, 2005;  
accepted 03 June, 2006.*

*Communicated by: S.S. Dragomir*

---

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

## Abstract

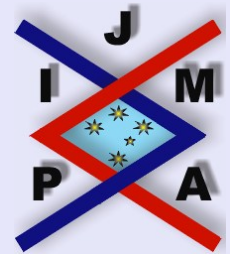
In this paper we give an approximation of  $B$ -continuous and  $B$ -differentiable functions by GBS operators of Bernstein bivariate polynomials.

*2000 Mathematics Subject Classification:* 41A10, 41A25, 41A35, 41A36, 41A63.

*Key words:* Linear positive operators, Bernstein bivariate polynomials, GBS operators,  $B$ -differentiable functions, approximation of  $B$ -differentiable functions by GBS operators, mixed modulus of smoothness.

## Contents

1	Preliminaries	3
2	Main Results	6
	References	



### Approximation of $B$ -Continuous and $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 2 of 18

# 1. Preliminaries

In this section, we recall some results which we will use in this article.

In the following, let  $X$  and  $Y$  be compact real intervals. A function  $f : X \times Y \rightarrow \mathbb{R}$  is called a  $B$ -continuous (Bögel-continuous) function in  $(x_0, y_0) \in X \times Y$  if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \Delta f((x, y), (x_0, y_0)) = 0.$$

Here

$$\Delta f((x, y), (x_0, y_0)) = f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0)$$

denotes a so-called mixed difference of  $f$ .

A function  $f : X \times Y \rightarrow \mathbb{R}$  is called a  $B$ -differentiable (Bögel-differentiable) function in  $(x_0, y_0) \in X \times Y$  if it exists and if the limit is finite:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{\Delta f((x, y), (x_0, y_0))}{(x - x_0)(y - y_0)}.$$

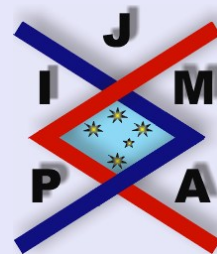
The limit is named the  $B$ -differential of  $f$  in the point  $(x_0, y_0)$  and is denoted by  $D_B f(x_0, y_0)$ .

The definitions of  $B$ -continuity and  $B$ -differentiability were introduced by K. Bögel in the papers [5] and [6].

The function  $f : X \times Y \rightarrow \mathbb{R}$  is  $B$ -bounded on  $X \times Y$  if there exists  $K > 0$  such that

$$|\Delta f((x, y), (s, t))| \leq K$$

for any  $(x, y), (s, t) \in X \times Y$ .



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 3 of 18

We shall use the function sets  $B(X \times Y) = \{f : X \times Y \rightarrow \mathbb{R} | f \text{ bounded on } X \times Y\}$  with the usual sup-norm  $\|\cdot\|_\infty$ ,  $B_b(X \times Y) = \{f : X \times Y \rightarrow \mathbb{R} | f \text{ } B\text{-bounded on } X \times Y\}$  and we set  $\|f\|_B = \sup_{(x,y),(s,t) \in X \times Y} |\Delta f((x,y),(s,t))|$ ,

where

$$f \in B_b(X \times Y),$$

$$C_b(X \times Y) = \{f : X \times Y \rightarrow \mathbb{R} | f \text{ } B\text{-continuous on } X \times Y\},$$

$$\text{and } D_b(X \times Y) = \{f : X \times Y \rightarrow \mathbb{R} | f \text{ } B\text{-differentiable on } X \times Y\}.$$

Let  $f \in B_b(X \times Y)$ . The function  $\omega_{\text{mixed}}(f; \cdot, \cdot) : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ , defined by

$$(1.1) \quad \omega_{\text{mixed}}(f; \delta_1, \delta_2) = \sup \{|\Delta f((x,y),(s,t))| : |x-s| \leq \delta_1, |y-t| \leq \delta_2\}$$

for any  $(\delta_1, \delta_2) \in [0, \infty) \times [0, \infty)$  is called the mixed modulus of smoothness.

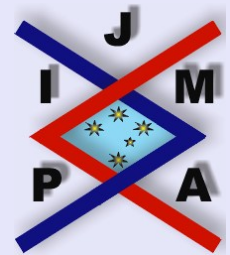
For related topics, see [1], [2], [3] and [10].

Let  $L : C_b(X \times Y) \rightarrow B(X \times Y)$  be a linear positive operator. The operator  $UL : C_b(X \times Y) \rightarrow B(X \times Y)$  defined for any function  $f \in C_b(X \times Y)$  and any  $(x,y) \in X \times Y$  by

$$(1.2) \quad (ULf)(x,y) = (L(f(\cdot, y) + f(x, *) - f(\cdot, *))) (x,y)$$

is called the GBS operator ("Generalized Boolean Sum" operator) associated to the operator  $L$ , where "." and "\*" stand for the first and second variable.

Let the functions  $e_{ij} : X \times Y \rightarrow \mathbb{R}$ ,  $(e_{ij})(x,y) = x^i y^j$  for any  $(x,y) \in X \times Y$ , where  $i, j \in \mathbb{N}$ . The following theorem is proved in [1].



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaş

Title Page

Contents



Go Back

Close

Quit

Page 4 of 18

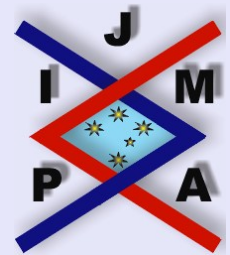
**Theorem 1.1.** Let  $L : C_b(X \times Y) \rightarrow B(X \times Y)$  be a linear positive operator and  $UL : C_b(X \times Y) \rightarrow B(X \times Y)$  the associated GBS operator. Then for any  $f \in C_b(X \times Y)$ , any  $(x, y) \in (X \times Y)$  and any  $\delta_1, \delta_2 > 0$ , we have

$$(1.3) \quad |f(x, y) - (ULf)(x, y)| \leq |f(x, y)| |1 - (Le_{00})(x, y)| \\ + \left[ (Le_{00})(x, y) + \delta_1^{-1} \sqrt{(L(\cdot - x)^2)(x, y)} + \delta_2^{-1} \sqrt{(L(* - y)^2)(x, y)} \right. \\ \left. + \delta_1^{-1} \delta_2^{-1} \sqrt{(L(\cdot - x)^2(* - y)^2)(x, y)} \right] \omega_{mixed}(f; \delta_1, \delta_2).$$

In the following, we need the following theorem for estimating the rate of the convergence of the  $B$ -differentiable functions (see [11]).

**Theorem 1.2.** Let  $L : C_b(X \times Y) \rightarrow B(X \times Y)$  be a linear positive operator and  $UL : C_b(X \times Y) \rightarrow B(X \times Y)$  the associated GBS operator. Then for any  $f \in D_b(X \times Y)$  with  $D_B f \in B(X \times Y)$ , any  $(x, y) \in X \times Y$  and any  $\delta_1, \delta_2 > 0$ , we have

$$(1.4) \quad |f(x, y) - (ULf)(x, y)| \\ \leq |f(x, y)| |1 - (Le_{00})(x, y)| + 3 \|D_B f\|_\infty \sqrt{(L(\cdot - x)^2(* - y)^2)(x, y)} \\ + \left[ \sqrt{(L(\cdot - x)^2(* - y)^2)(x, y)} + \delta_1^{-1} \sqrt{(L(\cdot - x)^4(* - y)^2)(x, y)} \right. \\ \left. + \delta_2^{-1} \sqrt{(L(\cdot - x)^2(* - y)^4)(x, y)} \right. \\ \left. + \delta_1^{-1} \delta_2^{-1} (L(\cdot - x)^2(* - y)^2)(x, y) \right] \omega_{mixed}(D_B f; \delta_1, \delta_2).$$



Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 5 of 18

## 2. Main Results

Let the sets  $\Delta_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x, y \geq 0, x + y \leq 1\}$  and  $\mathcal{F}(\Delta_2) = \{f | f : \Delta_2 \rightarrow \mathbb{R}\}$ . For  $m$  a non zero natural number, let the operators  $B_m : \mathcal{F}(\Delta_2) \rightarrow \mathcal{F}(\Delta_2)$ , defined for any function  $f \in \mathcal{F}(\Delta_2)$  by

$$(2.1) \quad (B_m f)(x, y) = \sum_{\substack{k, j=0 \\ k+j \leq m}} p_{m, k, j}(x, y) f\left(\frac{k}{m}, \frac{j}{m}\right)$$

for any  $(x, y) \in \Delta_2$ , where

$$(2.2) \quad p_{m, k, j}(x, y) = \frac{m!}{k!j!(m-k-j)!} x^k y^j (1-x-y)^{m-k-j}.$$

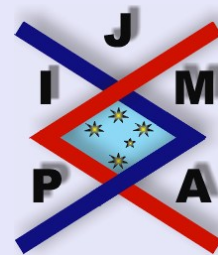
The operators are named Bernstein bivariate polynomials (see [8]).

**Lemma 2.1.** *The operators  $(B_m)_{m \geq 1}$  are linear and positive on  $\mathcal{F}(\Delta_2)$ .*

*Proof.* The proof follows immediately. □

For  $m$  a non zero natural number, let the GBS operator of Bernstein bivariate polynomials  $UB_m$  (see [1]),  $UB_m : C_b(\Delta_2) \rightarrow B(\Delta_2)$  defined for any function  $f \in C_b(\Delta_2)$  and any  $(x, y) \in \Delta_2$  by

$$(2.3) \quad \begin{aligned} (UB_m f)(x, y) &= (B_m(f(x, *) + f(\cdot, y) - f(\cdot, *))) (x, y) \\ &= \sum_{\substack{k, j=0 \\ k+j \leq m}} p_{m, k, j}(x, y) \left[ f\left(x, \frac{j}{m}\right) + f\left(\frac{k}{m}, y\right) - f\left(\frac{k}{m}, \frac{j}{m}\right) \right]. \end{aligned}$$



Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 6 of 18

**Lemma 2.2.** *The operators  $(B_m)_{m \geq 1}$  verify for any  $(x, y) \in \Delta_2$  the following:*

$$(2.4) \quad (B_m e_{00})(x, y) = 1;$$

$$(2.5) \quad (B_m(\cdot - x)^2)(x, y) = \frac{x(1-x)}{m};$$

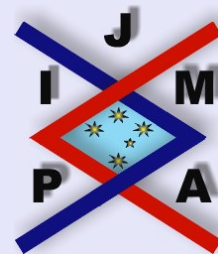
$$(2.6) \quad (B_m(* - y)^2)(x, y) = \frac{y(1-y)}{m};$$

$$(2.7) \quad (B_m(\cdot - x)^2(* - y)^2)(x, y) \\ = \frac{3(m-2)}{m^3} x^2 y^2 - \frac{m-2}{m^3} (x^2 y + x y^2) + \frac{m-1}{m^3} x y;$$

$$(2.8) \quad (B_m(\cdot - x)^4(* - y)^2)(x, y) \\ = -\frac{5(3m^2 - 26m + 24)}{m^5} x^4 y^2 + \frac{6(3m^2 - 26m + 24)}{m^5} x^3 y^2 \\ - \frac{6(m^2 - 7m + 6)}{m^5} x^3 y - \frac{3m^2 - 41m + 42}{m^5} x^2 y^2 \\ + \frac{3m^2 - 26m + 24}{m^5} x^4 y + \frac{3m^2 - 17m + 14}{m^5} x^2 y \\ - \frac{m-2}{m^5} x y^2 + \frac{m-1}{m^5} x y$$

and

$$(2.9) \quad (B_m(\cdot - x)^2(* - y)^4)(x, y) \\ = -\frac{5(m^2 - 26m + 24)}{m^5} x^2 y^4 + \frac{6(3m^2 - 26m + 24)}{m^5} x^2 y^3$$



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 7 of 18

$$\begin{aligned}
& - \frac{6(m^2 - 7m + 6)}{m^5} xy^3 - \frac{3m^2 - 41m + 42}{m^5} x^2 y^2 \\
& + \frac{3m^2 - 26m + 24}{m^5} xy^4 + \frac{3m^2 - 17m + 14}{m^5} xy^2 \\
& \quad - \frac{m - 2}{m^5} x^2 y + \frac{m - 1}{m^5} xy
\end{aligned}$$

for any non zero natural number  $m$ .

*Proof.* Let  $(x, y) \in \Delta_2$  and  $m$  be a non zero natural number. We have

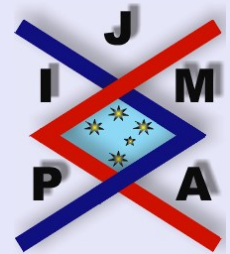
$$\begin{aligned}
(B_m e_{00})(x, y) &= \sum_{\substack{k, j=0 \\ k+j \leq m}} \frac{m!}{k!j!(m-k-j)!} x^k y^j (1-x-y)^{m-k-j} \\
&= (x+y+1-x-y)^m = 1,
\end{aligned}$$

so (2.4) holds,

$$\begin{aligned}
(B_m e_{10})(x, y) &= \sum_{\substack{k, j=0 \\ k+j \leq m}} \frac{m!}{k!j!(m-k-j)!} x^k y^j (1-x-y)^{m-k-j} \frac{k}{m} \\
&= x \sum_{\substack{k=1, j=0 \\ k+j \leq m}} \frac{(m-1)!}{(k-1)!j!(m-k-j)!} x^{k-1} y^j (1-x-y)^{m-k-j} \\
&= x,
\end{aligned}$$

it results that

$$(2.10) \quad (B_m e_{10})(x, y) = x$$



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 8 of 18



and similarly

$$(2.11) \quad (B_m e_{01})(x, y) = y.$$

In the same way, using the formulas

$$k^2 = k(k-1) + k,$$

$$k^3 = k(k-1)(k-2) + 3k(k-1) + k,$$

$$k^4 = k(k-1)(k-2)(k-3) + 6k(k-1)(k-2) + 7k(k-1) + k,$$

we obtain

$$(2.12) \quad (B_m e_{20})(x, y) = \frac{m-1}{m} x^2 + \frac{1}{m} x,$$

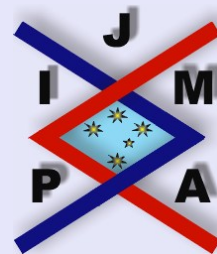
$$(2.13) \quad (B_m e_{30})(x, y) = \frac{(m-1)(m-2)}{m^2} x^3 + \frac{3(m-1)}{m^2} x^2 + \frac{1}{m^2} x,$$

$$(2.14) \quad (B_m e_{40})(x, y) = \frac{(m-1)(m-2)(m-3)}{m^3} x^4 + \frac{6(m-1)(m-2)}{m^3} x^3 + \frac{7(m-1)}{m^3} x^2 + \frac{1}{m^3} x$$

and similarly the relations  $(B_m e_{02})(x, y)$ ,  $(B_m e_{03})(x, y)$ ,  $(B_m e_{04})(x, y)$ .

We have

$$\begin{aligned} & (B_m e_{11})(x, y) \\ &= \frac{m-1}{m} y \sum_{\substack{k=0, j=1 \\ k+j \leq m}} \frac{(m-1)!}{k!(j-1)!(m-k-j)!} x^k y^{j-1} (1-x-y)^{m-k-j} \frac{k}{m-1} \\ &= \frac{m-1}{m} y (B_{m-1} e_{10})(x, y), \end{aligned}$$



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 9 of 18

$$\begin{aligned}
& (B_m e_{21})(x, y) \\
&= \left(\frac{m-1}{m}\right)^2 y \sum_{\substack{k=0, j=1 \\ k+j \leq m}} \frac{(m-1)!}{k!(j-1)!(m-k-j)!} x^k y^{j-1} (1-x-y)^{m-k-j} \left(\frac{k}{m-1}\right)^2 \\
&= \left(\frac{m-1}{m}\right)^2 y (B_{m-1} e_{20})(x, y),
\end{aligned}$$

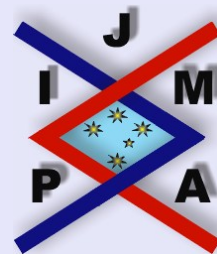
and in the same way, we write  $(B_m e_{31})(x, y)$ ,  $(B_m e_{41})(x, y)$ ,  $(B_m e_{32})(x, y)$ ,  $(B_m e_{42})(x, y)$ . Taking (2.12) - (2.14) into account, we obtain

$$(2.15) \quad (B_m e_{11})(x, y) = \frac{m-1}{m} xy,$$

$$(2.16) \quad (B_m e_{21})(x, y) = \frac{(m-1)(m-2)}{m^2} x^2 y + \frac{m-1}{m^2} xy,$$

$$(2.17) \quad (B_m e_{31})(x, y) = \frac{(m-1)(m-2)(m-3)}{m^3} x^3 y + \frac{3(m-1)(m-2)}{m^3} x^2 y + \frac{m-1}{m^3} xy,$$

$$(2.18) \quad (B_m e_{41})(x, y) = \frac{(m-1)(m-2)(m-3)(m-4)}{m^4} x^4 y + \frac{6(m-1)(m-2)(m-3)}{m^4} x^3 y + \frac{7(m-1)(m-2)}{m^4} x^2 y + \frac{m-1}{m^4} xy,$$



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

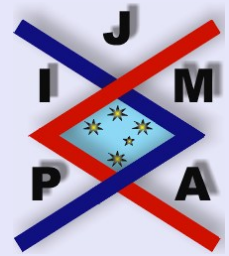
Quit

Page 10 of 18

$$(2.19) \quad (B_m e_{22})(x, y) = \frac{(m-1)(m-2)(m-3)}{m^3} x^2 y^2 + \frac{(m-1)(m-2)}{m^3} (x^2 y + x y^2) + \frac{m-1}{m^3} x y,$$

$$(2.20) \quad (B_m e_{32})(x, y) = \frac{(m-1)(m-2)(m-3)(m-4)}{m^4} x^3 y^2 + \frac{(m-1)(m-2)(m-3)}{m^4} x^3 y + \frac{3(m-1)(m-2)(m-3)}{m^4} x^2 y^2 + \frac{3(m-1)(m-2)}{m^4} x^2 y + \frac{(m-1)(m-2)}{m^4} x y^2 + \frac{m-1}{m^4} x y,$$

$$(2.21) \quad (B_m e_{42})(x, y) = \frac{(m-1)(m-2)(m-3)(m-4)(m-5)}{m^5} x^4 y^2 + \frac{(m-1)(m-2)(m-3)(m-4)}{m^5} x^4 y + \frac{6(m-1)(m-2)(m-3)(m-4)}{m^5} x^3 y^2 + \frac{6(m-1)(m-2)(m-3)}{m^5} x^3 y + \frac{7(m-1)(m-2)(m-3)}{m^5} x^2 y^2 + \frac{7(m-1)(m-2)}{m^5} x^2 y + \frac{(m-1)(m-2)}{m^5} x y^2 + \frac{m-1}{m^5} x y$$



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 11 of 18

and similarly the relations  $(B_m e_{12})(x, y)$ ,  $(B_m e_{13})(x, y)$ ,  $(B_m e_{14})(x, y)$ ,  $(B_m e_{23})(x, y)$ ,  $(B_m e_{24})(x, y)$ .

Now, we have

$$(B_m(\cdot - x)^2)(x, y) = (B_m e_{20})(x, y) - 2x(B_m e_{10})(x, y) + x^2(B_m e_{02})(x, y),$$

$$\begin{aligned} (B_m(\cdot - x)^2(* - y)^2)(x, y) &= (B_m e_{22})(x, y) - 2y(B_m e_{21})(x, y) + y^2(B_m e_{20})(x, y) \\ &\quad - 2x(B_m e_{12})(x, y) + 4xy(B_m e_{11})(x, y) - 2xy^2(B_m e_{10})(x, y) \\ &\quad + x^2(B_m e_{02})(x, y) - 2x^2y(B_m e_{01})(x, y) + x^2y^2(B_m e_{00})(x, y), \end{aligned}$$

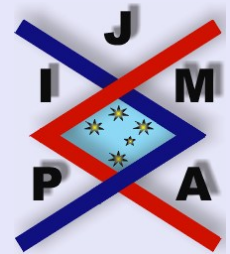
$$\begin{aligned} &(B_m(\cdot - x)^4(* - y)^2)(x, y) \\ &= (B_m e_{40})(x, y) - 2y(B_m e_{41})(x, y) + y^2(B_m e_{40})(x, y) \\ &\quad - 4x(B_m e_{32})(x, y) + 8xy(B_m e_{31})(x, y) - 4xy^2(B_m e_{30})(x, y) \\ &\quad + 6x^2(B_m e_{22})(x, y) - 12x^2y(B_m e_{21})(x, y) + 6x^2y^2(B_m e_{20})(x, y) \\ &\quad - 4x^3(B_m e_{12})(x, y) + 8x^3y(B_m e_{11})(x, y) - 4x^3y^2(B_m e_{10})(x, y) \\ &\quad + x^4(B_m e_{02})(x, y) - 2x^4y(B_m e_{01})(x, y) + x^4y^2(B_m e_{00})(x, y) \end{aligned}$$

and taking (2.9) – (2.21) into account, we obtain (2.5), (2.7) and (2.8). Similarly we obtain (2.9).  $\square$

**Lemma 2.3.** *The operators  $(B_m)_{m \geq 1}$  verify for any  $(x, y) \in \Delta_2$  the following inequalities:*

$$(2.22) \quad (B_m(\cdot - x)^2)(x, y) \leq \frac{1}{4m},$$

$$(2.23) \quad (B_m(* - y)^2)(x, y) \leq \frac{1}{4m},$$



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 12 of 18

for any non zero natural number  $m$ ,

$$(2.24) \quad (B_m(\cdot - x)^2(* - y)^2)(x, y) \leq \frac{9}{4m^2},$$

for any natural number  $m$ ,  $m \geq 2$ ,

$$(2.25) \quad (B_m(\cdot - x)^4(* - y)^2)(x, y) \leq \frac{9}{m^3},$$

$$(2.26) \quad (B_m(\cdot - x)^2(* - y)^4)(x, y) \leq \frac{9}{m^3},$$

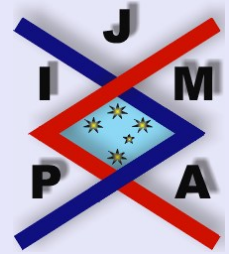
for any natural number  $m$ ,  $m \geq 8$ .

*Proof.* Because  $x(1 - x) \leq \frac{1}{4}$  for any  $x \in [0, 1]$ , (2.22) and (2.23) results.

From (2.7), we have

$$\begin{aligned} & (B_m(\cdot - x)^2(* - y)^2)(x, y) \\ &= \frac{2(m-2)}{m^3} x^2 y^2 + \frac{m-2}{m^3} x(1-x)y(1-y) + \frac{1}{m^3} xy \\ &\leq \frac{2(m-2)}{m^3} + \frac{m-2}{16m^3} + \frac{1}{m^3} \\ &= \frac{33m-50}{16m^3}, \end{aligned}$$

from where (2.24) results.



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 13 of 18

From (2.8), we have

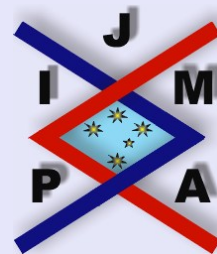
$$\begin{aligned}
 & (B_m(\cdot - x)^4(* - y)^2)(x, y) \\
 &= \frac{6(3m^2 - 26m + 24)}{m^5} x^3 y^2 (1 - x) + \frac{3m^2 - 26m + 24}{m^5} x^4 y (y + 1) \\
 &\quad - \frac{6(m^2 - 7m + 6)}{m^5} x^3 y + \frac{3m^2 - 17m + 14}{m^5} x^2 y (1 - y) \\
 &\quad + \frac{24m - 28}{m^5} x^2 y^2 + \frac{m - 2}{m^5} xy(1 - y) + xy.
 \end{aligned}$$

But

$$\begin{aligned}
 \frac{3m^2 - 26m + 24}{m^5} x^4 y (y + 1) &\leq 2 \frac{3m^2 - 26m + 24}{m^5} x^2 y \\
 &= \frac{6m^2 - 42m + 36}{m^5} x^2 y - \frac{10m - 12}{m^5} x^2 y \\
 &\leq \frac{6m^2 - 42m + 36}{m^5} x^2 y - \frac{10m - 12}{m^5} x^3 y^2
 \end{aligned}$$

and then, from the inequalities above, we obtain

$$\begin{aligned}
 (2.27) \quad & (B_m(\cdot - x)^4(* - y)^2)(x, y) \\
 &\leq \frac{6(3m^2 - 26m + 24)}{m^5} x^3 y^2 (1 - x) + \frac{6m^2 - 42m + 36}{m^5} x^2 y (1 - y) \\
 &\quad + \frac{3m^2 - 17m + 14}{m^5} x^2 y (1 - y) + \frac{10m - 12}{m^5} x^2 y^2 (1 - y) \\
 &\quad + \frac{14m - 16}{m^5} x^2 y^2 + \frac{m - 2}{m^5} xy(1 - y) + xy.
 \end{aligned}$$



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 14 of 18

Because  $x(1-x) \leq \frac{1}{4}$ ,  $y(1-y) \leq \frac{1}{4}$ ,  $xy \leq 1$  for any  $x, y \in [0, 1]$ , from (2.27) we have

$$\begin{aligned} & (B_m(\cdot - x)^4(* - y)^2)(x, y) \\ & \leq \frac{6(3m^2 - 26m + 24)}{4m^5} + \frac{6m^2 - 42m + 36}{4m^5} \\ & \quad + \frac{3m^2 - 17m + 14}{4m^5} + \frac{10m - 12}{4m^5} + \frac{14m - 16}{m^5} + \frac{m - 2}{4m^5} + 1 \\ & = \frac{27m^2 - 148m + 170}{m^5}, \end{aligned}$$

from where (2.25) results. □

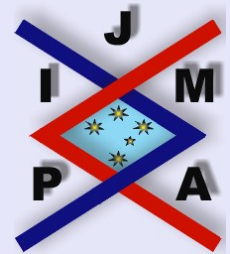
**Theorem 2.4.** Let the function  $f \in C_b(\Delta_2)$ . Then, for any  $(x, y) \in \Delta_2$ , any natural number  $m$ ,  $m \geq 2$ , we have

$$(2.28) \quad |f(x, y) - (UB_m f)(x, y)| \leq \left(1 + \delta_1^{-1} \frac{1}{2\sqrt{m}} + \delta_2^{-1} \frac{1}{2\sqrt{m}} + \delta_1^{-1} \delta_2^{-1} \frac{3}{2m}\right) \omega_{mixed}(f; \delta_1, \delta_2)$$

for any  $\delta_1, \delta_2 > 0$  and

$$(2.29) \quad |f(x, y) - (UB_m f)(x, y)| \leq \frac{7}{2} \omega_{mixed} \left( f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}} \right).$$

*Proof.* For the first inequality we apply Theorem 1.1 and Lemma 2.3. The inequality (2.29) is obtained from (2.28) by choosing  $\delta_1 = \delta_2 = \frac{1}{\sqrt{m}}$ . □



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

Quit

Page 15 of 18

**Corollary 2.5.** *If  $f \in C_b(\Delta_2)$ , then*

$$(2.30) \quad \lim_{m \rightarrow \infty} (UB_m f)(x, y) = f(x, y)$$

*uniformly on  $\Delta_2$ .*

*Proof.* Because  $f \in C_b(\Delta_2)$ , there results that  $f$  is uniform  $B$ -continuous on  $\Delta_2$  and then  $\lim_{m \rightarrow \infty} \omega_{\text{mixed}}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right) = 0$  (see [2] or [3]). From (2.29), there results the conclusion.  $\square$

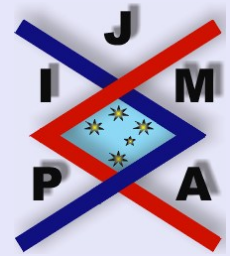
**Theorem 2.6.** *Let the function  $f \in D_b(\Delta_2)$  with  $D_B f \in B(\Delta_2)$ . Then for any  $(x, y) \in \Delta_2$ , any natural number  $m$ ,  $m \geq 8$ , we have*

$$(2.31) \quad |f(x, y) - (UB_m f)(x, y)| \leq \frac{9}{2m} \|D_b f\|_\infty + \left( \frac{3}{2m} + \delta_1^{-1} \frac{3}{m\sqrt{m}} + \delta_2^{-1} \frac{3}{m\sqrt{m}} + \delta_1^{-1} \delta_2^{-1} \frac{9}{4m^2} \right) \omega_{\text{mixed}}(D_B f; \delta_1, \delta_2)$$

*for any  $\delta_1, \delta_2 > 0$  and*

$$(2.32) \quad |f(x, y) - (UB_m f)(x, y)| \leq \frac{3}{4m} \left( 6 \|D_B f\|_\infty + 13 \omega_{\text{mixed}}\left(D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right) \right).$$

*Proof.* It results from Theorem 1.2 and Lemma 2.3.  $\square$



**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

Title Page

Contents



Go Back

Close

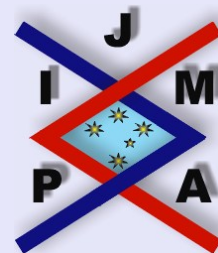
Quit

Page 16 of 18



## References

- [1] C. BADEA AND C. COTTIN, Korovkin-type theorems for generalised boolean sum operators, *Colloquia Mathematica Societatis János Bolyai*, 58, Approximation Theory, Kecskemét (Hungary), 1990, 51–67.
- [2] C. BADEA, Modul de continuitate în sens Bögel și unele aplicații în aproximarea printr-un operator Bernstein, *Studia Univ. "Babeș-Bolyai", Ser. Math.-Mech.*, **18(2)** (1973), 69–78 (Romanian).
- [3] C. BADEA, I. BADEA, C. COTTIN AND H.H. GONSKA, Notes on the degree of approximation of  $B$ -continuous and  $B$ -differentiable functions, *J. Approx. Theory Appl.*, **4** (1988), 95–108.
- [4] D. BĂRBOSU, Aproximarea funcțiilor de mai multe variabile prin sume booleene de operatori liniari de tip interpolator, Ed. Risoprint, Cluj-Napoca, 2002 (Romanian).
- [5] K. BÖGEL, Mehrdimensionale Differentiation von Funktionen mehrerer Veränderlicher, *J. Reine Angew. Math.*, **170** (1934), 197–217.
- [6] K. BÖGEL, Über die mehrdimensionale differentiation, integration und beschränkte variation, *J. Reine Angew. Math.*, **173** (1935), 5–29.
- [7] K. BÖGEL, Über die mehrdimensionale differentiation, *Jber. DMV*, **65** (1962), 45–71.
- [8] G.G. LORENTZ, *Bernstein Polynomials*, University of Toronto Press, Toronto, 1953.



---

Approximation of  $B$ -Continuous  
and  $B$ -Differentiable Functions  
by GBS Operators of Bernstein  
Bivariate Polynomials

Ovidiu T. Pop and Mircea Farcaș

---

Title Page

Contents



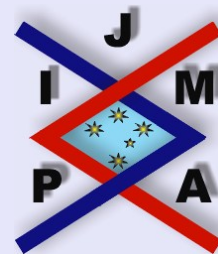
Go Back

Close

Quit

Page 17 of 18

- [9] M. NICOLESCU, Contribuții la o analiză de tip hiperbolic a planului, *St. Cerc. Mat.*, **III**, 1-2, 1952, 7–51 (Romanian).
- [10] M. NICOLESCU, *Analiză Matematică*, II, E. D. P. București, 1980 (Romanian).
- [11] O.T. POP, Approximation of  $B$ -differentiable functions by GBS operators (to appear in *Anal. Univ. Oradea*).
- [12] D.D. STANCU, Gh. COMAN, O. AGRATINI AND R. TRÎMBIȚAȘ, *Analiză Numerică și Teoria Aproximării*, I, Presa Universitară Clujeană, Cluj-Napoca, 2001 (Romanian).



---

**Approximation of  $B$ -Continuous and  $B$ -Differentiable Functions by GBS Operators of Bernstein Bivariate Polynomials**

Ovidiu T. Pop and Mircea Farcaș

---

Title Page

Contents



Go Back

Close

Quit

Page 18 of 18