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OPERATOR NORM INEQUALITIES OF MINKOWSKI TYPE

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ABSTRACT. Operator norm inequalities of Minkowski type are presented for unitarily invariant norm. Some of these inequalities generalize an earlier work of Hiai and Zhan.

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1. INTRODUCTION

Let $B(H)$ be the space of all bounded linear operators on a separable complex Hilbert space H. A unitarily invariant norm $|||\cdot|||$ is a norm on the space of operators satisfying $|||A||| = |||UAV|||$ for all A and all unitary operators U and V in $B(H)$. Except for the operator norm, which is defined on all of $B(H)$, each unitarily invariant norm $|||\cdot|||$ is defined on a norm ideal $C_{|||\cdot|||}$ contained in the ideal of compact operators. When we talk of $|||A|||$ we are implicitly assuming that A belongs to $C_{|||\cdot|||}$.

The absolute value of an operator $A \in B(H)$, denoted by $|A|$, is defined by $|A| = (A^*A)^{1/2}$. Let $s_1(A), s_2(A), \ldots$ be the singular values of the compact operator A, i.e., the eigenvalues of $|A|$, rearranged such that $s_1(A) \geq s_2(A) \geq \cdots$.

For $p > 0$ and for every unitarily invariant norm $|||\cdot|||$ on $B(H)$, define

$$|||A\,|||^{(p)} = |||\,|A|^p|||^{1/p}.$$

It is known that

$$(1.1) \qquad |||\,|A+B|^p|||^{1/p} \leq |||\,|A|^p|||^{1/p} + |||\,|B|^p|||^{1/p}$$

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for $p \geq 1$ and

$$|||\,|A+B|^p|||^{1/p} \leq 2^{1/p-1}\left(|||\,|A|^p|||^{1/p} + |||\,|B|^p|||^{1/p}\right) \tag{1.2}$$

for $0 < p < 1$ (see e.g., [1, p.p. 95,108]). Based on the definition of $|||\cdot|||^{(p)}$ and inequality (1.1), it can be easily seen that $|||\cdot|||^{(p)}$ is a unitarily invariant norm for $p \geq 1$.

For $0 < p < \infty$, let

$$\|A\|_p = \left(\sum_{i=1}^{\infty} s_i^p(A)\right)^{\frac{1}{p}}.$$

If $p \geq 1$, then $\|\cdot\|_p$ is a norm, called the Schatten p-norm. So,

$$\|A\|_p = \left(\operatorname{tr}|A|^p\right)^{1/p},$$

where tr is the usual trace functional. When $p = 1$, $\|A\|_1$ is called the trace norm of A. Note that for all positive real numbers r and p, we have

$$\|\,|A|^r\|_p = \|A\|_{rp}^r. \tag{1.3}$$

For the theory of unitarily invariant norms, the reader is referred to [1], [3], [8], [9], [10], and the references therein.

The Minkowski's inequality for scalars asserts that if a_i, b_i ($i = 1, 2, \ldots, n$) are complex numbers and $p \geq 1$, then

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}}.$$

Hiai and Zhan [4], proved that if A_1, A_2, B_1, B_2 are matrices of order n and $1 \leq p, r < \infty$, then

$$\begin{aligned} &|||\,|A_1+A_2|^p + |B_1+B_2|^p|||^{1/p} \\ &\qquad \leq 2^{|1/p-1/2|}\left(|||\,|A_1|^p + |B_1|^p|||^{1/p} + |||\,|A_2|^p + |B_2|^p|||^{1/p}\right), \end{aligned} \tag{1.4}$$

$$\begin{aligned} &\||A_1+A_2|^p + |B_1+B_2|^p\|_r^{1/p} \\ &\qquad \leq 2^{(1-1/r)/p}\left(\||A_1|^p + |B_1|^p\|_r^{1/p} + \||A_2|^p + |B_2|^p\|_r^{1/p}\right), \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\left\|\left(|A_1+A_2|^p + |B_1+B_2|^p\right)^{1/p}\right\|_r \\ &\qquad \leq 2^{|1/p-1/r|}\left(\left\|\left(|A_1|^p + |B_1|^p\right)^{1/p}\right\|_r + \left\|\left(|A_2|^p + |B_2|^p\right)^{1/p}\right\|_r\right). \end{aligned} \tag{1.6}$$

These inequalities are norm inequalities of Minkowski type.

The purpose of this paper is to establish new operator norm inequalities. Our inequalities generalize the inequalities (1.4), (1.5), and (1.6) for $n-$tuple of operators. Our analysis is based on some recent results on convexity and concavity of functions and on some operator inequalities.

2. NORM INEQUALITIES OF MINKOWSKI TYPE

In this section, we generalize inequality (1.4) for operators $A_i, B_i \in B(H)$ $(i = 1, 2, \dots, n)$, and other norm inequalities of Minkowski type. To achieve our goal we need the following two lemmas. The first lemma can be found in [2] and a stronger version of the second lemma can be found in [5].

Lemma 2.1. *Let* $A_1, \dots, A_n \in B(H)$ *be positive operators. Then, for every unitarily invariant norm,*

$$\left|\left|\left|\sum_{i=1}^{n} A_i^r\right|\right|\right| \le \left|\left|\left|\left(\sum_{i=1}^{n} A_i\right)^r\right|\right|\right| \tag{2.1}$$

for $r \ge 1$ *and*

$$\left|\left|\left|\left(\sum_{i=1}^{n} A_i\right)^r\right|\right|\right| \le \left|\left|\left|\sum_{i=1}^{n} A_i^r\right|\right|\right| \tag{2.2}$$

for $0 < r \le 1$.

Lemma 2.2. *Let* $A_1, \dots, A_n \in B(H)$ *be positive operators. Then, for every unitarily invariant norm,*

$$\left|\left|\left|\left(\sum_{i=1}^{n} A_i\right)^r\right|\right|\right| \le n^{r-1}\left|\left|\left|\sum_{i=1}^{n} A_i^r\right|\right|\right| \tag{2.3}$$

for $r \ge 1$ *and*

$$\left|\left|\left|\sum_{i=1}^{n} A_i^r\right|\right|\right| \le n^{1-r}\left|\left|\left|\left(\sum_{i=1}^{n} A_i\right)^r\right|\right|\right| \tag{2.4}$$

for $0 < r \le 1$.

Now, we are in a position to generalize (1.4).

Theorem 2.3. *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \dots, n)$ *and* $p \ge 1$. *Then, for every unitarily invariant norm,*

$$n^{-|1/p-1/2|}\left|\left|\left|\sum_{i=1}^{n} |A_i + B_i|^p\right|\right|\right|^{\frac{1}{p}} \le \left|\left|\left|\sum_{i=1}^{n} |A_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n} |B_i|^p\right|\right|\right|^{\frac{1}{p}} \tag{2.5}$$

and

$$\left|\left|\left|\sum_{i=1}^{n} |A_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n} |B_i|^p\right|\right|\right|^{\frac{1}{p}} \le n^{|1/p-1/2|}\left(\left|\left|\left|\sum_{i=1}^{n} |A_i + B_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n} |A_i - B_i|^p\right|\right|\right|^{\frac{1}{p}}\right). \tag{2.6}$$

Proof. Let

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_n & 0 & \cdots & 0 \end{bmatrix}$$

be operators in $B\left(\bigoplus_{i=1}^{n} H\right)$. Then

$$|A|^2 = \begin{bmatrix} \sum_{i=1}^n |A_i|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad |B|^2 = \begin{bmatrix} \sum_{i=1}^n |B_i|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$|A+B|^2 = \begin{bmatrix} \sum_{i=1}^n |A_i+B_i|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By applying (1.1) to the operators A and B, we get

$$\left|\left|\left|\left(\sum_{i=1}^n |A_i+B_i|^2\right)^{\frac{p}{2}}\right|\right|\right|^{\frac{1}{p}} \le \left|\left|\left|\left(\sum_{i=1}^n |A_i|^2\right)^{\frac{p}{2}}\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\left(\sum_{i=1}^n |B_i|^2\right)^{\frac{p}{2}}\right|\right|\right|^{\frac{1}{p}}. \tag{2.7}$$

For $1 \le p \le 2$, it follows, from (2.2) and (2.4), that

$$\left|\left|\left|\left(\sum_{i=1}^n |A_i|^2\right)^{\frac{p}{2}}\right|\right|\right| \le \left|\left|\left|\sum_{i=1}^n |A_i|^p\right|\right|\right|, \tag{2.8}$$

$$\left|\left|\left|\left(\sum_{i=1}^n |B_i|^2\right)^{\frac{p}{2}}\right|\right|\right| \le \left|\left|\left|\sum_{i=1}^n |B_i|^p\right|\right|\right|, \tag{2.9}$$

and

$$\left|\left|\left|\sum_{i=1}^n |A_i+B_i|^p\right|\right|\right| \le n^{1-p/2}\left|\left|\left|\left(\sum_{i=1}^n |A_i+B_i|^2\right)^{\frac{p}{2}}\right|\right|\right|. \tag{2.10}$$

Now, inequality (2.5) follows by combining (2.8), (2.9), and (2.10) by (2.7).

For $p > 2$, it follows, from (2.1) and (2.3), that

$$\left|\left|\left|\sum_{i=1}^n |A_i+B_i|^p\right|\right|\right| \le \left|\left|\left|\left(\sum_{i=1}^n |A_i+B_i|^2\right)^{\frac{p}{2}}\right|\right|\right|, \tag{2.11}$$

$$\left|\left|\left|\left(\sum_{i=1}^n |A_i|^2\right)^{\frac{p}{2}}\right|\right|\right| \le n^{p/2-1}\left|\left|\left|\sum_{i=1}^n |A_i|^p\right|\right|\right|, \tag{2.12}$$

and

$$\left|\left|\left|\left(\sum_{i=1}^n |B_i|^2\right)^{\frac{p}{2}}\right|\right|\right| \le n^{p/2-1}\left|\left|\left|\sum_{i=1}^n |B_i|^p\right|\right|\right|. \tag{2.13}$$

Consequently, inequality (2.5) follows, by combining (2.11), (2.12), and (2.13) by (2.7). This completes the proof of inequality (2.5).

For inequality (2.6), replacing A_i and B_i in (2.5) by $A_i + B_i$ and $A_i - B_i$, respectively, we have

$$(2.14)\qquad 2\left|\left|\left|\sum_{i=1}^{n}|A_i|^p\right|\right|\right|^{\frac{1}{p}} \le n^{|1/p-1/2|}\left(\left|\left|\left|\sum_{i=1}^{n}|A_i+B_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n}|A_i-B_i|^p\right|\right|\right|^{\frac{1}{p}}\right).$$

Again, replacing A_i and B_i in (2.5) by $A_i + B_i$ and $B_i - A_i$, respectively, we have

$$(2.15)\qquad 2\left|\left|\left|\sum_{i=1}^{n}|B_i|^p\right|\right|\right|^{\frac{1}{p}} \le n^{|1/p-1/2|}\left(\left|\left|\left|\sum_{i=1}^{n}|A_i+B_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n}|A_i-B_i|^p\right|\right|\right|^{\frac{1}{p}}\right).$$

Now, inequality (2.6) follows, by adding inequalities (2.14) and (2.15). This completes the proof of the theorem. □

Based on inequality (1.2) and using a procedure similar to that given in the proof of Theorem 2.3, we have the following result.

Theorem 2.4. *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ *and* $0 < p \le 1$. *Then, for every unitarily invariant norm,*

$$(2.16)\qquad 2^{1-1/p}n^{-|1/p-1/2|}\left|\left|\left|\sum_{i=1}^{n}|A_i+B_i|^p\right|\right|\right|^{\frac{1}{p}} \le \left(\left|\left|\left|\sum_{i=1}^{n}|A_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n}|B_i|^p\right|\right|\right|^{\frac{1}{p}}\right)$$

and

$$(2.17)\qquad \left|\left|\left|\sum_{i=1}^{n}|A_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n}|B_i|^p\right|\right|\right|^{\frac{1}{p}} \le 2^{1/p-1}n^{|1/p-1/2|}\left(\left|\left|\left|\sum_{i=1}^{n}|A_i+B_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n}|A_i-B_i|^p\right|\right|\right|^{\frac{1}{p}}\right).$$

For $p \ge 1$ inequalities (2.5) and (2.6) can be written in equivalent forms as follow:

$$(2.18)\qquad n^{-|1/p-1/2|}\left|\left|\left|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|^{(p)} \le \left|\left|\left|\left(\sum_{i=1}^{n}|A_i|^p\right)^{\frac{1}{p}}\right|\right|\right|^{(p)} + \left|\left|\left|\left(\sum_{i=1}^{n}|B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|^{(p)}$$

and

$$(2.19)\qquad \left|\left|\left|\left(\sum_{i=1}^{n}|A_i|^p\right)^{\frac{1}{p}}\right|\right|\right|^{(p)} + \left|\left|\left|\left(\sum_{i=1}^{n}|B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|^{(p)} \le n^{|1/p-1/2|}\left(\left|\left|\left|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|^{(p)} + \left|\left|\left|\left(\sum_{i=1}^{n}|A_i-B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|^{(p)}\right).$$

In the following theorem we give inequalities related to inequalities (2.18) and (2.19). In order to do that we need the following lemma, which is a particular case of Theorem 2 in [7].

Lemma 2.5. *Let $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ and $p \geq 2$. Then*

$$\left|\left|\left|\left(\sum_{i=1}^{n} |A_i|^2\right)^{\frac{1}{2}}\right|\right|\right| \leq n^{1/2-1/p}\left(\left|\left|\left|\left(\sum_{i=1}^{n} |A_i|^p\right)^{\frac{1}{p}}\right|\right|\right|\right) \tag{2.20}$$

for every unitarily invariant norm.

Theorem 2.6. *Let $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ and $p \geq 2$. Then, for every unitarily invariant norm,*

$$n^{-(1-1/p)}\left|\left|\left|\left(\sum_{i=1}^{n} |A_i + B_i|^p\right)^{\frac{1}{p}}\right|\right|\right| \leq \left|\left|\left|\left(\sum_{i=1}^{n} |A_i|^p\right)^{\frac{1}{p}}\right|\right|\right| + \left|\left|\left|\left(\sum_{i=1}^{n} |B_i|^p\right)^{\frac{1}{p}}\right|\right|\right| \tag{2.21}$$

and

$$\begin{aligned} &\left|\left|\left|\left(\sum_{i=1}^{n} |A_i|^p\right)^{\frac{1}{p}}\right|\right|\right| + \left|\left|\left|\left(\sum_{i=1}^{n} |B_i|^p\right)^{\frac{1}{p}}\right|\right|\right| \\ &\qquad \leq n^{1-1/p}\left(\left|\left|\left|\left(\sum_{i=1}^{n} |A_i + B_i|^p\right)^{\frac{1}{p}}\right|\right|\right| + \left|\left|\left|\left(\sum_{i=1}^{n} |A_i - B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|\right). \end{aligned} \tag{2.22}$$

Proof. By using (2.2), (2.4), (2.7), and (2.20), respectively, we have

$$\begin{aligned} \left|\left|\left|\left(\sum_{i=1}^{n} |A_i + B_i|^p\right)^{\frac{1}{p}}\right|\right|\right| &\leq \left|\left|\left|\sum_{i=1}^{n} |A_i + B_i|\right|\right|\right| \\ &\leq n^{1/2}\left|\left|\left|\left(\sum_{i=1}^{n} |A_i + B_i|^2\right)^{\frac{1}{2}}\right|\right|\right| \\ &\leq n^{1/2}\left(\left|\left|\left|\left(\sum_{i=1}^{n} |A_i|^2\right)^{\frac{1}{2}}\right|\right|\right| + \left|\left|\left|\left(\sum_{i=1}^{n} |B_i|^2\right)^{\frac{1}{2}}\right|\right|\right|\right) \\ &\leq n^{1-1/p}\left(\left|\left|\left|\left(\sum_{i=1}^{n} |A_i|^p\right)^{\frac{1}{p}}\right|\right|\right| + \left|\left|\left|\left(\sum_{i=1}^{n} |B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|\right). \end{aligned}$$

This proves inequality (2.21). Inequality (2.22) follows from inequality (2.21) by a proof similar to that given for inequality (2.6) in Theorem 2.3. The proof is complete. □

It is known that for a positive operator A and for $0 < r \leq 1$, we have

$$|||A|||^r \leq |||A^r||| \tag{2.23}$$

for every unitarily invariant norm; and the reverse inequality holds for $r \geq 1$.

Using inequality (2.23) we have the following application of Theorem 2.6.

Corollary 2.7. *Let $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ and $p \geq 2$. Then, for every unitarily invariant norm,*

$$n^{-(1-1/p)}\left|\left|\left|\sum_{i=1}^{n} |A_i + B_i|^p\right|\right|\right|^{\frac{1}{p}} \leq \left|\left|\left|\left(\sum_{i=1}^{n} |A_i|^p\right)^{\frac{1}{p}}\right|\right|\right| + \left|\left|\left|\left(\sum_{i=1}^{n} |B_i|^p\right)^{\frac{1}{p}}\right|\right|\right| \tag{2.24}$$

and

$$\left|\left|\left|\sum_{i=1}^{n}|A_i|^p\right|\right|\right|^{\frac{1}{p}}+\left|\left|\left|\sum_{i=1}^{n}|B_i|^p\right|\right|\right|^{\frac{1}{p}} \le n^{1-1/p}\left(\left|\left|\left|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|+\left|\left|\left|\left(\sum_{i=1}^{n}|A_i-B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|\right). \tag{2.25}$$

Remark 2.8. In view of (2.5), (2.21), and (2.23), one might conjecture that if $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$, then, for every unitarily invariant norm,

$$\left|\left|\left|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right|\right|\right| \le n^{|1/p-1/2|}\left(\left|\left|\left|\sum_{i=1}^{n}|A_i|^p\right|\right|\right|^{\frac{1}{p}}+\left|\left|\left|\sum_{i=1}^{n}|B_i|^p\right|\right|\right|^{\frac{1}{p}}\right) \tag{2.26}$$

for $p \ge 1$ and

$$\left|\left|\left|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right|\right|\right| \le n^{1-1/p}\left(\left|\left|\left|\sum_{i=1}^{n}|A_i|^p\right|\right|\right|^{\frac{1}{p}}+\left|\left|\left|\sum_{i=1}^{n}|B_i|^p\right|\right|\right|^{\frac{1}{p}}\right) \tag{2.27}$$

for $p \ge 2$.

Remark 2.9. Using the same procedure used in the proof of inequality (2.6) in Theorem 2.3, inequalities (1.1) and (1.2) imply that

$$|||\,|A|^p|||^{1/p} + |||\,|B|^p|||^{1/p} \le |||\,|A+B|^p|||^{1/p} + |||\,|A-B|^p|||^{1/p} \tag{2.28}$$

for $p \ge 1$ and

$$|||\,|A|^p|||^{1/p} + |||\,|B|^p|||^{1/p} \le 2^{1/p-1}(|||\,|A+B|^p|||^{1/p} + |||\,|A-B|^p|||^{1/p}) \tag{2.29}$$

for $0 < p \le 1$. For $p \ge 1$, it follows, from the triangle inequality for norms and a scalar inequality, that

$$|||\,|A+B|^p + |A-B|^p|||^{1/p} \le |||\,|A+B|^p|||^{1/p} + |||\,|A-B|^p|||^{1/p}. \tag{2.30}$$

For $p \ge 2$, the left hand side of (2.30) is the right hand side of the famous Clarkson inequality

$$2\,|||\,|A|^p + |B|^p||| \le |||\,|A+B|^p + |A+B|^p|||, \tag{2.31}$$

see e.g., [6]. In view of the inequalities (2.29) and (2.30) we may introduce the following question: For $p \ge 2$ are the following inequalities:

$$|||\,|A|^p|||^{1/p} + |||\,|B|^p|||^{1/p} \le |||\,|A+B|^p + |A-B|^p|||^{1/p} \tag{2.32}$$

and

$$2\,|||\,|A|^p + |B|^p||| \le \left(|||\,|A|^p|||^{1/p} + |||\,|B|^p|||^{1/p}\right)^p \tag{2.33}$$

true?

Inequalities (2.32) and (2.33), if true, form a refinement of the Clarkson inequality (2.31).

3. NORM INEQUALITIES OF MINKOWSKI TYPE FOR THE SCHATTEN $P-$NORM

In this section, we present some norm inequalities of Minkowski type for the Schatten $p-$norm. These inequalities generalize the inequalities (1.5) and (1.6) for an $n-$tuple of operators.

Theorem 3.1. *Let $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ and $1 \le p, r < \infty$. Then*

$$n^{-(1-1/r)/p} \left\| \sum_{i=1}^{n} |A_i + B_i|^p \right\|_r^{\frac{1}{p}} \le \left\| \sum_{i=1}^{n} |A_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^{n} |B_i|^p \right\|_r^{\frac{1}{p}} \tag{3.1}$$

and

$$\begin{aligned} \left\| \sum_{i=1}^{n} |A_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^{n} |B_i|^p \right\|_r^{\frac{1}{p}} \\ \le n^{(1-1/r)/p} \left(\left\| \sum_{i=1}^{n} |A_i + B_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^{n} |A_i - B_i|^p \right\|_r^{\frac{1}{p}} \right). \end{aligned} \tag{3.2}$$

Proof. It follows, from (1.3) and the triangle inequality, that

$$\begin{aligned} \left\| \sum_{i=1}^{n} |A_i + B_i|^{pr} \right\|_1^{\frac{1}{pr}} &= \left\| \begin{bmatrix} A_1+B_1 & 0 & \cdots & 0 \\ 0 & A_2+B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n+B_n \end{bmatrix} \right\|_{pr} \\ &= \left\| \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} + \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{bmatrix} \right\|_{pr} \\ &\le \left\| \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} \right\|_{pr} + \left\| \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{bmatrix} \right\|_{pr} \\ &= \left\| \sum_{i=1}^{n} |A_i|^{pr} \right\|_1^{\frac{1}{pr}} + \left\| \sum_{i=1}^{n} |B_i|^{pr} \right\|_1^{\frac{1}{pr}}. \end{aligned} \tag{3.3}$$

Now, by using (1.3), (2.3), (3.3), and (2.2), respectively, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n} |A_i + B_i|^p \right\|_r^{\frac{1}{p}} &= \left\| \left(\sum_{i=1}^{n} |A_i + B_i|^p \right)^r \right\|_1^{\frac{1}{pr}} \\ &\le n^{(r-1)/pr} \left\| \sum_{i=1}^{n} |A_i + B_i|^{pr} \right\|_1^{\frac{1}{pr}} \\ &\le n^{(1-1/r)/p} \left(\left\| \sum_{i=1}^{n} |A_i|^{pr} \right\|_1^{\frac{1}{pr}} + \left\| \sum_{i=1}^{n} |B_i|^{pr} \right\|_1^{\frac{1}{pr}} \right) \end{aligned}$$

$$= n^{(1-1/r)/p} \left(\left\| \left(\sum_{i=1}^{n} |A_i|^{pr} \right)^{\frac{1}{r}} \right\|_r^{\frac{1}{p}} + \left\| \left(\sum_{i=1}^{n} |B_i|^{pr} \right)^{\frac{1}{r}} \right\|_r^{\frac{1}{p}} \right)$$
$$\leq n^{(1-1/r)/p} \left(\left\| \sum_{i=1}^{n} |A_i|^{p} \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^{n} |B_i|^{p} \right\|_r^{\frac{1}{p}} \right).$$

This proves inequality (3.1). The proof of inequality (3.2) follows from (3.1) by a proof similar to that given for inequality (2.6) in Theorem 2.3. The proof is complete. □

The following is our final result.

Theorem 3.2. *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \dots, n)$ *and* $1 \leq p, r < \infty$. *Then*

$$n^{-|1/p-1/r|} \left\| \left(\sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right\|_r \leq \left\| \left(\sum_{i=1}^{n} |A_i|^p \right)^{\frac{1}{p}} \right\|_r + \left\| \left(\sum_{i=1}^{n} |B_i|^p \right)^{\frac{1}{p}} \right\|_r \tag{3.4}$$

and

$$\left\| \left(\sum_{i=1}^{n} |A_i|^p \right)^{\frac{1}{p}} \right\|_r + \left\| \left(\sum_{i=1}^{n} |B_i|^p \right)^{\frac{1}{p}} \right\|_r \leq n^{|1/p-1/r|} \left(\left\| \left(\sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right\|_r + \left\| \left(\sum_{i=1}^{n} |A_i - B_i|^p \right)^{\frac{1}{p}} \right\|_r \right). \tag{3.5}$$

Proof. First suppose that $r \leq p$. By using (1.3), (2.2), (3.3), and (2.4), respectively, we have

$$\left\| \left(\sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right\|_r = \left\| \left(\sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{r}{p}} \right\|_1^{\frac{1}{r}}$$
$$\leq \left\| \sum_{i=1}^{n} |A_i + B_i|^r \right\|_1^{\frac{1}{r}}$$
$$\leq \left\| \sum_{i=1}^{n} |A_i|^r \right\|_1^{\frac{1}{r}} + \left\| \sum_{i=1}^{n} |B_i|^r \right\|_1^{\frac{1}{r}}$$
$$\leq n^{1/r-1/p} \left(\left\| \left(\sum_{i=1}^{n} |A_i|^p \right)^{\frac{r}{p}} \right\|_1^{\frac{1}{r}} + \left\| \left(\sum_{i=1}^{n} |B_i|^p \right)^{\frac{r}{p}} \right\|_1^{\frac{1}{r}} \right)$$
$$= n^{1/r-1/p} \left(\left\| \left(\sum_{i=1}^{n} |A_i|^p \right)^{\frac{1}{p}} \right\|_r + \left\| \left(\sum_{i=1}^{n} |B_i|^p \right)^{\frac{1}{p}} \right\|_r \right).$$

Next, for $p < r$, by using (1.3) and (3.1), we have

$$\begin{aligned}\left\|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right\|_r &= \left\|\sum_{i=1}^{n}|A_i+B_i|^p\right\|_{\frac{r}{p}}^{\frac{1}{p}} \\ &\leq n^{1/p(1-p/r)}\left(\left\|\sum_{i=1}^{n}|A_i|^p\right\|_{\frac{r}{p}}^{\frac{1}{p}}+\left\|\sum_{i=1}^{n}|B_i|^p\right\|_{\frac{r}{p}}^{\frac{1}{p}}\right) \\ &= n^{1/p-1/r}\left(\left\|\left(\sum_{i=1}^{n}|A_i|^p\right)^{\frac{1}{p}}\right\|_r+\left\|\left(\sum_{i=1}^{n}|B_i|^p\right)^{\frac{1}{p}}\right\|_r\right).\end{aligned}$$

This proves inequality (3.4). The proof of inequality (3.5) follows from (3.4) by a proof similar to that given for inequality (2.6) in Theorem 2.3. The proof is complete. □

Remark 3.3. For the Schatten $p-$norm, (3.4) is better than (2.21), and if $rp \leq 2$ or $r(4-p) \leq 2$, then (3.1) is better than (2.5).

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