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A MULTIPLE HARDY-HILBERT INTEGRAL INEQUALITY WITH THE BEST CONSTANT FACTOR

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ABSTRACT. In this paper, by introducing the norm $\|x\|_\alpha (x \in \mathbb{R}^n)$, we give a multiple Hardy-Hilbert's integral inequality with a best constant factor and two parameters α, λ.

Key words and phrases: Multiple Hardy-Hilbert integral inequality, the Γ-function, Best constant factor.

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1. INTRODUCTION

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \geq 0$, $g \geq 0$, $0 < \int_0^\infty f^p(x)dx < +\infty$, $0 < \int_0^\infty g^q(x)dx < +\infty$, then the well known Hardy-Hilbert integral inequality is given by (see [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible. Its equivalent form is:

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^p dy < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \int_0^\infty f^p(x)dx, \tag{1.2}$$

where the constant factor $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p$ is also the best possible.

Hardy-Hilbert's inequality is valuable in harmonic analysis, real analysis and operator theory. In recent years, many valuable results (see [4] – [10]) have been obtained in the form of

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generalizations and improvements of Hardy-Hilbert's inequality. In 1999, Kuang [5] gave a generalization with a parameter λ of (1.1) as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy < h_\lambda(p)\left(\int_0^\infty x^{1-\lambda}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{1-\lambda}g^q(x)dx\right)^{\frac{1}{q}}, \tag{1.3}$$

where $\max\left\{\frac{1}{p},\frac{1}{q}\right\}<\lambda\le 1$, $h_\lambda(p)=\pi\left[\lambda\sin^{\frac{1}{p}}\left(\frac{\pi}{p\lambda}\right)\sin^{\frac{1}{q}}\left(\frac{\pi}{q\lambda}\right)\right]^{-1}$. Because of the constant factor $h_\lambda(p)$ being not the best possible, Yang [8] gave a new generalization of (1.1) in 2002 as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy < \frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\left(\int_0^\infty x^{(1-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{(1-\lambda)(q-1)}g^q(x)dx\right)^{\frac{1}{q}}, \tag{1.4}$$

its equivalent form is:

$$\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{f(x)}{x^\lambda+y^\lambda}dx\right)^p dy < \left[\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\right]^p\int_0^\infty x^{(1-\lambda)(p-1)}f^p(x)dx, \tag{1.5}$$

where the constant factors $\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}$ in (1.4) and $\left[\frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\right]^p$ in (1.5) are all the best possible.

At present, because of the requirement of higher-dimensional harmonic analysis and higher-dimensional operator theory, multiple Hardy-Hilbert integral inequalities have been studied. Hong [11] obtained: If $a>0$, $\sum_{i=1}^n\frac{1}{p_i}=1$, $p_i>1$, $f_i\ge 0$, $r_i=\frac{1}{p_i}\prod_{j=1}^n p_j$, $\lambda>\frac{1}{a}\left(n-1-\frac{1}{r_i}\right)$, $i=1,2,\dots,n$, then

$$\int_\alpha^\infty\cdots\int_\alpha^\infty \frac{1}{\left[\sum_{i=1}^n(x_i-\alpha_i)^a\right]^\lambda}\prod_{i=1}^n f_i(x)dx_1\cdots dx_n \le \frac{\Gamma^{n-2}\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma(\lambda)} \times\prod_{i=1}^n\left[\Gamma\left(\frac{1}{a}\left(1-\frac{1}{r_i}\right)\right)\Gamma\left(\lambda-\frac{1}{a}\left(n-1-\frac{1}{r_i}\right)\right)\int_\alpha^\infty(t-\alpha)^{n-1-\alpha\lambda}f_i^{p_i}dt\right]^{\frac{1}{p_i}}. \tag{1.6}$$

Afterwards, Bicheng Yang and Kuang Jichang etc. obtained some multiple Hardy-Hilbert integral inequalities (see [9, 6]).

In this paper, by introducing the Γ-function, we generalize (1.3) and (1.4) into multiple Hardy-Hilbert integral inequalities with the best constant factors.

2. SOME LEMMAS

First of all, we introduce the signs as:

$$\mathbb{R}_+^n=\{x=(x_1,\dots,x_n):x_1,\dots,x_n>0\},$$

$$\|x\|_\alpha=(x_1^\alpha+\cdots+x_n^\alpha)^{\frac{1}{\alpha}},\quad \alpha>0.$$

Lemma 2.1 (see [1]). *If* $p_i > 0$, $a_i > 0$, $\alpha_i > 0$, $i = 1, 2, \dots, n$, $\Psi(u)$ *is a measurable function, then*

$$\int \cdots \int_{x_1,\dots,x_n>0;\left(\frac{x_1}{a_1}\right)^{\alpha_1}+\cdots+\left(\frac{x_n}{a_n}\right)^{\alpha_n}\le 1} \Psi\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1}+\cdots+\left(\frac{x_n}{a_n}\right)^{\alpha_n}\right) \times x_1^{p_1-1}\cdots x_n^{p_n-1}dx_1\cdots dx_n = \frac{a_1^{p_1}\cdots a_n^{p_n}\Gamma\left(\frac{p_1}{\alpha_1}\right)\cdots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1\cdots\alpha_n\Gamma\left(\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}\right)}\int_0^1 \Psi(u)u^{\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}-1}du. \tag{2.1}$$

where $\Gamma(\cdot)$ *is the* Γ*-function.*

Lemma 2.2. *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $n \in \mathbb{Z}_+$, $\alpha > 0$, $\lambda > 0$, *setting the weight function* $\omega_{\alpha,\lambda}(x,p,q)$ *as:*

$$\omega_{\alpha,\lambda}(x,p,q) = \int_{\mathbb{R}_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda}\left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{(n-\lambda)p}\left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{q}}dy,$$

then

$$\omega_{\alpha,\lambda}(x,p,q) = \|x\|_\alpha^{(n-\lambda)(p-1)}\frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}. \tag{2.2}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
\omega_{\alpha,\lambda}(x,p,q) &= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}}\int_{\mathbb{R}_+^n}\frac{1}{\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda}\|y\|_\alpha^{-\left(n-\lambda+\frac{\lambda}{q}\right)}dy \\
&= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}}\lim_{r\to+\infty}\int\cdots\int_{y_1,\dots,y_n>0;y_1^\alpha+\cdots+y_n^\alpha<r^\alpha} \\
&\quad\times\frac{\left[r\left(\left(\frac{y_1}{r}\right)^\alpha+\cdots+\left(\frac{y_n}{r}\right)^\alpha\right)^{\frac{1}{\alpha}}\right]^{-(n-\lambda+\frac{\lambda}{q})}}{\|x\|_\alpha^\lambda+\left[r\left(\left(\frac{y_1}{r}\right)^\alpha+\cdots+\left(\frac{y_n}{r}\right)^\alpha\right)^{\frac{1}{\alpha}}\right]^\lambda}y_1^{1-1}\cdots y_n^{1-1}dy_1\cdots dy_n \\
&= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}}\lim_{r\to+\infty}\frac{r^n\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^n\Gamma\left(\frac{n}{\alpha}\right)}\int_0^1\frac{\left(ru^{\frac{1}{\alpha}}\right)^{-\left(n-\lambda+\frac{\lambda}{q}\right)}}{\|x\|_\alpha^\lambda+\left(ru^{\frac{1}{\alpha}}\right)^\lambda}u^{\frac{n}{\alpha}-1}du \\
&= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}}\frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\lim_{r\to+\infty}\int_0^r\frac{1}{\|x\|_\alpha^\lambda+u^\lambda}u^{\lambda-\frac{\lambda}{q}-1}du \\
&= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}}\frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\int_0^\infty\frac{1}{\|x\|_\alpha^\lambda+u^\lambda}u^{\lambda-\frac{\lambda}{q}-1}du \\
&= \|x\|_\alpha^{(n-\lambda)(p-1)}\frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\int_0^\infty\frac{1}{1+u}u^{\frac{1}{p}-1}du \\
&= \|x\|_\alpha^{(n-\lambda)(p-1)}\frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)
\end{aligned}$$

$$= \|x\|_\alpha^{(n-\lambda)(p-1)} \frac{\pi \Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right) \lambda \alpha^{n-1} \Gamma\left(\frac{n}{\alpha}\right)},$$

hence (2.2) is valid. □

Lemma 2.3. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{Z}_+$, $\alpha > 0$, $\lambda > 0$, $0 < \varepsilon < \lambda(q-1)$, *setting* $\tilde{\omega}_{\alpha,\lambda}(x, q, \varepsilon)$ *as:*

$$\tilde{\omega}_{\alpha,\lambda}(x, q, \varepsilon) = \int_{\mathbb{R}_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}} dy,$$

then we have

$$\tilde{\omega}_{\alpha,\lambda}(x, q, \varepsilon) = \|x\|_\alpha^{-\frac{\lambda}{q} - \frac{\varepsilon}{q}} \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\lambda \alpha^{n-1} \Gamma\left(\frac{n}{\alpha}\right)} \Gamma\left(\frac{1}{p} - \frac{\varepsilon}{\lambda q}\right) \Gamma\left(\frac{1}{q} + \frac{\varepsilon}{\lambda q}\right). \tag{2.3}$$

Proof. Lemma 2.3 can be proved in the same manner as Lemma 2.2. □

3. MAIN RESULTS

Theorem 3.1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{Z}_+$, $\alpha > 0$, $\lambda > 0$, $f \geq 0$, $g \geq 0$, *and*

$$0 < \int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x) dx < \infty, \qquad 0 < \int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(q-1)} g^q(x) dx < \infty, \tag{3.1}$$

then

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \frac{f(x) g(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx dy < \frac{\pi \Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right) \lambda \alpha^{n-1} \Gamma\left(\frac{n}{\alpha}\right)} \times \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x) dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(q-1)} g^q(x) dx\right)^{\frac{1}{q}}; \tag{3.2}$$

$$\int_{\mathbb{R}_+^n} \|y\|_\alpha^{\lambda-n} \left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx\right)^p dy < \left[\frac{\pi \Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right) \lambda \alpha^{n-1} \Gamma\left(\frac{n}{\alpha}\right)}\right]^p \int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x) dx, \tag{3.3}$$

where the constant factors $\frac{\pi \Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right) \lambda \alpha^{n-1} \Gamma\left(\frac{n}{\alpha}\right)}$ *and* $\left[\frac{\pi \Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right) \lambda \alpha^{n-1} \Gamma\left(\frac{n}{\alpha}\right)}\right]^p$ *are all the best possible.*

Proof. By Hölder's inequality, we have

$$\begin{aligned}
A &:= \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{f(x)g(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dxdy \\
&= \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{f(x)}{(\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda)^{\frac{1}{p}}} \left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{n-\lambda} \left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{pq}} \\
&\qquad \times \frac{g(y)}{(\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda)^{\frac{1}{q}}} \left(\frac{\|y\|_\alpha^{\frac{1}{p}}}{\|x\|_\alpha^{\frac{1}{q}}}\right)^{n-\lambda} \left(\frac{\|y\|_\alpha}{\|x\|_\alpha}\right)^{\frac{\lambda}{pq}} dxdy \\
&\le \left[\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{f^p(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{(n-\lambda)p} \left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{q}} dxdy\right]^{\frac{1}{p}} \\
&\qquad \times \left[\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{g^q(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|y\|_\alpha^{\frac{1}{p}}}{\|x\|_\alpha^{\frac{1}{q}}}\right)^{(n-\lambda)q} \left(\frac{\|y\|_\alpha}{\|x\|_\alpha}\right)^{\frac{\lambda}{p}} dxdy\right]^{\frac{1}{q}} \\
&= \left[\int_{\mathbb{R}_+^n} f^p(x) \left(\int_{\mathbb{R}_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{(n-\lambda)p} \left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{q}} dy\right) dx\right]^{\frac{1}{p}} \\
&\qquad \times \left[\int_{\mathbb{R}_+^n} g^q(y) \left(\int_{\mathbb{R}_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|y\|_\alpha^{\frac{1}{p}}}{\|x\|_\alpha^{\frac{1}{q}}}\right)^{(n-\lambda)q} \left(\frac{\|y\|_\alpha}{\|x\|_\alpha}\right)^{\frac{\lambda}{p}} dx\right) dy\right]^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}_+^n} f^p(x)\omega_{\alpha,\lambda}(x,p,q)dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^n} g^q(y)\omega_{\alpha,\lambda}(y,q,p)dy\right)^{\frac{1}{q}},
\end{aligned}$$

according to the condition of taking equality in Hölder's inequality, if this inequality takes the form of an equality, then there exist constants C_1 and C_2, such that they are not all zero, and

$$\begin{aligned}
&\frac{C_1 f^p(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{(n-\lambda)p} \left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{q}} \\
&\qquad = \frac{C_2 g^q(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|y\|_\alpha^{\frac{1}{p}}}{\|x\|_\alpha^{\frac{1}{q}}}\right)^{(n-\lambda)q} \left(\frac{\|y\|_\alpha}{\|x\|_\alpha}\right)^{\frac{\lambda}{p}}, \qquad \text{a.e. in } \mathbb{R}_+^n \times \mathbb{R}_+^n,
\end{aligned}$$

it follows that

$$C_1\|x\|_\alpha^{(n-\lambda)(p-1)+n} f^p(x) = C_2\|y\|_\alpha^{(n-\lambda)(q-1)+n} g^q(y) = C \text{ (constant)}, \qquad \text{a.e. in } \mathbb{R}_+^n \times \mathbb{R}_+^n,$$

which contradicts (3.1), hence we have

$$A < \left(\int_{\mathbb{R}_+^n} f^p(x)\omega_{\alpha,\lambda}(x,p,q)dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^n} g^q(y)\omega_{\alpha,\lambda}(y,q,p)dy\right)^{\frac{1}{q}}.$$

By Lemma 2.2 and $\sin\left(\frac{\pi}{p}\right) = \sin(\frac{\pi}{q})$, we have

$$\begin{aligned} A &< \left[\frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\right]^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right)^{\frac{1}{p}} \\ &\quad \times \left[\frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin(\frac{\pi}{q})\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\right]^{\frac{1}{q}} \left(\int_{\mathbb{R}_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} g^q(y)dy\right)^{\frac{1}{q}} \\ &= \frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(q-1)} g^q(x)dx\right)^{\frac{1}{q}}. \end{aligned}$$

Hence (3.2) is valid.

For $0 < a < b < \infty$, setting

$$g_{a,b}(y) = \begin{cases} \|y\|_\alpha^{\lambda-n}\left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda}dx\right)^{p-1}, & a < \|y\|_\alpha < b, \\ 0, & 0 < \|y\|_\alpha \le a \quad \text{or} \quad \|y\|_\alpha \ge b, \end{cases}$$

$$\tilde{g}(y) = \|y\|_\alpha^{\lambda-n}\left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda}dx\right)^{p-1}, \qquad y \in \mathbb{R}_+^n,$$

by (3.1), for sufficiently small $a > 0$ and sufficiently large $b > 0$, we have

$$0 < \int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{(n-\lambda)(q-1)} g_{a,b}^q(y)dy < \infty.$$

Hence by (3.2), we have

$$\begin{aligned} &\int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{(n-\lambda)(q-1)} \tilde{g}^q(y)dy \\ &= \int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{\lambda-n}\left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda}dx\right)^p dy \\ &= \int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{\lambda-n}\left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda}dx\right)^{p-1}\left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda}dx\right)dy \\ &= \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{f(x)g_{a,b}(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda}dxdy \end{aligned}$$

$$< \frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\left(\int_{\mathbb{R}^n_+}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\mathbb{R}^n_+}\|y\|_\alpha^{(n-\lambda)(q-1)}g^q_{a,b}(y)dy\right)^{\frac{1}{q}}$$
$$= \frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\left(\int_{\mathbb{R}^n_+}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}$$
$$\times\left(\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{(n-\lambda)(q-1)}\tilde{g}^q(y)dy\right)^{\frac{1}{q}},$$

it follows that

$$\int_{a<\|y\|_\alpha<b}\|y\|_\alpha^{(n-\lambda)(q-1)}\tilde{g}^q(y)dy < \left[\frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\right]^p\int_{\mathbb{R}^n_+}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx.$$

For $a\to 0^+$, $b\to+\infty$, by (3.1), we obtain

$$0<\int_{\mathbb{R}^n_+}\|y\|_\alpha^{(n-\lambda)(q-1)}\tilde{g}^q(y)dy$$
$$\le\left[\frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\right]^p\int_{\mathbb{R}^n_+}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx<\infty,$$

hence by (3.2), we have

$$\int_{\mathbb{R}^n_+}\|y\|_\alpha^{\lambda-n}\left(\int_{\mathbb{R}^n_+}\frac{f(x)}{\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda}dx\right)^p dy$$
$$=\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{f(x)\tilde{g}(y)}{\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda}dxdy$$
$$<\frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\left(\int_{\mathbb{R}^n_+}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n_+}\|y\|_\alpha^{(n-\lambda)(q-1)}\tilde{g}^q(y)dy\right)^{\frac{1}{q}}$$
$$=\frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\left(\int_{\mathbb{R}^n_+}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}$$
$$\times\left[\int_{\mathbb{R}^n_+}\|y\|_\alpha^{\lambda-n}\left(\int_{\mathbb{R}^n_+}\frac{f(x)}{\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda}dx\right)^p dy\right]^{\frac{1}{q}},$$

it follows that

$$\int_{\mathbb{R}_+^n} \|y\|_\alpha^{\lambda-n} \left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx \right)^p dy < \left[\frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \right]^p \int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx.$$

Hence (3.3) is valid. ☐

Remark 3.2. By (3.3), we can also obtain (3.2), hence (3.3) and (3.2) are equivalent.

If the constant factor $C_{n,\alpha}(\lambda,p) := \frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}$ in (3.2) is not the best possible, then there exists a positive constant $K < C_{n,\alpha}(\lambda,p)$, such that

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{f(x)g(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dxdy < K \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} g^q(y)dy \right)^{\frac{1}{q}}. \tag{3.4}$$

In particular, setting

$$f(x) = \|x\|_\alpha^{-\frac{(n-\lambda)(p-1)+n+\varepsilon}{p}}, \quad g(y) = \|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}}, \quad 0 < \varepsilon < \lambda(q-1),$$

(3.4) is still true. By the properties of limit, there exists a sufficiently small $a > 0$, such that

$$\begin{aligned} &\int_{\|x\|_\alpha > a}\int_{\mathbb{R}_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \|x\|_\alpha^{-\frac{(n-\lambda)(p-1)+n+\varepsilon}{p}} \|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}} dxdy \\ &< K \left(\int_{\|x\|_\alpha > a} \|x\|_\alpha^{(n-\lambda)(p-1)} \|x\|_\alpha^{-(n-\lambda)(p-1)-n-\varepsilon} dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\|y\|_\alpha > a} \|y\|_\alpha^{(n-\lambda)(q-1)} \|y\|_\alpha^{-(n-\lambda)(q-1)-n-\varepsilon} dy \right)^{\frac{1}{q}} \\ &= K \int_{\|x\|_\alpha > a} \|x\|_\alpha^{-n-\varepsilon} dx. \end{aligned}$$

On the other hand, by Lemma 2.3, we have

$$\begin{aligned} &\int_{\|x\|_\alpha > a}\int_{\mathbb{R}_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \|x\|_\alpha^{-\frac{(n-\lambda)(p-1)+n+\varepsilon}{p}} \|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}} dxdy \\ &= \int_{\|x\|_\alpha > a} \|x\|_\alpha^{-n+\frac{\lambda}{q}-\frac{\varepsilon}{p}} \int_{\mathbb{R}_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}} dydx \\ &= \int_{\|x\|_\alpha > a} \|x\|_\alpha^{-n+\frac{\lambda}{q}-\frac{\varepsilon}{p}} \tilde{\omega}_{\alpha,\lambda}(x,q,\varepsilon)dx \\ &= \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \Gamma\left(\frac{1}{p} - \frac{\varepsilon}{\lambda q}\right) \Gamma\left(\frac{1}{q} + \frac{\varepsilon}{\lambda q}\right) \int_{\|x\|_\alpha > a} \|x\|_\alpha^{-n-\varepsilon} dx, \end{aligned}$$

hence we obtain

$$\frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \Gamma\left(\frac{1}{p} - \frac{\varepsilon}{\lambda q}\right) \Gamma\left(\frac{1}{q} + \frac{\varepsilon}{\lambda q}\right) < K.$$

For $\varepsilon \to 0^+$, we have

$$C_{n,\alpha}(\lambda,p) = \frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)\lambda\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \le K,$$

this contradicts the fact that $K < C_{n,\alpha}(\lambda,p)$.Hence the constant factor in (3.2) is the best possible.

Since (3.3) and (3.2) are equivalent, the constant factor in (3.3) is also the best possible.

Corollary 3.3. *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $n \in \mathbb{Z}_+$, $\alpha > 0$, $f \ge 0$, $g \ge 0$, *and*

$$0 < \int_{\mathbb{R}_+^n} f^p(x)dx < \infty, \qquad 0 < \int_{\mathbb{R}_+^n} g^q(x)dx < \infty, \tag{3.5}$$

then

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{f(x)g(y)}{\|x\|_\alpha^n + \|y\|_\alpha^n}dxdy < \frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)n\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\left(\int_{\mathbb{R}_+^n} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}_+^n} g^q(x)dx\right)^{\frac{1}{q}}; \tag{3.6}$$

$$\int_{\mathbb{R}_+^n}\left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^n + \|y\|_\alpha^n}dx\right)^p dy < \left[\frac{\pi\Gamma^n\left(\frac{1}{\alpha}\right)}{\sin\left(\frac{\pi}{p}\right)n\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)}\right]^p \int_{\mathbb{R}_+^n} f^p(x)dx, \tag{3.7}$$

where the constant factors in (3.6) and (3.7) are all the best possible.

Proof. By taking $\lambda = n$ in Theorem 3.1, (3.6) and (3.7) can be obtained. □

Corollary 3.4. *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $n \in \mathbb{Z}_+$, $f \ge 0$, $g \ge 0$, *and*

$$0 < \int_{\mathbb{R}_+^n} f^p(x)dx < \infty, \qquad 0 < \int_{\mathbb{R}_+^n} g^q(x)dx < \infty, \tag{3.8}$$

then

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{f(x)g(y)}{\left(\sum_{i=1}^n x_i\right)^n + \left(\sum_{i=1}^n y_i\right)^n}dxdy < \frac{\pi}{n!\sin\left(\frac{\pi}{p}\right)}\left(\int_{\mathbb{R}_+^n} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}_+^n} g^q(x)dx\right)^{\frac{1}{q}}; \tag{3.9}$$

$$\int_{\mathbb{R}_+^n}\left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\left(\sum_{i=1}^n x_i\right)^n + \left(\sum_{i=1}^n y_i\right)^n}dx\right)^p dy < \left[\frac{\pi}{n!\sin\left(\frac{\pi}{p}\right)}\right]^p \int_{\mathbb{R}_+^n} f^p(x)dx, \tag{3.10}$$

where the constant factors in (3.9) and (3.10) are all the best possible.

Proof. By taking $\alpha = 1$ in Corollary 3.3, (3.9) and (3.10) can be obtained. □

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