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RATE OF CONVERGENCE OF CHLODOWSKY TYPE DURRMEYER OPERATORS

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Abstract

In the present paper, we estimate the rate of pointwise convergence of the Chlodowsky type Durrmeyer Operators $D_n(f,x)$ for functions, defined on the interval $[0,b_n]$, $(b_n \to \infty)$, extending infinity, of bounded variation. To prove our main result, we have used some methods and techniques of probability theory.

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Contents

1. Introduction

Very recently, some authors studied some linear positive operators and obtained the rate of convergence for functions of bounded variation. For example, Bojanic R. and Vuilleumier M. [3] estimated the rate of convergence of Fourier Legendre series of functions of bounded variation on the interval $[0,1]$, Cheng F. [4] estimated the rate of convergence of Bernstein polynomials of functions bounded variation on the interval $[0,1]$, Zeng and Chen [9] estimated the rate of convergence of Durrmeyer type operators for functions of bounded variation on the interval $[0,1]$.

Durrmeyer operators M_n introduced by Durrmeyer [1]. Also let us note that these operators were introduced by Lupaş [2]. The polynomial $M_n f$ defined by

$$M_n(f;x) = (n+1)\sum_{k=0}^{n} P_{n,k}(x)\int_0^1 f(t)P_{n,k}(t)\,dt, \qquad 0 \le x \le 1,$$

where

$$P_{n,k}(x) = \binom{n}{k}(x)^k(1-x)^{n-k}.$$

These operators are the integral modification of Bernstein polynomials so as to approximate Lebesgue integrable functions on the interval $[0,1]$. The operators M_n were studied by several authors. Also, Guo S. [5] investigated Durmeyer operators M_n and estimated the rate of convergence of operators M_n for functions of bounded variation on the interval $[0,1]$.

Chlodowsky polynomials are given [6] by

$$C_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}, \quad 0 \le x \le b_n,$$

where (b_n) is a positive increasing sequence with the properties $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} \frac{b_n}{n} = 0$.

Works on Chlodowsky operators are fewer, since they are defined on an unbounded interval $[0,\infty)$.

This paper generalizes Chlodowsky polynomials by incorporating Durrmeyer operators, hence the name Chlodowsky-Durrmeyer operators: $D_n : BV[0,\infty) \to \mathcal{P}$,

$$D_n(f;x) = \frac{(n+1)}{b_n} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \int_0^{b_n} f(t) P_{n,k}\left(\frac{t}{b_n}\right) dt, \quad 0 \le x \le b_n$$

where $\mathcal{P} := \{P : [0,\infty) \to \mathbb{R}\}$, is a polynomial functions set, (b_n) is a positive increasing sequence with the properties,

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{n} = 0$$

and

$$P_{n,k}(x) = \binom{n}{k} (x)^k (1-x)^{n-k}$$

is the Bernstein basis.

In this paper, by means of the techniques of probability theory, we shall estimate the rate of convergence of operators D_n, for functions of bounded variation in terms of the Chanturiya's modulus of variation. At the points which one sided limit exist, we shall prove that operators D_n converge to the limit $\frac{1}{2}[f(x+) + f(x-)]$ on the interval $[0, b_n]$, $(n \to \infty)$ extending infinity, for functions of bounded variation on the interval $[0,\infty)$.

For the sake of brevity, let the auxiliary function g_x be defined by

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq b_n; \\ 0, & t = x; \\ f(t) - f(x+), & 0 \leq t < x. \end{cases}$$

The main theorem of this paper is as follows.

Theorem 1.1. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$. Then for every $x \in (0,\infty)$, and n sufficiently large, we have,*

$$\left| D_n(f;x) - \frac{1}{2}\left(f(x+) + f(x-)\right) \right| \leq \frac{3A_n(x)b_n^2}{x^2(b_n - x)^2} \left\{ \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n - x}{\sqrt{k}}} (g_x) \right\} + \frac{2}{\sqrt{\frac{nx}{b_n}\left(1 - \frac{x}{b_n}\right)}} \left|\{f(x+) - f(x-)\}\right|, \tag{1.1}$$

where $A_n(x) = \left[\frac{2\,n\,x(b_n - x) + 2b_n^2}{n^2}\right]$ and $\bigvee_a^b (g_x)$ is the total variation of g_x on $[a,b]$.

2. Auxiliary Results

In this section we give certain results, which are necessary to prove our main theorem.

Lemma 2.1. *If* $s \in \mathbb{N}$ *and* $s \leq n$, *then*

$$D_n(t^s; x) = \frac{(n+1)!b_n^s}{(n+s+1)!} \sum_{r=0}^{s} \binom{s}{r} \frac{s!}{r!} \cdot \frac{n!}{(n-r)!} (xb_n)^r.$$

Proof.

$$\begin{aligned} D_n(t^s; x) &= \frac{n+1}{b_n} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \left[\int_0^{b_n} P_{n,k}\left(\frac{t}{b_n}\right) t^s dt\right] \\ &= \frac{n+1}{b_n} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \left[\int_0^{b_n} \binom{n}{k} \left(\frac{t}{b_n}\right)^k \left(1 - \frac{t}{b_n}\right)^{n-k} t^s dt\right] \\ &= \frac{n+1}{b_n} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) b_n^{s+1} \binom{n}{k} \int_0^1 (u)^{k+s} (1-u)^{n-k} du, \text{ set } u = \frac{t}{b_n} \\ &= \frac{n+1}{b_n} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) b_n^{s+1} \frac{(k+s)!}{k!} \cdot \frac{n!}{(n+s+1)!}. \end{aligned}$$

Thus

$$D_n(t^s; x) = \frac{(n+1)!b_n^s}{(n+s+1)!} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \frac{(k+s)!}{k!}.$$

For $s \leq n$, we have

$$\frac{\partial^s}{\partial x^s}\left[\left(\frac{x}{b_n}\right)^s\left(\frac{x+y}{b_n}\right)^n\right] = \frac{1}{b_n^s}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(\frac{y}{b_n}\right)^{n-k}\frac{(k+s)!}{k!}$$

and from the Leibnitz formula

$$\begin{aligned}\frac{\partial^s}{\partial x^s}\left[\left(\frac{x}{b_n}\right)^s\left(\frac{x+y}{b_n}\right)^n\right] &= \sum_{r=0}^{s}\binom{s}{r}\frac{s!}{r!}\cdot\frac{n!}{(n-r)!}(xb_n)^r\left(\frac{x+y}{b_n}\right)^{n-r}\frac{1}{b_n^s}\\ &= \frac{1}{b_n^s}\sum_{r=0}^{s}\binom{s}{r}\frac{s!}{r!}\cdot\frac{n!}{(n-r)!}(xb_n)^r\left(\frac{x+y}{b_n}\right)^{n-r}\end{aligned}$$

Let $x+y=b_n$, we have

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(\frac{y}{b_n}\right)^{n-k}\frac{(k+s)!}{k!} = \sum_{r=0}^{s}\binom{s}{r}\frac{s!}{r!}\cdot\frac{n!}{(n-r)!}(xb_n)^r\left(\frac{x+y}{b_n}\right)^{n-r}$$

Thus the proof is complete. □

By the Lemma 2.1, we get

$$\begin{aligned}D_n(1;x) &= 1\\ D_n(t;x) &= x + \frac{b_n - 2x}{n+2}\\ D_n(t^2;x) &= x^2 + \frac{[4nb_n - 6(n+1)x]}{(n+2)(n+3)}x + \frac{2b_n^2}{(n+2)(n+3)}.\end{aligned} \tag{2.1}$$

By direct computation, we get

$$D_n((t-x)^2;x) = \frac{2(n-3)(b_n-x)x}{(n+2)(n+3)} + \frac{2b_n^2}{(n+2)(n+3)}$$

and hence,

$$D_n((t-x)^2;x) \le \frac{2nx(b_n-x)+2b_n^2}{n^2}. \tag{2.2}$$

Lemma 2.2. *For all* $x \in (0,\infty)$ *, we have*

$$\begin{aligned} \lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) &= \int_0^t K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right)du \\ &\le \frac{1}{(x-t)^2}\cdot\frac{2nx(b_n-x)+2b_n^2}{n^2}, \end{aligned} \tag{2.3}$$

where

$$K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right) = \frac{n+1}{b_n}\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right)P_{n,k}\left(\frac{u}{b_n}\right).$$

Proof.

$$\begin{aligned} \lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right) &= \int_0^t K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right)du \\ &\le \int_0^t K_n\left(\frac{x}{b_n},\frac{u}{b_n}\right)\left(\frac{x-u}{x-t}\right)^2 du \end{aligned}$$

$$= \frac{1}{(x-t)^2} \int_0^t K_n\left(\frac{x}{b_n}, \frac{u}{b_n}\right)(x-u)^2 du$$
$$= \frac{1}{(x-t)^2} D_n((u-x)^2; x)$$

By the (2.2), we have,

$$\lambda_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) \le \frac{1}{(x-t)^2} \cdot \frac{2(n-3)(b_n-x)x}{(n+2)(n+3)} + \frac{2b_n^2}{(n+2)(n+3)}$$
$$\le \frac{1}{(x-t)^2} \cdot \frac{2nx(b_n-x)+2b_n^2}{n^2}.$$

□

Set

$$J_{n,j}^{\alpha}\left(\frac{x}{b_n}\right) = \left(\sum_{k=j}^{n} P_{n,k}\left(\frac{x}{b_n}\right)\right)^{\alpha}, \quad \left(J_{n,n+1}^{\alpha}\left(\frac{x}{b_n}\right) = 0\right), \tag{2.4}$$

where $\alpha \ge 1$.

Lemma 2.3. *For all $x \in (0,1)$ and $j = 0, 1, 2, \ldots, n$, we have*

$$\left|J_{n,j}^{\alpha}(x) - J_{n+1,j+1}^{\alpha}(x)\right| \le \frac{2\alpha}{\sqrt{nx(1-x)}}$$

and

$$\left|J_{n,j}^{\alpha}(x) - J_{n+1,j}^{\alpha}(x)\right| \le \frac{2\alpha}{\sqrt{nx(1-x)}}.$$

Proof. The proof of this lemma is given in [9]. □

For $\alpha = 1$, replacing the variable x with $\frac{x}{b_n}$ in Lemma 2.3 we get the following lemma:

Lemma 2.4. *For all $x \in (0, b_n)$ and $j = 0, 1, 2, \ldots, n$, we have*

$$\left| J_{n,j}\left(\frac{x}{b_n}\right) - J_{n+1,j+1}\left(\frac{x}{b_n}\right) \right| \leq \frac{2}{\sqrt{n\frac{x}{b_n}\left(1 - \frac{x}{b_n}\right)}}$$

and

$$\left| J_{n,j}\left(\frac{x}{b_n}\right) - J_{n+1,j}\left(\frac{x}{b_n}\right) \right| \leq \frac{2}{\sqrt{n\frac{x}{b_n}\left(1 - \frac{x}{b_n}\right)}}. \tag{2.5}$$

3. Proof Of The Main Result

Now, we can prove the Theorem 1.1.

Proof. For any $f \in BV[0,\infty)$, we can decompose f into four parts on $[0, b_n]$ for sufficiently large n,

$$f(t) = \frac{1}{2}\left(f(x+) + f(x-)\right) + g_x(t) + \frac{f(x+) - f(x-)}{2}\operatorname{sgn}(t-x) + \delta_x(t)\left[f(x) - \frac{1}{2}\left(f(x+) + f(x-)\right)\right] \tag{3.1}$$

where

$$\delta_x(t) = \begin{cases} 1, & x = t \\ 0, & x \neq t. \end{cases} \tag{3.2}$$

If we applying the operator D_n the both side of equality (3.1), we have

$$D_n(f;x) = \frac{1}{2}\left(f(x+) + f(x-)\right) D_n(1;x) + D_n(g_x;x) + \frac{f(x+) - f(x-)}{2} D_n(\operatorname{sgn}(t-x);x) + \left[f(x) - \frac{1}{2}\left(f(x+) + f(x-)\right)\right] D_n(\delta_x;x).$$

Hence, since (2.1) $D_n(1;x)=1$, we get,

$$\left|D_n(f;x)-\frac{1}{2}\left(f(x+)+f(x-)\right)\right| \leq |D_n(g_x;x)| + \left|\frac{f(x+)-f(x-)}{2}\right| |D_n(\operatorname{sgn}(t-x);x)| + \left|f(x)-\frac{1}{2}\left(f(x+)+f(x-)\right)\right| |D_n(\delta_x;x)|.$$

For operators D_n, using (3.2) we can see that $D_n(\delta_x;x)=0$.

Hence we have

$$\left|D_n(f;x)-\frac{1}{2}\left(f(x+)+f(x-)\right)\right| \leq |D_n(g_x;x)| + \left|\frac{f(x+)-f(x-)}{2}\right| |D_n(\operatorname{sgn}(t-x);x)|$$

In order to prove above inequality, we need the estimates for $D_n(g_x;x)$ and $D_n(\operatorname{sgn}(t-x);x)$.

We first estimate $|D_n(g_x;x)|$ as follows:

$$|D_n(g_x;x)| = \left|\frac{n+1}{b_n}\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right)\left[\int_0^{b_n} P_{n,k}\left(\frac{t}{b_n}\right) g_x(t)dt\right]\right|$$

$$= \left| \frac{n+1}{b_n} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \left[\left(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} + \int_{x+\frac{b_n-x}{\sqrt{n}}}^{b_n} \right) P_{n,k}\left(\frac{t}{b_n}\right) g_x(t)dt \right] \right|$$

$$\leq \left| \frac{n+1}{b_n} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \int_0^{x-\frac{x}{\sqrt{n}}} P_{n,k}\left(\frac{t}{b_n}\right) g_x(t)dt \right|$$

$$+ \left| \frac{n+1}{b_n} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} P_{n,k}\left(\frac{t}{b_n}\right) g_x(t)dt \right|$$

$$+ \left| \frac{n+1}{b_n} \sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \int_{x+\frac{b_n-x}{\sqrt{n}}}^{b_n} P_{n,k}\left(\frac{t}{b_n}\right) g_x(t)dt \right|$$

$$= |I_1(n,x)| + |I_2(n,x)| + |I_3(n,x)|$$

We shall evaluate $I_1(n,x)$, $I_2(n,x)$ and $I_3(n,x)$. To do this we first observe that $I_1(n,x)$, $I_2(n,x)$ and $I_3(n,x)$ can be written as Lebesque-Stieltjes integral,

$$|I_1(n,x)| = \left| \int_0^{x-\frac{x}{\sqrt{n}}} g_x(t) d_t \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right|$$

$$|I_2(n,x)| = \left| \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} g_x(t) d_t \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right|$$

$$|I_3(n,x)| = \left| \int_{x+\frac{b_n-x}{\sqrt{n}}}^{b_n} g_x(t) d_t \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right|,$$

where

$$\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) = \int_0^t K_n \left(\frac{x}{b_n}, \frac{u}{b_n} \right) du$$

and

$$K_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)=\frac{n+1}{b_n}\sum_{k=0}^{n}P_{n,k}\left(\frac{x}{b_n}\right)P_{n,k}\left(\frac{t}{b_n}\right).$$

First we estimate $I_2(n,x)$. For $t\in\left[x-\frac{x}{\sqrt{n}},x+\frac{b_n-x}{\sqrt{n}}\right]$, we have

$$\begin{aligned}|I_2(n,x)|&=\left|\int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}}(g_x(t)-g_x(x))\,d_t\left(\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)\right)\right|\\&\le\int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}}|g_x(t)-g_x(x)|\left|d_t\left(\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)\right)\right|\\&\le\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}}(g_x)\le\frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}}(g_x).\end{aligned}\tag{3.3}$$

Next, we estimate $I_1(n,x)$. Using partial Lebesque-Stieltjes integration, we obtain

$$\begin{aligned}I_1(n,x)&=\int_0^{x-\frac{x}{\sqrt{n}}}g_x(t)d_t\left(\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)\right)\\&=g_x\left(x-\frac{x}{\sqrt{n}}\right)\lambda_n\left(\frac{x}{b_n},\frac{x-\frac{x}{\sqrt{n}}}{b_n}\right)\\&\quad-g_x(0)\lambda_n\left(\frac{x}{b_n},0\right)-\int_0^{x-\frac{x}{\sqrt{n}}}\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)d_t\left(g_x(t)\right).\end{aligned}$$

Since

$$\left|g_x\left(x-\frac{x}{\sqrt{n}}\right)\right| = \left|g_x\left(x-\frac{x}{\sqrt{n}}\right) - g_x(x)\right| \le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(g_x),$$

it follows that

$$|I_1(n,x)| \le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(g_x)\left|\lambda_n\left(\frac{x}{b_n},\frac{x-\frac{x}{\sqrt{n}}}{b_n}\right)\right| + \int_0^{x-\frac{x}{\sqrt{n}}}\lambda_n\left(\frac{x}{b_n},\frac{t}{b_n}\right)d_t\left(-\bigvee_t^x(g_x)\right).$$

From (2.3), it is clear that

$$\lambda_n\left(\frac{x}{b_n},\frac{x-\frac{x}{\sqrt{n}}}{b_n}\right) \le \frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2}\left\{\frac{2\,n\,x(b_n-x)+2b_n^2}{n^2}\right\}.$$

It follows that

$$\begin{aligned}|I_1(n,x)| &\le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(g_x)\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2}\left\{\frac{2nx(b_n-x)+2b_n^2}{n^2}\right\} \\ &\quad + \int_0^{x-\frac{x}{\sqrt{n}}}\frac{1}{(x-t)^2}\left\{\frac{2nx(b_n-x)+2b_n^2}{n^2}\right\}d_t\left(-\bigvee_t^x(g_x)\right) \\ &= \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(g_x)\frac{A_n(x)}{\left(\frac{x}{\sqrt{n}}\right)^2} + A_n(x)\int_0^{x-\frac{x}{\sqrt{n}}}\frac{1}{(x-t)^2}d_t\left(-\bigvee_t^x(g_x)\right).\end{aligned}$$

Furthermore, since

$$\int_0^{x-\frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2} dt \left(-\bigvee_t^x (g_x) \right)$$

$$= -\frac{1}{(x-t)^2} \bigvee_t^x (g_x) \Bigg|_0^{x-\frac{x}{\sqrt{n}}} + \int_0^{x-\frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3} \bigvee_t^x (g_x) dt$$

$$= -\frac{1}{(\frac{x}{\sqrt{n}})^2} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (g_x) + \frac{1}{x^2} \bigvee_0^x (g_x) + \int_0^{x-\frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3} \bigvee_t^x (g_x) dt.$$

Putting $t = x - \frac{x}{\sqrt{u}}$ in the last integral, we get

$$\int_0^{x-\frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3} \bigvee_t^x (g_x) dt = \frac{1}{x^2} \int_1^n \bigvee_{x-\frac{x}{\sqrt{u}}}^x (g_x) du = \frac{1}{x^2} \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x (g_x).$$

Consequently,

$$|I_1(n,x)| \leq \bigvee_{x-\frac{x}{\sqrt{n}}}^x (g_x) \frac{A_n(x)}{(\frac{x}{\sqrt{n}})^2}$$

$$+ A_n(x) \left\{ -\frac{1}{(\frac{x}{\sqrt{n}})^2} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (g_x) + \frac{1}{x^2} \bigvee_0^x (g_x) + \frac{1}{x^2} \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x (g_x) \right\}$$

$$= A_n(x)\left\{\frac{1}{x^2}\bigvee_0^x(g_x)+\frac{1}{x^2}\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x)\right\}$$

$$= \frac{A_n(x)}{x^2}\left\{\bigvee_0^{b_n}(g_x)+\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x)\right\}. \tag{3.4}$$

Using the similar method for estimating $|I_3(n,x)|$,we get

$$|I_3(n,x)| \leq \frac{A_n(x)}{(b_n-x)^2}\left\{\bigvee_x^{b_n}(g_x)+\sum_{k=1}^{n}\bigvee_{x}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x)\right\}$$

$$\leq \frac{A_n(x)}{(b_n-x)^2}\left\{\bigvee_0^{b_n}(g_x)+\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x)\right\}. \tag{3.5}$$

Hence from (3.3), (3.4) and (3.5), it follows that

$$|D_n(g_x;x)| \leq |I_1(n,x)|+|I_2(n,x)|+|I_3(n,x)|$$

$$\leq \frac{A_n(x)}{x^2}\left\{\bigvee_0^{b_n}(g_x)+\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x)\right\}$$

$$+\frac{A_n(x)}{(b_n-x)^2}\left\{\bigvee_0^{b_n}(g_x)+\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x)\right\}+\frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{n}}}(g_x).$$

Obviously,

$$\frac{1}{x^2}+\frac{1}{(b_n-x)^2}=\frac{b_n^2}{x^2(b_n-x)^2},$$

for $\frac{x}{b_n}\in[0,1]$ and

$$\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}(g_x)\le\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x).$$

Hence,

$$\begin{aligned}|D_n(g_x;x)|&\le\left(\frac{A_n(x)}{x^2}+\frac{A_n(x)}{(b_n-x)^2}\right)\left\{\bigvee_0^{b_n}(g_x)+\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x)\right\}\\&\quad+\frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x)\\&=\frac{A_n(x)\,b_n^2}{x^2(b_n-x)^2}\left\{\bigvee_0^{b_n}(g_x)+\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x)\right\}+\frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x)\\&=\frac{A_n(x)\,b_n^2}{x^2(b_n-x)^2}\left\{\bigvee_0^{b_n}(g_x)+\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x)\right\}+\frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}}(g_x).\end{aligned}$$

On the other hand, note that

$$\bigvee_{0}^{b_n}(g_x) \leq \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (g_x).$$

By (2.3), we have

$$|D_n(g_x;x)| \leq \frac{2\,A_n(x)\,b_n^2}{x^2(b_n-x)^2}\left\{\sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (g_x)\right\} + \frac{1}{n-1}\sum_{k=2}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (g_x).$$

Note that $\frac{1}{n-1} \leq \frac{A_n(x)\,b_n^2}{x^2(b_n-x)^2}$, for $n>1$, $\frac{x}{b_n} \in [0,1]$. Consequently

$$|D_n(g_x;x)| \leq \frac{3\,A_n(x)\,b_n^2}{x^2(b_n-x)^2}\left\{\sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (g_x)\right\}. \tag{3.6}$$

Now secondly, we can estimate $D_n(\operatorname{sgn}(t-x);x)$. If we apply operator D_n to the signum function, we get

$$\begin{aligned}
&D_n(\operatorname{sgn}(t-x);x)\\
&= \frac{n+1}{b_n}\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right)\left[\int_x^{b_n} P_{n,k}\left(\frac{t}{b_n}\right)dt - \int_0^x P_{n,k}\left(\frac{t}{b_n}\right)dt\right]\\
&= \frac{n+1}{b_n}\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right)\left[\int_0^{b_n} P_{n,k}\left(\frac{t}{b_n}\right)dt - 2\int_0^x P_{n,k}\left(\frac{t}{b_n}\right)dt\right]
\end{aligned}$$

using (2.1), we have

$$D_n(\operatorname{sgn}(t-x);x) = 1 - 2\frac{n+1}{b_n}\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right)\int_0^x P_{n,k}\left(\frac{t}{b_n}\right)dt. \tag{3.7}$$

Now, we differentiate both side of the following equality

$$J_{n+1,k+1}\left(\frac{x}{b_n}\right) = \sum_{j=k+1}^{n+1} P_{n+1,j}\left(\frac{x}{b_n}\right).$$

For $k = 0, 1, 2, \dots, n$ we get,

$$\begin{aligned}\frac{d}{dx}J_{n+1,k+1}\left(\frac{x}{b_n}\right) &= \frac{d}{dx}\sum_{j=k+1}^{n+1} P_{n+1,j}\left(\frac{x}{b_n}\right)\\ &= \frac{d}{dx}P_{n+1,k+1}\left(\frac{x}{b_n}\right) + \frac{d}{dx}P_{n+1,k+2}\left(\frac{x}{b_n}\right)\\ &\quad + \cdots + \frac{d}{dx}P_{n+1,n+1}\left(\frac{x}{b_n}\right)\end{aligned}$$

$$\begin{aligned}&\frac{d}{dx}J_{n+1,k+1}\left(\frac{x}{b_n}\right)\\ &= \frac{(n+1)}{b_n}\left\{\left[P_{n,k}\left(\frac{x}{b_n}\right) - P_{n,k+1}\left(\frac{x}{b_n}\right)\right] + \left[P_{n,k+1}\left(\frac{x}{b_n}\right) - P_{n,k+2}\left(\frac{x}{b_n}\right)\right]\right.\\ &\quad\left. + \cdots + \left[P_{n,n-1}\left(\frac{x}{b_n}\right) - P_{n,n}\left(\frac{x}{b_n}\right)\right] + \left[P_{n,n}\left(\frac{x}{b_n}\right)\right]\right\}\end{aligned}$$

$$= \frac{(n+1)}{b_n} \sum_{j=k+1}^{n+1} \left[P_{n,j-1}\left(\frac{x}{b_n}\right) - P_{n,j}\left(\frac{x}{b_n}\right) \right] = \frac{(n+1)}{b_n} P_{n,k}\left(\frac{x}{b_n}\right)$$

and $J_{n+1,k+1}(0) = 0$. Taking the integral from zero to x, we have

$$\frac{(n+1)}{b_n} \int_0^x P_{n,k}\left(\frac{t}{b_n}\right) dt = J_{n+1,k+1}\left(\frac{x}{b_n}\right)$$

and therefore from (2.4)

$$\begin{aligned} J_{n+1,k+1}\left(\frac{x}{b_n}\right) &= \sum_{j=k+1}^{n+1} P_{n+1,j}\left(\frac{x}{b_n}\right) \\ &= \sum_{j=0}^{n+1} P_{n+1,j}\left(\frac{x}{b_n}\right) - \sum_{j=0}^{k} P_{n+1,j}\left(\frac{x}{b_n}\right) \\ &= 1 - \sum_{j=0}^{k} P_{n+1,j}\left(\frac{x}{b_n}\right). \end{aligned}$$

Hence

$$\frac{(n+1)}{b_n} \int_0^x P_{n,k}\left(\frac{t}{b_n}\right) dt = 1 - \sum_{j=0}^{k} P_{n+1,j}\left(\frac{x}{b_n}\right).$$

From (3.7), we get

$$D_n(\operatorname{sgn}(t-x); x) = 1 - 2\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) \left[1 - \sum_{j=0}^{k} P_{n+1,j}\left(\frac{x}{b_n}\right) \right]$$

$$= 1 - 2\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right) + 2\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right)\sum_{j=0}^{k} P_{n+1,j}\left(\frac{x}{b_n}\right)$$

$$= -1 + 2\sum_{k=0}^{n} P_{n,k}\left(\frac{x}{b_n}\right)\sum_{j=0}^{k} P_{n+1,j}\left(\frac{x}{b_n}\right).$$

Set

$$Q^{(2)}_{n+1,j}\left(\frac{x}{b_n}\right) = J^2_{n+1,j}\left(\frac{x}{b_n}\right) - J^2_{n+1,j+1}\left(\frac{x}{b_n}\right).$$

Also note that

$$\sum_{k=0}^{n}\sum_{j=0}^{k} * = \sum_{j=0}^{n}\sum_{k=j}^{n} *,$$

$$\sum_{k=j}^{n+1} Q^{(2)}_{n+1,k}\left(\frac{x}{b_n}\right) = J^2_{n+1,j}\left(\frac{x}{b_n}\right) \quad \text{and} \quad J_{n,n+1}\left(\frac{x}{b_n}\right) = 0,$$

we have

$$D_n(\operatorname{sgn}(t-x);x) = -1 + 2\sum_{j=0}^{n} P_{n+1,j}\left(\frac{x}{b_n}\right)\sum_{k=j}^{n} P_{n,k}\left(\frac{x}{b_n}\right)$$

$$= -1 + 2\sum_{j=0}^{n} P_{n+1,j}\left(\frac{x}{b_n}\right) J_{n,j}\left(\frac{x}{b_n}\right)$$

$$= -1 + 2\sum_{j=0}^{n+1} P_{n+1,j}\left(\frac{x}{b_n}\right) J_{n,j}\left(\frac{x}{b_n}\right)$$

$$= 2\sum_{j=0}^{n+1} P_{n+1,j}\left(\frac{x}{b_n}\right) J_{n,j}\left(\frac{x}{b_n}\right) - 1.$$

Since $\sum_{j=0}^{n+1} Q_{n+1,j}^{(2)}\left(\frac{x}{b_n}\right) = 1$, thus

$$D_n(\operatorname{sgn}(t-x);x) = 2\sum_{j=0}^{n+1} P_{n+1,j}\left(\frac{x}{b_n}\right) J_{n,j}\left(\frac{x}{b_n}\right) - \sum_{j=0}^{n+1} Q_{n+1,j}^{(2)}\left(\frac{x}{b_n}\right).$$

By the mean value theorem, we have

$$\begin{aligned} Q_{n+1,j}^{(2)}\left(\frac{x}{b_n}\right) &= J_{n+1,j}^2\left(\frac{x}{b_n}\right) - J_{n+1,j+1}^2\left(\frac{x}{b_n}\right) \\ &= 2P_{n+1,j}\left(\frac{x}{b_n}\right) \gamma_{n,j}\left(\frac{x}{b_n}\right) \end{aligned}$$

where

$$J_{n+1,j+1}\left(\frac{x}{b_n}\right) < \gamma_{n,j}\left(\frac{x}{b_n}\right) < J_{n+1,j}\left(\frac{x}{b_n}\right).$$

Hence it follows from (2.5) that

$$\begin{aligned} |D_n(\operatorname{sgn}(t-x);x)| &= \left| 2\sum_{j=0}^{n+1} P_{n+1,j}\left(\frac{x}{b_n}\right)\left(J_{n,j}\left(\frac{x}{b_n}\right) - \gamma_{n,j}\left(\frac{x}{b_n}\right)\right)\right| \\ &\le 2\sum_{j=0}^{n+1} P_{n+1,j}\left(\frac{x}{b_n}\right)\left| J_{n,j}\left(\frac{x}{b_n}\right) - \gamma_{n,j}\left(\frac{x}{b_n}\right)\right| \end{aligned}$$

$$\left|J_{n,j}\left(\frac{x}{b_n}\right)-J_{n+1,j+1}\left(\frac{x}{b_n}\right)\left(\frac{x}{b_n}\right)\right|$$
$$=\left|J_{n,j}\left(\frac{x}{b_n}\right)-\gamma_{n,j}\left(\frac{x}{b_n}\right)+\gamma_{n,j}\left(\frac{x}{b_n}\right)-J_{n+1,j+1}\left(\frac{x}{b_n}\right)\right|,$$

since $\gamma_{n,j}\left(\frac{x}{b_n}\right)-J_{n+1,j+1}\left(\frac{x}{b_n}\right)>0$,then we have

$$\left|J_{n,j}\left(\frac{x}{b_n}\right)-\gamma_{n,j}\left(\frac{x}{b_n}\right)\right|\leq\left|J_{n,j}\left(\frac{x}{b_n}\right)-J_{n+1,j+1}\left(\frac{x}{b_n}\right)\right|.$$

Hence

$$\begin{aligned}|D_n(\operatorname{sgn}(t-x);x)|&\leq 2\sum_{j=0}^{n+1}P_{n+1,j}\left(\frac{x}{b_n}\right)\left|J_{n,j}\left(\frac{x}{b_n}\right)-J_{n+1,j+1}\left(\frac{x}{b_n}\right)\right|\\&\leq 2\sum_{j=0}^{n+1}P_{n+1,j}\left(\frac{x}{b_n}\right)\frac{2}{\sqrt{n\frac{x}{b_n}(1-\frac{x}{b_n})}}\\&=\frac{4}{\sqrt{n\frac{x}{b_n}(1-\frac{x}{b_n})}}.\end{aligned}\tag{3.8}$$

Combining (3.6) and (3.8) we get (1.1). Thus, the proof of the theorem is completed. □

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