

SOME APPLICATIONS OF THE GENERALIZED BERNARDI–LIBERA–LIVINGSTON INTEGRAL OPERATOR ON UNIVALENT FUNCTIONS

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Abstract: In this paper by making use of the generalized Bernardi–Libera–Livingston integral operator we introduce and study some new subclasses of univalent functions. Also we investigate the relations between those classes and the classes which are studied by Jin–Lin Liu.

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[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 1 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
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mathematics

issn: 1443-5756

Contents

1 Introduction

3

2 Main Results

7



Bernardi-Libera-Livingston

Integral Operator

M. Eshaghi Gordji,

D. Alimohammadi and A. Ebadian

vol. 10, iss. 4, art. 100, 2009

Title Page

Contents



Page 2 of 12

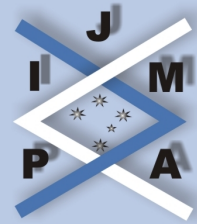
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 3 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

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1. Introduction

Let A be the class of functions of the form, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $U = \{z : |z| < 1\}$. Also, let S denote the subclass of A consisting of all univalent functions in U . Suppose λ is a real number with $0 \leq \lambda < 1$. A function $f \in S$ is said to be starlike of order λ if and only if $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \lambda$, $z \in U$.

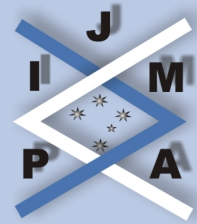
Also, $f \in S$ is said to be convex of order λ if and only if $\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \lambda$, $z \in U$. We denote by $S^*(\lambda)$, $C(\lambda)$ the classes of starlike and convex functions of order λ respectively. It is well known that $f \in C(\lambda)$ if and only if $zf' \in S^*(\lambda)$. If $f \in A$, then $f \in K(\beta, \lambda)$ if and only if there exists a function $g \in S^*(\lambda)$ such that $\operatorname{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > \beta$, $z \in U$, where $0 \leq \beta < 1$. These functions are called close-to-convex functions of order β type λ . A function $f \in A$ is called quasi-convex of order β type λ if there exists a function $g \in C(\lambda)$ such that $\operatorname{Re} \left\{ \frac{(zf'(z))'}{g'(z)} \right\} > \beta$. We denote this class by $K^*(\beta, \lambda)$ [10]. It is easy to see that $f \in K^*(\beta, \gamma)$ if and only if $zf' \in K(\beta, \gamma)$ [9]. For $f \in A$ if for some $\lambda (0 \leq \lambda < 1)$ and $\eta (0 < \eta \leq 1)$ we have

$$(1.1) \quad \left| \arg \left(\frac{zf'(z)}{f(z)} - \lambda \right) \right| < \frac{\pi}{2} \eta, \quad (z \in U),$$

then $f(z)$ is said to be strongly starlike of order η and type λ in U and we denote this class by $S^*(\eta, \lambda)$. If $f \in A$ satisfies the condition

$$(1.2) \quad \left| \arg \left(1 + \frac{zf''(z)}{f'(z)} - \lambda \right) \right| < \frac{\pi}{2} \eta, \quad (z \in U)$$

for some λ and η as above, then we say that $f(z)$ is strongly convex of order η and type λ in U and we denote this class by $C(\eta, \lambda)$. Clearly $f \in C(\eta, \lambda)$ if and only if $zf' \in S^*(\eta, \lambda)$, and in particular, we have $S^*(1, \lambda) = S^*(\lambda)$ and $C(1, \lambda) = C(\lambda)$.



Title Page

Contents



Page 4 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

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For $c > -1$ and $f \in A$ the generalized Bernardi–Libera–Livingston integral operator $L_c f$ is defined as follows

$$(1.3) \quad L_c f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt.$$

This operator for $c \in \mathbb{N} = \{1, 2, 3, \dots\}$ was studied by Bernardi [1] and for $c = 1$ by Libera [4] (see also [8]). The classes $ST_c(\eta, \lambda)$ and $CV_c(\eta, \lambda)$ were introduced by Liu [7], where

$$ST_c(\eta, \lambda) = \left\{ f \in A : L_c f \in S^*(\eta, \lambda), \frac{z(L_c f(z))'}{L_c f(z)} \neq \lambda, z \in U \right\},$$
$$CV_c(\eta, \lambda) = \left\{ f \in A : L_c f \in C(\eta, \lambda), \frac{(z(L_c f(z)))'}{(L_c f(z))'} \neq \lambda, z \in U \right\}.$$

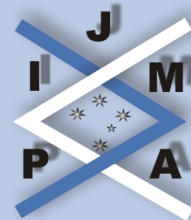
Now by making use of the operator given by (1.3) we introduce the following classes.

$$S_c^*(\lambda) = \{f \in A : L_c f \in S^*(\lambda)\},$$
$$C_c(\lambda) = \{f \in A : L_c f \in C(\lambda)\}.$$

Obviously $f \in CV_c(\eta, \lambda)$ if and only if $z f' \in ST_c(\eta, \lambda)$. J. L. Liu [5] and [6] introduced and similarly investigated the classes $S_\sigma^*(\lambda)$, $C_\sigma(\lambda)$, $K_\sigma(\beta, \lambda)$, $K_\sigma^*(\beta, \lambda)$, $ST_\sigma(\eta, \lambda)$, $CV_\sigma(\eta, \lambda)$ by making use of the integral operator $I^\sigma f$ given by

$$(1.4) \quad I^\sigma f(z) = \frac{2^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t}\right)^{\sigma-1} f(t) dt, \quad \sigma > 0, f \in A.$$

The operator I^σ was introduced by Jung, Kim and Srivastava [2] and then investigated by Uraleghaddi and Somanatha [13], Li [3] and Liu [5]. For the integral opera-



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 5 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

tors given by (1.3) and (1.4) we have verified following relationships.

$$(1.5) \quad I^\sigma f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1} \right)^\sigma a_n z^n,$$

$$(1.6) \quad L_c f(z) = z + \sum_{n=2}^{\infty} \frac{c+1}{n+c} a_n z^n,$$

$$(1.7) \quad z(I^\sigma L_c f(z))'^\sigma f(z) - cI^\sigma L_c f(z),$$

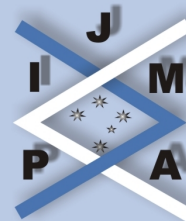
$$(1.8) \quad z(L_c I^\sigma f(z))'^\sigma f(z) - cL_c I^\sigma f(z).$$

It follows from (1.5) that one can define the operator I^σ for any real number σ . In this paper we investigate the properties of the classes $S_c^*(\lambda)$, $C_c(\lambda)$, $K_c(\beta, \lambda)$, $K_c^*(\beta, \lambda)$, $ST_c(\eta, \lambda)$ and $CV_c(\eta, \lambda)$. We also study the relations between these classes by the classes which are introduced by Liu in [5] and [6]. For our purposes we need the following lemmas.

Lemma 1.1 ([9]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and let $\psi(u, v)$ be a complex function $\psi : D \subset \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$. Suppose that ψ satisfies the following conditions*

- (i) $\psi(u, v)$ is continuous in D ;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\psi(1, 0)\} > 0$;
- (iii) $\operatorname{Re}\{\psi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ with $v_1 \leq -\frac{1+u_2^2}{2}$.
 Let $p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$ be analytic in U so that $(p(z), zp'(z)) \in D$ for all $z \in U$. If $\operatorname{Re}\{\psi(p(z), zp'(z))\} > 0$, $z \in U$ then $\operatorname{Re}\{p(z)\} > 0$, $z \in U$.

Lemma 1.2 ([11]). *Let the function $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in U and $p(z) \neq 0$, $z \in U$. If there exists a point $z_0 \in U$ such that $|\arg(p(z))| < \frac{\pi}{2}\eta$ for $|z| < |z_0|$ and $\arg p(z_0) = \frac{\pi}{2}\eta$ where $0 < \eta \leq 1$, then $\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta$ and $k \geq \frac{1}{2}(r + \frac{1}{r})$ when $\arg p(z_0) = \frac{\pi}{2}\eta$, Also, $k \leq \frac{-1}{2}(r + \frac{1}{r})$ when $\arg p(z_0) = \frac{-\pi}{2}\eta$, and $p(z_0)^{1/\eta} = \pm ir$ ($r > 0$).*



Bernardi-Libera-Livingston

Integral Operator

M. Eshaghi Gordji,

D. Alimohammadi and A. Ebadian

vol. 10, iss. 4, art. 100, 2009

Title Page

Contents



Page 6 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 7 of 12

Go Back

Full Screen

Close

2. Main Results

In this section we obtain some inclusion theorems by following the method of proof adopted in [12].

Theorem 2.1.

- (i) For $f \in A$ if $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \frac{z(L_c f(z))'}{L_c f(z)} \right\} > 0$ and $\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}$ is an analytic function, then $S_c^*(\lambda) \subset S_{c+1}^*(\lambda)$.
- (ii) Let $c > -\lambda$. For $f \in A$ if $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} \right\} > 0$ and $\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}$ is an analytic function, then $S_{c+1}^*(\lambda) \subset S_c^*(\lambda)$.

Proof. (i) Suppose that $f \in S_c^*(\lambda)$ and set

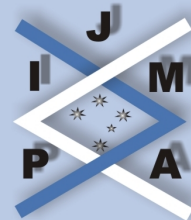
$$(2.1) \quad \frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} - \lambda = (1 - \lambda)p(z),$$

where $p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$. An easy calculation shows that

$$(2.2) \quad \frac{\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} \left[2 + c + \frac{z(L_{c+1}f(z))''}{(L_{c+1}f(z))'} \right]}{\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} + c + 1} = \frac{zf'(z)}{f(z)}.$$

By setting $H(z) = \frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}$ we have

$$(2.3) \quad 1 + \frac{z(L_{c+1}f(z))''}{(L_{c+1}f(z))'} = H(z) + \frac{zH'(z)}{H(z)}.$$



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 8 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

By making use of (2.3) in (2.2), since $H(z) = \lambda + (1 - \lambda)p(z)$, we obtain

$$(2.4) \quad (1 - \lambda)p(z) + \frac{(1 - \lambda)zp'(z)}{\lambda + c + 1 + (1 - \lambda)p(z)} = \frac{zf'(z)}{f(z)} - \lambda.$$

If we consider

$$\psi(u, v) = (1 - \lambda)u + \frac{(1 - \lambda)v}{\lambda + c + 1 + (1 - \lambda)u},$$

then $\psi(u, v)$ is a continuous function in $D = \{\mathbb{C} - \frac{\lambda+c+1}{\lambda-1}\} \times \mathbb{C}$ and $(1, 0) \in D$.

Also, $\psi(1, 0) > 0$ and for all $(iu_2, v_1) \in D$ with $v_1 \leq -\frac{1+u_2^2}{2}$ we have

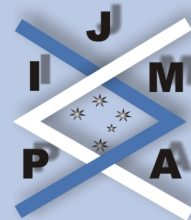
$$\begin{aligned} \operatorname{Re} \psi(iu_2, v_1) &= \frac{(1 - \lambda)(\lambda + c + 1)v_1}{(1 - \lambda)^2 u_2^2 + (\lambda + c + 1)^2} \\ &\leq \frac{-(1 - \lambda)(\lambda + c + 1)(1 + u_2^2)}{2[(1 - \lambda)^2 u_2^2 + (\lambda + c + 1)^2]} < 0. \end{aligned}$$

Therefore the function $\psi(u, v)$ satisfies the conditions of Lemma 1.1 and since in view of the assumption, by considering (2.4), we have $\operatorname{Re}\{\psi(p(z), zp'(z))\} > 0$, Lemma 1.1 implies that $\operatorname{Re}p(z) > 0, z \in U$ and this completes the proof of (i).

(ii) For proving this part of the theorem, we use the same method and a easily verified formula similar to (2.2). By replacing $c + 1$ with c we get the desired result. \square

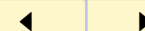
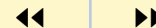
Theorem 2.2.

(i) For $f \in A$ if $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \frac{z(L_c f(z))'}{L_c f(z)} \right\} > 0$ and $\frac{z(L_{c+1} f(z))'}{L_{c+1} f(z)}$ is an analytic function, then $C_c(\lambda) \subset C_{c+1}(\lambda)$.



Title Page

Contents



Page 9 of 12

Go Back

Full Screen

Close

(ii) Let $c > -\lambda$. For $f \in A$ if $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)} \right\} > 0$ and $\frac{z(L_{c+1}f(z))'}{L_{c+1}f(z)}$ is an analytic function, then $C_{c+1}(\lambda) \subset C_c(\lambda)$.

Proof. (i) In view of part (i) of Theorem 2.1 we can write

$$\begin{aligned}
 f \in C_c(\lambda) &\Leftrightarrow L_c f \in C(\lambda) \Leftrightarrow z(L_c f)'(\lambda) \Leftrightarrow L_c z f'(\lambda) \Leftrightarrow z f_c'(\lambda) \Rightarrow z f_{c+1}'(\lambda) \\
 &\Leftrightarrow L_{c+1} z f'(\lambda) \Leftrightarrow z(L_{c+1} f)'(\lambda) \Leftrightarrow L_{c+1} f \in C(\lambda) \Leftrightarrow f \in C_{c+1}(\lambda).
 \end{aligned}$$

Part (ii) of the theorem can be proved in a similar manner. \square

Theorem 2.3. If $c \geq -\lambda$ and $\frac{zf'(z)}{f(z)}$ is an analytic function, then $f \in S^*(\lambda)$ implies $f \in S_c^*(\lambda)$.

Proof. By differentiating logarithmically both sides of (1.3) with respect to z we obtain

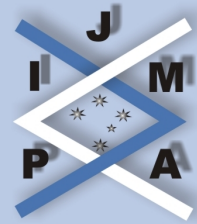
$$(2.5) \quad \frac{z(L_c f(z))'}{L_c f(z)} + c = \frac{(c+1)f(z)}{L_c f(z)}.$$

Again differentiating logarithmically both sides of (2.5) we have

$$(2.6) \quad p(z) + \frac{zp'(z)}{c + \lambda + p(z)} = \frac{zf'(z)}{f(z)} - \lambda,$$

where $p(z) = \frac{z(L_c f(z))'}{L_c f(z)} - \lambda$. Let us consider $\psi(u, v) = u + \frac{v}{u+c+\lambda}$. Then ψ is a continuous function in $D = \{\mathbb{C} - (-c - \lambda)\} \times \mathbb{C}$, $(1, 0) \in D$ and $\operatorname{Re} \psi(1, 0) > 0$. If $(iu_2, v_1) \in D$ with $v_1 \leq -\frac{1+u_2^2}{2}$, then

$$\operatorname{Re} \psi(iu_2, v_1) = \frac{v_1(c + \lambda)}{u_2^2 + (c + \lambda)^2} \leq 0.$$



Since $f \in S^*(\lambda)$, then (2.6) gives

$$\operatorname{Re}(\psi(p(z), zp'(z))) = \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \lambda \right\} > 0.$$

Therefore Lemma 1.1 concludes that $\operatorname{Re}\{p(z)\} > 0$ and this completes the proof. \square

Corollary 2.4. *If $c \geq \lambda$ and $\frac{zf'(z)}{f(z)}$ is an analytic function, then $f \in C(\lambda)$ implies $f \in C_c(\lambda)$.*

Proof. We have

$$\begin{aligned} f \in C(\lambda) &\Leftrightarrow zf'^*(\lambda) \Lambda zf'_c(\lambda) \Leftrightarrow L_c zf' \in S^*(\lambda) \\ &\Leftrightarrow z(L_c f)^{l*}(\lambda) \Leftrightarrow L_c f \in C(\lambda) \Leftrightarrow f \in C_c(\lambda). \end{aligned}$$

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Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 10 of 12

Go Back

Full Screen

Close

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Bernardi-Libera-Livingston

Integral Operator

M. Eshaghi Gordji,

D. Alimohammadi and A. Ebadian

vol. 10, iss. 4, art. 100, 2009

[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 11 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
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mathematics

issn: 1443-5756



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**Bernardi-Libera-Livingston
Integral Operator**
M. Eshaghi Gordji,
D. Alimohammadi and A. Ebadian
vol. 10, iss. 4, art. 100, 2009

[Title Page](#)

[Contents](#)



Page 12 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
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mathematics

issn: 1443-5756