

Journal of Inequalities in Pure and Applied Mathematics

RATE OF CONVERGENCE OF CHLODOWSKY TYPE DURMEYER OPERATORS

ERTAN IBIKLI AND HARUN KARSLI

Ankara University, Faculty of Sciences,
Department of Mathematics,
06100 Tandogan - Ankara/Turkey.

EMail: ibikli@science.ankara.edu.tr

EMail: karsli@science.ankara.edu.tr

©2000 Victoria University
ISSN (electronic): 1443-5756
276-05



volume 6, issue 4, article 106,
2005.

*Received 16 September, 2005;
accepted 23 September, 2005.*

Communicated by: A. Lupaş

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In the present paper, we estimate the rate of pointwise convergence of the Chlodowsky type Durrmeyer Operators $D_n(f, x)$ for functions, defined on the interval $[0, b_n]$, $(b_n \rightarrow \infty)$, extending infinity, of bounded variation. To prove our main result, we have used some methods and techniques of probability theory.

2000 Mathematics Subject Classification: 41A25, 41A35, 41A36.

Key words: Approximation, Bounded variation, Chlodowsky polynomials, Durrmeyer Operators, Chanturiya's modulus of variation, Rate of convergence.

Contents

1	Introduction	3
2	Auxiliary Results	6
3	Proof Of The Main Result	11
	References	



Rate of Convergence of Chlodowsky Type Durrmeyer Operators

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 2 of 26

1. Introduction

Very recently, some authors studied some linear positive operators and obtained the rate of convergence for functions of bounded variation. For example, Bojanic R. and Vuilleumier M. [3] estimated the rate of convergence of Fourier Legendre series of functions of bounded variation on the interval $[0, 1]$, Cheng F. [4] estimated the rate of convergence of Bernstein polynomials of functions bounded variation on the interval $[0, 1]$, Zeng and Chen [9] estimated the rate of convergence of Durrmeyer type operators for functions of bounded variation on the interval $[0, 1]$.

Durrmeyer operators M_n introduced by Durrmeyer [1]. Also let us note that these operators were introduced by Lupaş [2]. The polynomial $M_n f$ defined by

$$M_n(f; x) = (n + 1) \sum_{k=0}^n P_{n,k}(x) \int_0^1 f(t) P_{n,k}(t) dt, \quad 0 \leq x \leq 1,$$

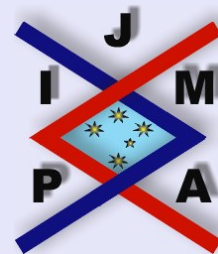
where

$$P_{n,k}(x) = \binom{n}{k} (x)^k (1 - x)^{n-k}.$$

These operators are the integral modification of Bernstein polynomials so as to approximate Lebesgue integrable functions on the interval $[0, 1]$. The operators M_n were studied by several authors. Also, Guo S. [5] investigated Durrmeyer operators M_n and estimated the rate of convergence of operators M_n for functions of bounded variation on the interval $[0, 1]$.

Chlodowsky polynomials are given [6] by

$$C_n(f; x) = \sum_{k=0}^n f\left(\frac{k}{n} b_n\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}, \quad 0 \leq x \leq b_n,$$



Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 3 of 26

where (b_n) is a positive increasing sequence with the properties $\lim_{n \rightarrow \infty} b_n = \infty$ and $\lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$.

Works on Chlodowsky operators are fewer, since they are defined on an unbounded interval $[0, \infty)$.

This paper generalizes Chlodowsky polynomials by incorporating Durrmeyer operators, hence the name Chlodowsky-Durrmeyer operators: $D_n: BV[0, \infty) \rightarrow \mathcal{P}$,

$$D_n(f; x) = \frac{(n+1)}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \int_0^{b_n} f(t) P_{n,k} \left(\frac{t}{b_n} \right) dt, \quad 0 \leq x \leq b_n$$

where $\mathcal{P} := \{P : [0, \infty) \rightarrow \mathbb{R}\}$, is a polynomial functions set, (b_n) is a positive increasing sequence with the properties,

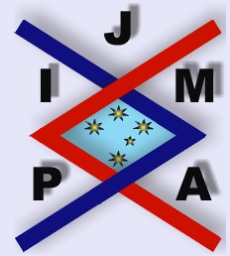
$$\lim_{n \rightarrow \infty} b_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$$

and

$$P_{n,k}(x) = \binom{n}{k} (x)^k (1-x)^{n-k}$$

is the Bernstein basis.

In this paper, by means of the techniques of probability theory, we shall estimate the rate of convergence of operators D_n , for functions of bounded variation in terms of the Chanturiya's modulus of variation. At the points which one sided limit exist, we shall prove that operators D_n converge to the limit $\frac{1}{2} [f(x+) + f(x-)]$ on the interval $[0, b_n]$, $(n \rightarrow \infty)$ extending infinity, for functions of bounded variation on the interval $[0, \infty)$.



Rate of Convergence of Chlodowsky Type Durrmeyer Operators

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 4 of 26

For the sake of brevity, let the auxiliary function g_x be defined by

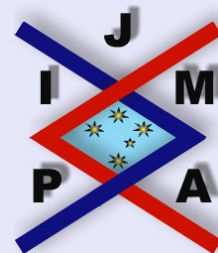
$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq b_n; \\ 0, & t = x; \\ f(t) - f(x+), & 0 \leq t < x. \end{cases}$$

The main theorem of this paper is as follows.

Theorem 1.1. *Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$. Then for every $x \in (0, \infty)$, and n sufficiently large, we have,*

$$(1.1) \quad \left| D_n(f; x) - \frac{1}{2} (f(x+) + f(x-)) \right| \leq \frac{3A_n(x)b_n^2}{x^2(b_n - x)^2} \left\{ \sum_{k=1}^n \bigvee_{x - \frac{x}{\sqrt{k}} \quad x + \frac{b_n - x}{\sqrt{k}}} (g_x) \right\} + \frac{2}{\sqrt{\frac{nx}{b_n} \left(1 - \frac{x}{b_n}\right)}} |f(x+) - f(x-)|,$$

where $A_n(x) = \left[\frac{2nx(b_n - x) + 2b_n^2}{n^2} \right]$ and $\bigvee_a^b (g_x)$ is the total variation of g_x on $[a, b]$.



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 5 of 26

2. Auxiliary Results

In this section we give certain results, which are necessary to prove our main theorem.

Lemma 2.1. *If $s \in \mathbb{N}$ and $s \leq n$, then*

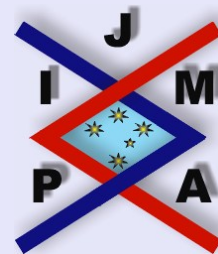
$$D_n(t^s; x) = \frac{(n+1)!b_n^s}{(n+s+1)!} \sum_{r=0}^s \binom{s}{r} \frac{s!}{r!} \cdot \frac{n!}{(n-r)!} (xb_n)^r.$$

Proof.

$$\begin{aligned} D_n(t^s; x) &= \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \left[\int_0^{b_n} P_{n,k} \left(\frac{t}{b_n} \right) t^s dt \right] \\ &= \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \left[\int_0^{b_n} \binom{n}{k} \left(\frac{t}{b_n} \right)^k \left(1 - \frac{t}{b_n} \right)^{n-k} t^s dt \right] \\ &= \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) b_n^{s+1} \binom{n}{k} \int_0^1 (u)^{k+s} (1-u)^{n-k} du, \text{ set } u = \frac{t}{b_n} \\ &= \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) b_n^{s+1} \frac{(k+s)!}{k!} \cdot \frac{n!}{(n+s+1)!}. \end{aligned}$$

Thus

$$D_n(t^s; x) = \frac{(n+1)!b_n^s}{(n+s+1)!} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \frac{(k+s)!}{k!}.$$



Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 6 of 26

For $s \leq n$, we have

$$\frac{\partial^s}{\partial x^s} \left[\left(\frac{x}{b_n} \right)^s \left(\frac{x+y}{b_n} \right)^n \right] = \frac{1}{b_n^s} \sum_{k=0}^n \binom{n}{k} \left(\frac{x}{b_n} \right)^k \left(\frac{y}{b_n} \right)^{n-k} \frac{(k+s)!}{k!}$$

and from the Leibnitz formula

$$\begin{aligned} \frac{\partial^s}{\partial x^s} \left[\left(\frac{x}{b_n} \right)^s \left(\frac{x+y}{b_n} \right)^n \right] &= \sum_{r=0}^s \binom{s}{r} \frac{s!}{r!} \cdot \frac{n!}{(n-r)!} (xb_n)^r \left(\frac{x+y}{b_n} \right)^{n-r} \frac{1}{b_n^s} \\ &= \frac{1}{b_n^s} \sum_{r=0}^s \binom{s}{r} \frac{s!}{r!} \cdot \frac{n!}{(n-r)!} (xb_n)^r \left(\frac{x+y}{b_n} \right)^{n-r} \end{aligned}$$

Let $x + y = b_n$, we have

$$\sum_{k=0}^n \binom{n}{k} \left(\frac{x}{b_n} \right)^k \left(\frac{y}{b_n} \right)^{n-k} \frac{(k+s)!}{k!} = \sum_{r=0}^s \binom{s}{r} \frac{s!}{r!} \cdot \frac{n!}{(n-r)!} (xb_n)^r \left(\frac{x+y}{b_n} \right)^{n-r}$$

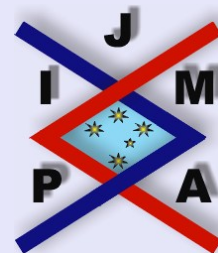
Thus the proof is complete. \square

By the Lemma 2.1, we get

$$(2.1) \quad D_n(1; x) = 1$$

$$D_n(t; x) = x + \frac{b_n - 2x}{n + 2}$$

$$D_n(t^2; x) = x^2 + \frac{[4nb_n - 6(n+1)x]}{(n+2)(n+3)}x + \frac{2b_n^2}{(n+2)(n+3)}.$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karlı

Title Page

Contents



Go Back

Close

Quit

Page 7 of 26

By direct computation, we get

$$D_n((t-x)^2; x) = \frac{2(n-3)(b_n-x)x}{(n+2)(n+3)} + \frac{2b_n^2}{(n+2)(n+3)}$$

and hence,

$$(2.2) \quad D_n((t-x)^2; x) \leq \frac{2nx(b_n-x) + 2b_n^2}{n^2}.$$

Lemma 2.2. For all $x \in (0, \infty)$, we have

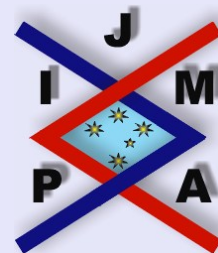
$$(2.3) \quad \begin{aligned} \lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) &= \int_0^t K_n \left(\frac{x}{b_n}, \frac{u}{b_n} \right) du \\ &\leq \frac{1}{(x-t)^2} \cdot \frac{2nx(b_n-x) + 2b_n^2}{n^2}, \end{aligned}$$

where

$$K_n \left(\frac{x}{b_n}, \frac{u}{b_n} \right) = \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) P_{n,k} \left(\frac{u}{b_n} \right).$$

Proof.

$$\begin{aligned} \lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) &= \int_0^t K_n \left(\frac{x}{b_n}, \frac{u}{b_n} \right) du \\ &\leq \int_0^t K_n \left(\frac{x}{b_n}, \frac{u}{b_n} \right) \left(\frac{x-u}{x-t} \right)^2 du \end{aligned}$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 8 of 26

$$\begin{aligned}
 &= \frac{1}{(x-t)^2} \int_0^t K_n \left(\frac{x}{b_n}, \frac{u}{b_n} \right) (x-u)^2 du \\
 &= \frac{1}{(x-t)^2} D_n((u-x)^2; x)
 \end{aligned}$$

By the (2.2), we have,

$$\begin{aligned}
 \lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) &\leq \frac{1}{(x-t)^2} \cdot \frac{2(n-3)(b_n-x)x}{(n+2)(n+3)} + \frac{2b_n^2}{(n+2)(n+3)} \\
 &\leq \frac{1}{(x-t)^2} \cdot \frac{2nx(b_n-x) + 2b_n^2}{n^2}.
 \end{aligned}$$

□

Set

$$(2.4) \quad J_{n,j}^\alpha \left(\frac{x}{b_n} \right) = \left(\sum_{k=j}^n P_{n,k} \left(\frac{x}{b_n} \right) \right)^\alpha, \quad \left(J_{n,n+1}^\alpha \left(\frac{x}{b_n} \right) = 0 \right),$$

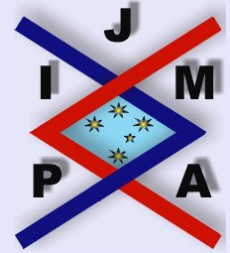
where $\alpha \geq 1$.

Lemma 2.3. For all $x \in (0, 1)$ and $j = 0, 1, 2, \dots, n$, we have

$$|J_{n,j}^\alpha(x) - J_{n+1,j+1}^\alpha(x)| \leq \frac{2\alpha}{\sqrt{nx(1-x)}}$$

and

$$|J_{n,j}^\alpha(x) - J_{n+1,j}^\alpha(x)| \leq \frac{2\alpha}{\sqrt{nx(1-x)}}.$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 9 of 26

Proof. The proof of this lemma is given in [9]. □

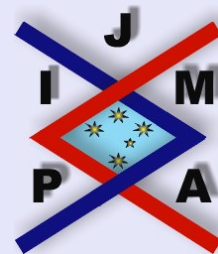
For $\alpha = 1$, replacing the variable x with $\frac{x}{b_n}$ in Lemma 2.3 we get the following lemma:

Lemma 2.4. For all $x \in (0, b_n)$ and $j = 0, 1, 2, \dots, n$, we have

$$\left| J_{n,j} \left(\frac{x}{b_n} \right) - J_{n+1,j+1} \left(\frac{x}{b_n} \right) \right| \leq \frac{2}{\sqrt{n \frac{x}{b_n} \left(1 - \frac{x}{b_n} \right)}}$$

and

$$(2.5) \quad \left| J_{n,j} \left(\frac{x}{b_n} \right) - J_{n+1,j} \left(\frac{x}{b_n} \right) \right| \leq \frac{2}{\sqrt{n \frac{x}{b_n} \left(1 - \frac{x}{b_n} \right)}}.$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 10 of 26

3. Proof Of The Main Result

Now, we can prove the Theorem 1.1.

Proof. For any $f \in BV[0, \infty)$, we can decompose f into four parts on $[0, b_n]$ for sufficiently large n ,

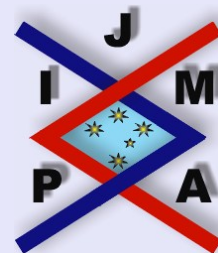
$$(3.1) \quad f(t) = \frac{1}{2} (f(x+) + f(x-)) \\ + g_x(t) + \frac{f(x+) - f(x-)}{2} \operatorname{sgn}(t - x) \\ + \delta_x(t) \left[f(x) - \frac{1}{2} (f(x+) + f(x-)) \right]$$

where

$$(3.2) \quad \delta_x(t) = \begin{cases} 1, & x = t \\ 0, & x \neq t. \end{cases}$$

If we applying the operator D_n the both side of equality (3.1), we have

$$D_n(f; x) = \frac{1}{2} (f(x+) + f(x-)) D_n(1; x) + D_n(g_x; x) \\ + \frac{f(x+) - f(x-)}{2} D_n(\operatorname{sgn}(t - x); x) \\ + \left[f(x) - \frac{1}{2} (f(x+) + f(x-)) \right] D_n(\delta_x; x).$$



Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 11 of 26

Hence, since (2.1) $D_n(1; x) = 1$, we get,

$$\begin{aligned} & \left| D_n(f; x) - \frac{1}{2} (f(x+) + f(x-)) \right| \\ & \leq |D_n(g_x; x)| + \left| \frac{f(x+) - f(x-)}{2} \right| |D_n(\text{sgn}(t - x); x)| \\ & \quad + \left| f(x) - \frac{1}{2} (f(x+) + f(x-)) \right| |D_n(\delta_x; x)|. \end{aligned}$$

For operators D_n , using (3.2) we can see that $D_n(\delta_x; x) = 0$.

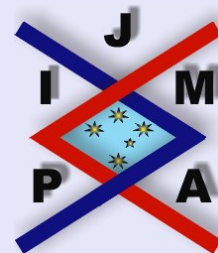
Hence we have

$$\begin{aligned} & \left| D_n(f; x) - \frac{1}{2} (f(x+) + f(x-)) \right| \\ & \leq |D_n(g_x; x)| + \left| \frac{f(x+) - f(x-)}{2} \right| |D_n(\text{sgn}(t - x); x)| \end{aligned}$$

In order to prove above inequality, we need the estimates for $D_n(g_x; x)$ and $D_n(\text{sgn}(t - x); x)$.

We first estimate $|D_n(g_x; x)|$ as follows:

$$|D_n(g_x; x)| = \left| \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \left[\int_0^{b_n} P_{n,k} \left(\frac{t}{b_n} \right) g_x(t) dt \right] \right|$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 12 of 26

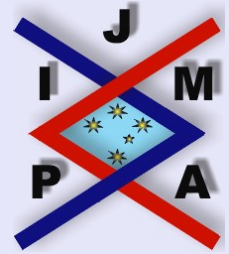
$$\begin{aligned}
&= \left| \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \left[\left(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} + \int_{x+\frac{b_n-x}{\sqrt{n}}}^{b_n} \right) P_{n,k} \left(\frac{t}{b_n} \right) g_x(t) dt \right] \right| \\
&\leq \left| \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \int_0^{x-\frac{x}{\sqrt{n}}} P_{n,k} \left(\frac{t}{b_n} \right) g_x(t) dt \right| \\
&\quad + \left| \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} P_{n,k} \left(\frac{t}{b_n} \right) g_x(t) dt \right| \\
&\quad + \left| \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \int_{x+\frac{b_n-x}{\sqrt{n}}}^{b_n} P_{n,k} \left(\frac{t}{b_n} \right) g_x(t) dt \right| \\
&= |I_1(n, x)| + |I_2(n, x)| + |I_3(n, x)|
\end{aligned}$$

We shall evaluate $I_1(n, x)$, $I_2(n, x)$ and $I_3(n, x)$. To do this we first observe that $I_1(n, x)$, $I_2(n, x)$ and $I_3(n, x)$ can be written as Lebesgue-Stieltjes integral,

$$\begin{aligned}
|I_1(n, x)| &= \left| \int_0^{x-\frac{x}{\sqrt{n}}} g_x(t) d_t \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right| \\
|I_2(n, x)| &= \left| \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} g_x(t) d_t \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right| \\
|I_3(n, x)| &= \left| \int_{x+\frac{b_n-x}{\sqrt{n}}}^{b_n} g_x(t) d_t \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right|,
\end{aligned}$$

where

$$\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) = \int_0^t K_n \left(\frac{x}{b_n}, \frac{u}{b_n} \right) du$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 13 of 26

and

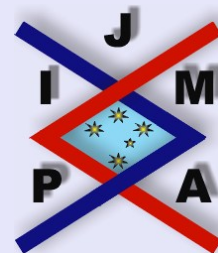
$$K_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) = \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) P_{n,k} \left(\frac{t}{b_n} \right).$$

First we estimate $I_2(n, x)$. For $t \in \left[x - \frac{x}{\sqrt{n}}, x + \frac{b_n - x}{\sqrt{n}} \right]$, we have

$$\begin{aligned} |I_2(n, x)| &= \left| \int_{x - \frac{x}{\sqrt{n}}}^{x + \frac{b_n - x}{\sqrt{n}}} (g_x(t) - g_x(x)) d_t \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right| \\ &\leq \int_{x - \frac{x}{\sqrt{n}}}^{x + \frac{b_n - x}{\sqrt{n}}} |g_x(t) - g_x(x)| \left| d_t \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right| \\ &\leq \bigvee_{x - \frac{x}{\sqrt{n}}}^{x + \frac{b_n - x}{\sqrt{n}}} (g_x) \leq \frac{1}{n-1} \sum_{k=2}^n \bigvee_{x - \frac{x}{\sqrt{n}}}^{x + \frac{b_n - x}{\sqrt{n}}} (g_x). \end{aligned} \tag{3.3}$$

Next, we estimate $I_1(n, x)$. Using partial Lebesgue-Stieltjes integration, we obtain

$$\begin{aligned} I_1(n, x) &= \int_0^{x - \frac{x}{\sqrt{n}}} g_x(t) d_t \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \\ &= g_x \left(x - \frac{x}{\sqrt{n}} \right) \lambda_n \left(\frac{x}{b_n}, \frac{x - \frac{x}{\sqrt{n}}}{b_n} \right) \\ &\quad - g_x(0) \lambda_n \left(\frac{x}{b_n}, 0 \right) - \int_0^{x - \frac{x}{\sqrt{n}}} \lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) d_t (g_x(t)). \end{aligned}$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karşli

Title Page

Contents



Go Back

Close

Quit

Page 14 of 26

Since

$$\left| g_x \left(x - \frac{x}{\sqrt{n}} \right) \right| = \left| g_x \left(x - \frac{x}{\sqrt{n}} \right) - g_x(x) \right| \leq \bigvee_{x - \frac{x}{\sqrt{n}}}^x (g_x),$$

it follows that

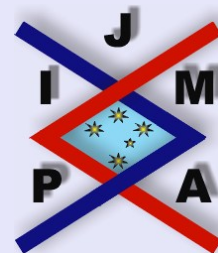
$$|I_1(n, x)| \leq \bigvee_{x - \frac{x}{\sqrt{n}}}^x (g_x) \left| \lambda_n \left(\frac{x}{b_n}, \frac{x - \frac{x}{\sqrt{n}}}{b_n} \right) \right| + \int_0^{x - \frac{x}{\sqrt{n}}} \lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) d_t \left(- \bigvee_t^x (g_x) \right).$$

From (2.3), it is clear that

$$\lambda_n \left(\frac{x}{b_n}, \frac{x - \frac{x}{\sqrt{n}}}{b_n} \right) \leq \frac{1}{\left(\frac{x}{\sqrt{n}} \right)^2} \left\{ \frac{2nx(b_n - x) + 2b_n^2}{n^2} \right\}.$$

It follows that

$$\begin{aligned} |I_1(n, x)| &\leq \bigvee_{x - \frac{x}{\sqrt{n}}}^x (g_x) \frac{1}{\left(\frac{x}{\sqrt{n}} \right)^2} \left\{ \frac{2nx(b_n - x) + 2b_n^2}{n^2} \right\} \\ &\quad + \int_0^{x - \frac{x}{\sqrt{n}}} \frac{1}{(x - t)^2} \left\{ \frac{2nx(b_n - x) + 2b_n^2}{n^2} \right\} d_t \left(- \bigvee_t^x (g_x) \right) \\ &= \bigvee_{x - \frac{x}{\sqrt{n}}}^x (g_x) \frac{A_n(x)}{\left(\frac{x}{\sqrt{n}} \right)^2} + A_n(x) \int_0^{x - \frac{x}{\sqrt{n}}} \frac{1}{(x - t)^2} d_t \left(- \bigvee_t^x (g_x) \right). \end{aligned}$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 15 of 26

Furthermore, since

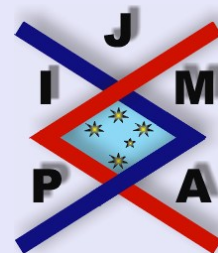
$$\begin{aligned} & \int_0^{x-\frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2} dt \left(-\bigvee_t^x (g_x) \right) \\ &= -\frac{1}{(x-t)^2} \bigvee_t^x (g_x) \Big|_0^{x-\frac{x}{\sqrt{n}}} + \int_0^{x-\frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3} \bigvee_t^x (g_x) dt \\ &= -\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (g_x) + \frac{1}{x^2} \bigvee_0^x (g_x) + \int_0^{x-\frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3} \bigvee_t^x (g_x) dt. \end{aligned}$$

Putting $t = x - \frac{x}{\sqrt{u}}$ in the last integral, we get

$$\int_0^{x-\frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3} \bigvee_t^x (g_x) dt = \frac{1}{x^2} \int_1^n \bigvee_{x-\frac{x}{\sqrt{u}}}^x (g_x) du = \frac{1}{x^2} \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x (g_x).$$

Consequently,

$$\begin{aligned} |I_1(n, x)| &\leq \bigvee_{x-\frac{x}{\sqrt{n}}}^x (g_x) \frac{A_n(x)}{\left(\frac{x}{\sqrt{n}}\right)^2} \\ &+ A_n(x) \left\{ -\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (g_x) + \frac{1}{x^2} \bigvee_0^x (g_x) + \frac{1}{x^2} \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x (g_x) \right\} \end{aligned}$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 16 of 26

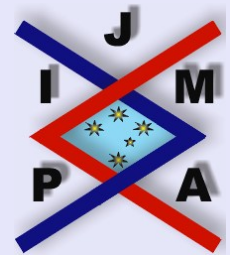
$$\begin{aligned}
 &= A_n(x) \left\{ \frac{1}{x^2} \bigvee_0^x(g_x) + \frac{1}{x^2} \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x(g_x) \right\} \\
 (3.4) \quad &= \frac{A_n(x)}{x^2} \left\{ \bigvee_0^{b_n}(g_x) + \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x(g_x) \right\}.
 \end{aligned}$$

Using the similar method for estimating $|I_3(n, x)|$, we get

$$\begin{aligned}
 |I_3(n, x)| &\leq \frac{A_n(x)}{(b_n - x)^2} \left\{ \bigvee_x^{b_n}(g_x) + \sum_{k=1}^n \bigvee_x^{x+\frac{b_n-x}{\sqrt{k}}}(g_x) \right\} \\
 (3.5) \quad &\leq \frac{A_n(x)}{(b_n - x)^2} \left\{ \bigvee_0^{b_n}(g_x) + \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x) \right\}.
 \end{aligned}$$

Hence from (3.3), (3.4) and (3.5), it follows that

$$\begin{aligned}
 |D_n(g_x; x)| &\leq |I_1(n, x)| + |I_2(n, x)| + |I_3(n, x)| \\
 &\leq \frac{A_n(x)}{x^2} \left\{ \bigvee_0^{b_n}(g_x) + \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x(g_x) \right\} \\
 &\quad + \frac{A_n(x)}{(b_n - x)^2} \left\{ \bigvee_0^{b_n}(g_x) + \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x) \right\} + \frac{1}{n-1} \sum_{k=2}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{n}}}(g_x).
 \end{aligned}$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 17 of 26

Obviously,

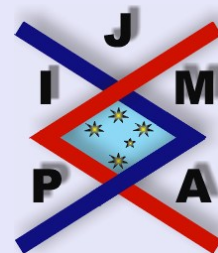
$$\frac{1}{x^2} + \frac{1}{(b_n - x)^2} = \frac{b_n^2}{x^2(b_n - x)^2},$$

for $\frac{x}{b_n} \in [0, 1]$ and

$$\bigvee_{x - \frac{x}{\sqrt{k}}}^x (g_x) \leq \bigvee_{x - \frac{x}{\sqrt{k}}}^{x + \frac{b_n - x}{\sqrt{k}}} (g_x).$$

Hence,

$$\begin{aligned} |D_n(g_x; x)| &\leq \left(\frac{A_n(x)}{x^2} + \frac{A_n(x)}{(b_n - x)^2} \right) \left\{ \bigvee_0^{b_n} (g_x) + \sum_{k=1}^n \bigvee_{x - \frac{x}{\sqrt{k}}}^{x + \frac{b_n - x}{\sqrt{k}}} (g_x) \right\} \\ &\quad + \frac{1}{n-1} \sum_{k=2}^n \bigvee_{x - \frac{x}{\sqrt{k}}}^{x + \frac{b_n - x}{\sqrt{k}}} (g_x) \\ &= \frac{A_n(x) b_n^2}{x^2(b_n - x)^2} \left\{ \bigvee_0^{b_n} (g_x) + \sum_{k=1}^n \bigvee_{x - \frac{x}{\sqrt{k}}}^{x + \frac{b_n - x}{\sqrt{k}}} (g_x) \right\} + \frac{1}{n-1} \sum_{k=2}^n \bigvee_{x - \frac{x}{\sqrt{k}}}^{x + \frac{b_n - x}{\sqrt{k}}} (g_x) \\ &= \frac{A_n(x) b_n^2}{x^2(b_n - x)^2} \left\{ \bigvee_0^{b_n} (g_x) + \sum_{k=1}^n \bigvee_{x - \frac{x}{\sqrt{k}}}^{x + \frac{b_n - x}{\sqrt{k}}} (g_x) \right\} + \frac{1}{n-1} \sum_{k=2}^n \bigvee_{x - \frac{x}{\sqrt{k}}}^{x + \frac{b_n - x}{\sqrt{k}}} (g_x). \end{aligned}$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 18 of 26

On the other hand, note that

$$\bigvee_0^{b_n}(g_x) \leq \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x).$$

By (2.3), we have

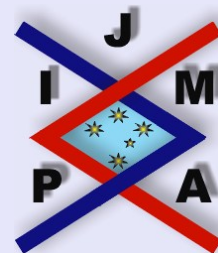
$$|D_n(g_x; x)| \leq \frac{2 A_n(x) b_n^2}{x^2(b_n - x)^2} \left\{ \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x) \right\} + \frac{1}{n-1} \sum_{k=2}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x).$$

Note that $\frac{1}{n-1} \leq \frac{A_n(x) b_n^2}{x^2(b_n-x)^2}$, for $n > 1$, $\frac{x}{b_n} \in [0, 1]$. Consequently

$$(3.6) \quad |D_n(g_x; x)| \leq \frac{3 A_n(x) b_n^2}{x^2(b_n - x)^2} \left\{ \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x) \right\}.$$

Now secondly, we can estimate $D_n(\text{sgn}(t-x); x)$. If we apply operator D_n to the signum function, we get

$$\begin{aligned} & D_n(\text{sgn}(t-x); x) \\ &= \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \left[\int_x^{b_n} P_{n,k} \left(\frac{t}{b_n} \right) dt - \int_0^x P_{n,k} \left(\frac{t}{b_n} \right) dt \right] \\ &= \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \left[\int_0^{b_n} P_{n,k} \left(\frac{t}{b_n} \right) dt - 2 \int_0^x P_{n,k} \left(\frac{t}{b_n} \right) dt \right] \end{aligned}$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 19 of 26

using (2.1), we have

$$(3.7) \quad D_n(\operatorname{sgn}(t-x); x) = 1 - 2 \frac{n+1}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \int_0^x P_{n,k} \left(\frac{t}{b_n} \right) dt.$$

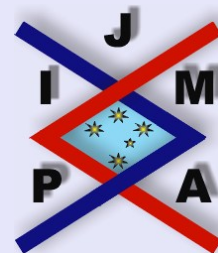
Now, we differentiate both side of the following equality

$$J_{n+1,k+1} \left(\frac{x}{b_n} \right) = \sum_{j=k+1}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right).$$

For $k = 0, 1, 2, \dots, n$ we get,

$$\begin{aligned} \frac{d}{dx} J_{n+1,k+1} \left(\frac{x}{b_n} \right) &= \frac{d}{dx} \sum_{j=k+1}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) \\ &= \frac{d}{dx} P_{n+1,k+1} \left(\frac{x}{b_n} \right) + \frac{d}{dx} P_{n+1,k+2} \left(\frac{x}{b_n} \right) \\ &\quad + \dots + \frac{d}{dx} P_{n+1,n+1} \left(\frac{x}{b_n} \right) \end{aligned}$$

$$\begin{aligned} &\frac{d}{dx} J_{n+1,k+1} \left(\frac{x}{b_n} \right) \\ &= \frac{(n+1)}{b_n} \left\{ \left[P_{n,k} \left(\frac{x}{b_n} \right) - P_{n,k+1} \left(\frac{x}{b_n} \right) \right] + \left[P_{n,k+1} \left(\frac{x}{b_n} \right) - P_{n,k+2} \left(\frac{x}{b_n} \right) \right] \right. \\ &\quad \left. + \dots + \left[P_{n,n-1} \left(\frac{x}{b_n} \right) - P_{n,n} \left(\frac{x}{b_n} \right) \right] + \left[P_{n,n} \left(\frac{x}{b_n} \right) \right] \right\} \end{aligned}$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 20 of 26

$$= \frac{(n+1)}{b_n} \sum_{j=k+1}^{n+1} \left[P_{n,j-1} \left(\frac{x}{b_n} \right) - P_{n,j} \left(\frac{x}{b_n} \right) \right] = \frac{(n+1)}{b_n} P_{n,k} \left(\frac{x}{b_n} \right)$$

and $J_{n+1,k+1}(0) = 0$. Taking the integral from zero to x , we have

$$\frac{(n+1)}{b_n} \int_0^x P_{n,k} \left(\frac{t}{b_n} \right) dt = J_{n+1,k+1} \left(\frac{x}{b_n} \right)$$

and therefore from (2.4)

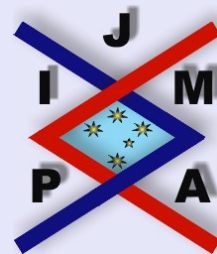
$$\begin{aligned} J_{n+1,k+1} \left(\frac{x}{b_n} \right) &= \sum_{j=k+1}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) \\ &= \sum_{j=0}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) - \sum_{j=0}^k P_{n+1,j} \left(\frac{x}{b_n} \right) \\ &= 1 - \sum_{j=0}^k P_{n+1,j} \left(\frac{x}{b_n} \right). \end{aligned}$$

Hence

$$\frac{(n+1)}{b_n} \int_0^x P_{n,k} \left(\frac{t}{b_n} \right) dt = 1 - \sum_{j=0}^k P_{n+1,j} \left(\frac{x}{b_n} \right).$$

From (3.7), we get

$$D_n(\operatorname{sgn}(t-x); x) = 1 - 2 \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \left[1 - \sum_{j=0}^k P_{n+1,j} \left(\frac{x}{b_n} \right) \right]$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 21 of 26

$$\begin{aligned}
 &= 1 - 2 \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) + 2 \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \sum_{j=0}^k P_{n+1,j} \left(\frac{x}{b_n} \right) \\
 &= -1 + 2 \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \sum_{j=0}^k P_{n+1,j} \left(\frac{x}{b_n} \right).
 \end{aligned}$$

Set

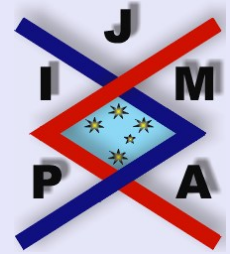
$$Q_{n+1,j}^{(2)} \left(\frac{x}{b_n} \right) = J_{n+1,j}^2 \left(\frac{x}{b_n} \right) - J_{n+1,j+1}^2 \left(\frac{x}{b_n} \right).$$

Also note that

$$\begin{aligned}
 \sum_{k=0}^n \sum_{j=0}^k * &= \sum_{j=0}^n \sum_{k=j}^n *, \\
 \sum_{k=j}^{n+1} Q_{n+1,k}^{(2)} \left(\frac{x}{b_n} \right) &= J_{n+1,j}^2 \left(\frac{x}{b_n} \right) \quad \text{and} \quad J_{n,n+1} \left(\frac{x}{b_n} \right) = 0,
 \end{aligned}$$

we have

$$\begin{aligned}
 D_n(\text{sgn}(t-x); x) &= -1 + 2 \sum_{j=0}^n P_{n+1,j} \left(\frac{x}{b_n} \right) \sum_{k=j}^n P_{n,k} \left(\frac{x}{b_n} \right) \\
 &= -1 + 2 \sum_{j=0}^n P_{n+1,j} \left(\frac{x}{b_n} \right) J_{n,j} \left(\frac{x}{b_n} \right) \\
 &= -1 + 2 \sum_{j=0}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) J_{n,j} \left(\frac{x}{b_n} \right)
 \end{aligned}$$



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 22 of 26

$$= 2 \sum_{j=0}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) J_{n,j} \left(\frac{x}{b_n} \right) - 1.$$

Since $\sum_{j=0}^{n+1} Q_{n+1,j}^{(2)} \left(\frac{x}{b_n} \right) = 1$, thus

$$D_n(\operatorname{sgn}(t-x); x) = 2 \sum_{j=0}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) J_{n,j} \left(\frac{x}{b_n} \right) - \sum_{j=0}^{n+1} Q_{n+1,j}^{(2)} \left(\frac{x}{b_n} \right).$$

By the mean value theorem, we have

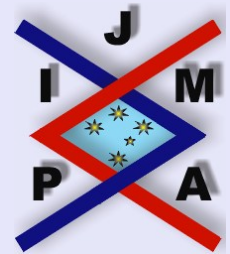
$$\begin{aligned} Q_{n+1,j}^{(2)} \left(\frac{x}{b_n} \right) &= J_{n+1,j}^2 \left(\frac{x}{b_n} \right) - J_{n+1,j+1}^2 \left(\frac{x}{b_n} \right) \\ &= 2P_{n+1,j} \left(\frac{x}{b_n} \right) \gamma_{n,j} \left(\frac{x}{b_n} \right) \end{aligned}$$

where

$$J_{n+1,j+1} \left(\frac{x}{b_n} \right) < \gamma_{n,j} \left(\frac{x}{b_n} \right) < J_{n+1,j} \left(\frac{x}{b_n} \right).$$

Hence it follows from (2.5) that

$$\begin{aligned} |D_n(\operatorname{sgn}(t-x); x)| &= \left| 2 \sum_{j=0}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) \left(J_{n,j} \left(\frac{x}{b_n} \right) - \gamma_{n,j} \left(\frac{x}{b_n} \right) \right) \right| \\ &\leq 2 \sum_{j=0}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) \left| J_{n,j} \left(\frac{x}{b_n} \right) - \gamma_{n,j} \left(\frac{x}{b_n} \right) \right| \end{aligned}$$



Rate of Convergence of Chlodowsky Type Durrmeyer Operators

Ertan Ibikli and Harun Karsli

Title Page

Contents

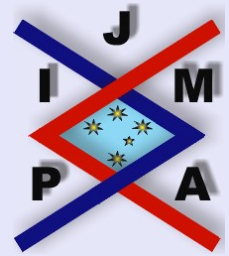


Go Back

Close

Quit

Page 23 of 26



**Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators**

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 24 of 26

$$\begin{aligned} & \left| J_{n,j} \left(\frac{x}{b_n} \right) - J_{n+1,j+1} \left(\frac{x}{b_n} \right) \left(\frac{x}{b_n} \right) \right| \\ &= \left| J_{n,j} \left(\frac{x}{b_n} \right) - \gamma_{n,j} \left(\frac{x}{b_n} \right) + \gamma_{n,j} \left(\frac{x}{b_n} \right) - J_{n+1,j+1} \left(\frac{x}{b_n} \right) \right|, \end{aligned}$$

since $\gamma_{n,j} \left(\frac{x}{b_n} \right) - J_{n+1,j+1} \left(\frac{x}{b_n} \right) > 0$, then we have

$$\left| J_{n,j} \left(\frac{x}{b_n} \right) - \gamma_{n,j} \left(\frac{x}{b_n} \right) \right| \leq \left| J_{n,j} \left(\frac{x}{b_n} \right) - J_{n+1,j+1} \left(\frac{x}{b_n} \right) \right|.$$

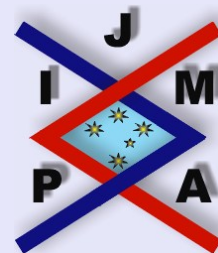
Hence

$$\begin{aligned} |D_n(\text{sgn}(t-x); x)| &\leq 2 \sum_{j=0}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) \left| J_{n,j} \left(\frac{x}{b_n} \right) - J_{n+1,j+1} \left(\frac{x}{b_n} \right) \right| \\ &\leq 2 \sum_{j=0}^{n+1} P_{n+1,j} \left(\frac{x}{b_n} \right) \frac{2}{\sqrt{n \frac{x}{b_n} \left(1 - \frac{x}{b_n} \right)}} \\ (3.8) \quad &= \frac{4}{\sqrt{n \frac{x}{b_n} \left(1 - \frac{x}{b_n} \right)}}. \end{aligned}$$

Combining (3.6) and (3.8) we get (1.1). Thus, the proof of the theorem is completed. \square

References

- [1] J.L. DURRMEYER, Une formule d'inversion de la transformee de Laplace: Applicationsa la theorie de moments , these de 3e cycle, Faculte des sciences de l'universite de Paris,1971.
- [2] A. LUPAŞ, Die Folge Der Betaoperatoren, Dissertation, Stuttgart Universität, 1972.
- [3] R. BOJANIC AND M. VUILLEUMIER, On the rate of convergence of Fourier Legendre series of functions of bounded variation, *J. Approx. Theory*, **31** (1981), 67–79.
- [4] F. CHENG, On the rate of convergence of Bernstein polynomials of functions of bounded variation, *J. Approx. Theory*, **39** (1983), 259–274.
- [5] S. GUO, On the rate of convergence of Durrmeyer operator for functions of bounded variation, *J. Approx. Theory*, **51** (1987), 183–197.
- [6] I. CHLODOWSKY, Sur le développement des fonctions définies dans un interval infini en séries de polynômes de S.N. Bernstein, *Compositio Math.*, **4** (1937), 380–392.
- [7] XIAO-MING ZENG, Bounds for Bernstein basis functions and Meyer-König-Zeller basis functions, *J. Math. Anal. Appl.*, **219** (1998), 364–376.
- [8] XIAO-MING ZENG AND A. PIRIOU, On the rate of convergence of two Bernstein-Bezier type operators for bounded variation functions, *J. Approx. Theory*, **95** (1998), 369–387.



Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators

Ertan Ibikli and Harun Karsli

Title Page

Contents



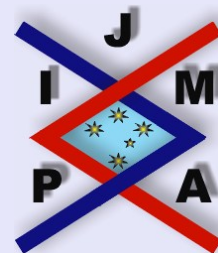
Go Back

Close

Quit

Page 25 of 26

- [9] XIAO-MING ZENG AND W. CHEN, On the rate of convergence of the generalized Durrmeyer type operators for functions of bounded variation, *J. Approx. Theory*, **102** (2000), 1–12.
- [10] A.N. SHIRYAYEV, *Probability*, Springer-Verlag, New York, 1984.
- [11] G.G. LORENTZ, *Bernstein Polynomials*, Univ. of Toronto Press, Toronto, 1953.
- [12] Z.A. CHANTURIYA, Modulus of variation of function and its application in the theory of Fourier series, *Dokl. Akad. Nauk. SSSR*, **214** (1974), 63–66.



Rate of Convergence of
Chlodowsky Type Durrmeyer
Operators

Ertan Ibikli and Harun Karsli

Title Page

Contents



Go Back

Close

Quit

Page 26 of 26