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ON A PROBLEM FOR ISOMETRIC MAPPINGS OF $\mathbb{S}^n$ POSED BY TH. M. RASSIAS

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ABSTRACT. In this article we prove the problem on isometric mappings of $\mathbb{S}^n$ posed by Th. M. Rassias. We prove that any map $f : \mathbb{S}^n \to \mathbb{S}^p$, $p \geq n > 1$, preserving two angles θ and $m\theta$ ($m\theta < \pi$, $m > 1$) is an isometry. With the assumption of continuity we prove that any map $f : \mathbb{S}^n \to \mathbb{S}^n$ preserving an irrational angle is an isometry.

Key words and phrases: $n-$sphere, isometry.

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1. INTRODUCTION

Given two metric spaces X and Y what are the minimum requirements for a map $f : X \to Y$ to be an isometry? There is a rich literature in this direction when the domain and range of the mapping have the same dimension and the map preserves only one distance. In fact it has been shown that a map with this property is indeed an isometry. But if the co-domain has a dimension larger than the domain then no satisfactory results are available except some partial results with more assumptions on f.

Let $\mathbb{S}^n$ denote the unit n-sphere in $\mathbb{R}^{n+1}$. In this article we are interested in a map $f : \mathbb{S}^n \to \mathbb{S}^p$, $p \geq n > 1$, that preserves two distances involving an angle. This problem was posed by T. M. Rassias in [8]. The general proof for $\mathbb{R}^n$ does not work in this setup as the proof uses the properties of an equilateral triangle and a rhombus in a plane. In this paper we give a proof for the problem posed by T. M. Rassias. Assuming the continuity of f, we prove that if it preserves one irrational angular distance then it is an isometry. We shall follow the notations $A, B, C, \ldots$ for points in a domain and $A', B', C', \ldots$ for the corresponding images under f. Also we shall use the notation π for the angle $180°$.

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2. RESULTS

Theorem 2.1. *Any isometry $f : \mathbb{S}^n \to \mathbb{S}^n$ is a composition of rotations and a reflection.*

Proof. Consider first $n = 1$. It is clear that f is one-one, continuous and open. Therefore it is trivial to note that f is onto. Since the antipodal points are mapped to the antipodal points, it is possible give a rotation with suitable angle such that $(0, 1)$ and $(0, -1)$ are mapped to themselves. Call this rotation R_1. Now if $(1, 0)$ and $(-1, 0)$ are mapped to themselves then we are done, otherwise give a reflection about the line joining $(0, 1)$ and $(0, -1)$. Call the reflection R_2. Now these four points uniquely determine any point on the circle via angles. Therefore the resulting map is an identity and hence $f = R_1^{-1} \circ R_2^{-1}$.

Now consider the case $n = 2$. Again f is one-one, continuous and onto. First, we give two successive rotations, say R_1 and R_2, to map the points $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, $(0, 0, -1)$ to themselves. Now if $(1, 0, 0), (-1, 0, 0)$ are mapped to themselves then we are done, otherwise give a reflection R_3 about the YZ plane to map the points to themselves. After these operations f has been transformed to an identity map and hence $f = R_1^{-1} \circ R_2^{-1} \circ R_3^{-1}$.

Similarly, the above may be applied for any $n \geq 3$. $\square$

A map $f : \mathbb{S}^1 \to \mathbb{S}^1$ that preserves two angles $(\theta, 2\theta)$ need not be an isometry. Let f be defined as follows (see Figure 2.1)

$$f(x) = \begin{cases} x & \text{if } x \neq Ai \\ A(i-1) & \text{if } x = Ai, i \geq 2 \\ A6 & \text{if } x = A1 \end{cases}$$

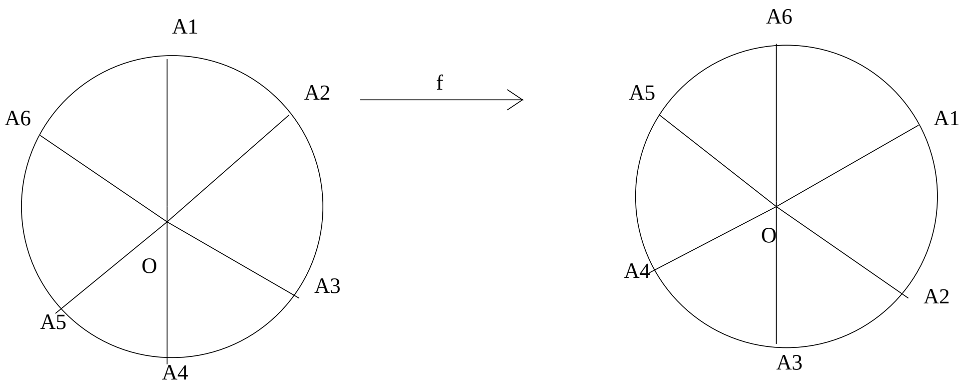

Figure 2.1:

where the angle between the consecutive points Ai's is $\pi/3$. It is easy to check that f preserves the distances $(\pi/3, 2\pi/3)$ but it is not an isometry.

Theorem 2.2. *Let $f : \mathbb{S}^2 \to \mathbb{S}^2$, be a function that preserves angles $(\theta, m\theta)$ where $m\theta < \pi$ and m is a positive integer greater than 1. Then f is an isometry i.e. f preserves all angles.*

Proof. First consider $m = 2$. For simplicity of geometry we will consider $\theta \leq \frac{\pi}{4}$. First we show that the points at a distance of θ on a great circle are mapped on a great circle. Consider Figure 2.2 below.

Let A, B and C be three points on a great circle such that $\angle AOB = \theta$ and $\angle AOC = 2\theta$. Let $f(A) = A', f(B) = B'$ and $f(C) = C'$. Consider the great circle through A' and B'. Since

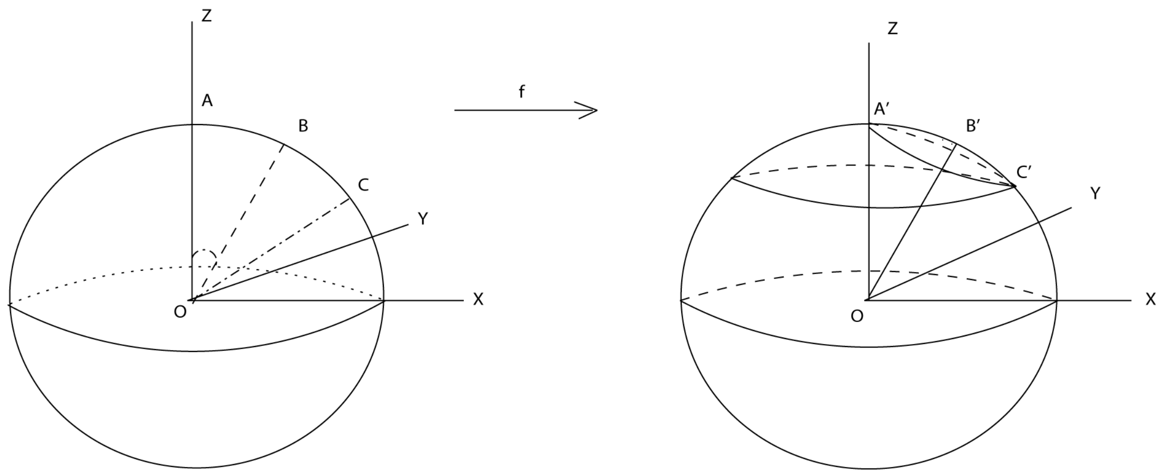

Figure 2.2:

C' has to maintain an angle of θ with B' the possible positions for C' are on the smaller circle which makes angle of θ with B'. Again w.r.t. A' the possibilities of C' are on the lower circle which makes an angle of 2θ with A'. But these two circles intersect at only one point on the great circle. Hence C' will be on the great circle. Hence the points are mapped on the great circle and so any angular distance $p\theta$ is preserved for any integer $p \geq 1$.

Now we consider the following spherical triangle $\triangle ACE$ (See Figure 2.3).

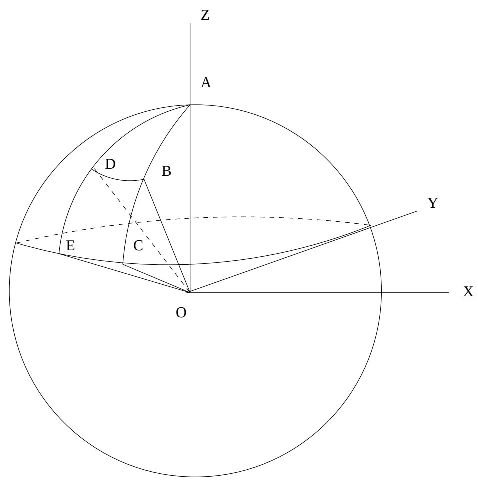

Figure 2.3:

where A, B, C and A, D, E lie on two great circles. Also

$$\angle AOC = 2\theta = \angle AOE,$$
$$\angle AOB = \angle BOC = \angle COE = \angle DOE = \angle AOD = \theta.$$

From the above statement and the assumption on f it follows that the angle $\angle BOD$ is preserved under f. Similarly, by taking $\angle COE = 2\theta$ we note that $2\angle BOD$ is preserved under f. So from above, f preserves $p\angle BOD$ for any positive integer p. Let $\angle BOD = \theta_1$. Therefore $\theta_1 < \theta$.

Now repeating the same argument we will get a decreasing sequence of angles $\{\theta_n\}$ such that

$$\theta_{n+1} < \theta_n \ \ \forall n,$$
$$f \text{ preserves } p\theta_n \quad \forall n, p \geq 1.$$

From our construction it is trivial to note that $\lim_{n\to\infty} \theta_n = 0$.

Let A, B be two arbitary points on $\mathbb{S}^2$. Consider the great circle passing through A, B. Now we can choose a sequence of points $\{C_n\}$ on the great circle such that $\angle AOC_n = p(n)\theta_n$ and $p(n)\theta_n \to \angle AOB$ with $C_n \to B$. Since $C_n \to B$ we can always choose a point on the sphere such that C_n and B will be on a circle of small angular radius, say $\theta_{r(n)}$, about that point. Again θ_r is preserved by f.

So C'_n and B' will be on a circle in $\mathbb{S}^2$ that makes a solid angle of order $2\theta_{r(n)}$ at the center of $\mathbb{S}^2$. Thus as $n \to \infty$, $\angle C_nOB \to 0$. Hence $\angle A'OB' = \angle AOB$. This completes the proof for $m = 2$.

Now consider $m = 3$. Let A, B, D be three points on a great circle such that $\angle AOB = \theta$ and $\angle AOD = 3\theta$. Consider the great circle in the co-domain passing through A' and D'.

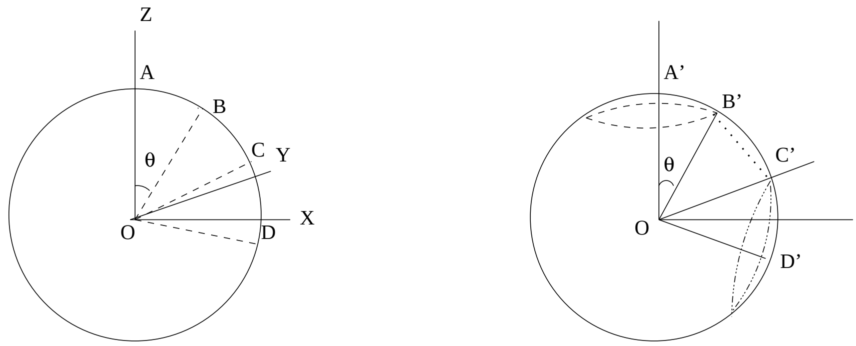

Figure 2.4:

Let C be a point on the great circle in the domain such that $\angle AOC = 2\theta$. From the figure the options for B' are on the circle above that makes an angle θ with A' and options for C' are on the lower circle that makes an angle θ with D'. Hence it is easy to see that the only way that f can preserve an angle of θ is if B' and C' is on the great circle through A', D'. Therefore f preserves an angle of 2θ and so by similar arguments as above f preserves all the angles.

We can use similar arguments as above to prove the case for any $m > 3$. $\square$

Remark 1. Note that the above proof holds with suitable modifications if we replace the co-domain $\mathbb{S}^2$ by $\mathbb{S}^p$, $p \geq 2$. Let f preserve the angles θ and 2θ. If we fix the image of A, B in the X_1X_p plane with A as the north pole and $B = (\sin\theta, 0, ..., 0, \cos\theta)$ where $\angle AOB = \angle BOC = \theta$ and $\angle AOC = 2\theta$ as above, then a possible position for C' would be the intersection of the following $(p-1)$-spheres,

$$x_1^2 + x_2^2 + \cdots + x_{p-1}^2 = \sin^2 2\theta, \quad x_p = \cos 2\theta$$
$$\text{and} \quad (x_1 - \sin\theta\cos\theta)^2 + x_2^2 + \cdots + (x_p - \cos^2\theta)^2 = \sin^2\theta.$$

A simple calculation shows that $x_1 = \sin 2\theta$ and hence $C' = (\sin 2\theta, 0, \ldots, 0, \cos 2\theta)$, i.e., C' lies on the X_1X_p plane. Therefore f preserves any angle of the form $m\theta$ for positive integers $m \geq 1$ and points at a distance θ on the great circle are mapped to some great circle. We consider a similar spherical triangle as in the proof above to obtain a decreasing sequence of angles. A similar attribute can be deduced if f preserves angles of the form $(\theta, m\theta)$ for some positive integer $m > 1$.

Now we consider $f : \mathbb{S}^n \to \mathbb{S}^p, p \geq n > 1$ that preserves angles $(\theta, m\theta)$ for some positive integer $m > 1$. Using the above argument one can show that points at a distance θ on some great circle are mapped to some great circle. Instead of considering a 2-dimensional spherical triangle we have to consider spherical simplexes of $(n-1)$-dimension with sides of length θ and 2θ (side of length θ means that the side makes an angle of θ at the center). By similar arguments as those above, we will obtain a decreasing sequence and complete the proof along the same lines.

Thus as a generalization we have the following theorem.

Theorem 2.3. *Let $f : \mathbb{S}^n \to \mathbb{S}^p$, $p \geq n > 1$, be a continuous mapping that preserves angle $(\theta, m\theta)$ where $m > 1$ and $m\theta < \pi$. Then f is an isometry*

Now it is quite reasonable to ask "would f be an isometry if f preserves one angle?". We further assume that f is continuous. Note that it is possible to give continuous $f : \mathbb{S}^1 \to \mathbb{S}^1$ that preserve a distance of $\pi/3$ but not isometry. For example, we can map $f(Ai) = Ai$ (see Figure 2.1) and change the speed of the arc$\{A1A2\}$ by mapping the first half of arc$\{A1A2\}$ into the first $\frac{3}{4}$ of arc$\{A1A2\}$ in the image and the second half of arc$\{A1A2\}$ into the next $\frac{1}{4}$ of arc$\{A1A2\}$. Similarly map for other arcs.

Theorem 2.4. *Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a mapping that preserves the angle θ. Let $\cos^{-1}\left(\frac{1}{m+\sec\theta}\right)$ be irrational for $0 \leq m \leq n-1$. Then f is an isometry.*

Proof. First consider $n = 1$. Let A, B be two points on the circle such that $\angle AOB = \phi$ (say) is irrational and $f(A) = f(B)$. Consider two points C, D on the circle in the anti-clockwise direction of $\{A, B\}$ such that $\angle COD = \phi$, $\angle AOC = \theta$ and $\angle BOD = \theta$ (Figure 2.5).

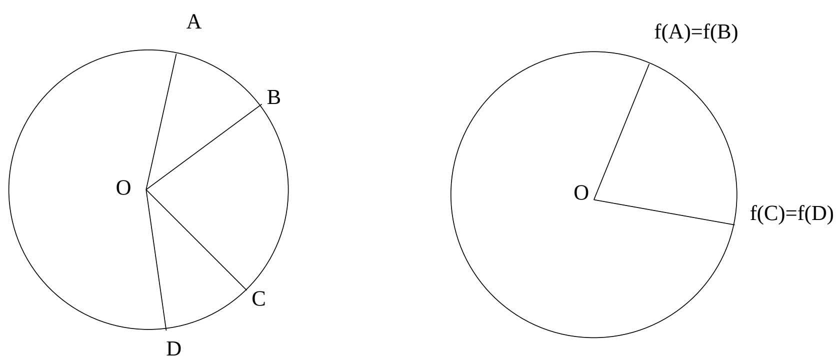

Figure 2.5:

Since $f(A) = f(B)$ then there are only two options for $f(C)$ in the image. Let us fix one of the possibilities as $f(C)$. Since f is continuous and preserves the angle θ, the image of arc$\{CD\}$ would behave as the image of arc$\{AB\}$. This will give $f(C) = f(D)$. Now consider the next tuple of two points in the anti-clockwise direction of $\{C, D\}$ that make an angle of θ with C, D respectively and the angle between the points in the tuple is ϕ. By the same argument as above, these two points will be mapped in the same point. If we continue with the same procedure as mentioned above we will get a dense set of tuples on the circle with the property that each tuple is mapped in the same point since θ is irrational. Also apply the same procedure in the clockwise direction. Therefore, by the continuity of f, any two points that make an angle of ϕ among themselves are mapped to the same point.

Now we start with two points A, B such that $\angle AOB = \phi$, then $f(A) = f(B) = p$ (say). Therefore, from above, the next point what makes an angle of ϕ with B is mapped to p. Since ϕ is irrational, by simply repeating the procedure we will obtain a dense set of points that will be mapped to p and thus $f(\mathbb{S}^1) = p$. This is a contradiction. Therefore $f(A) \neq f(B)$.

Let A_1, A_2, A_3 be points on circle such that $\angle A_1OA_2 = \angle A_2OA_3 = \theta$ and $\angle A_1OA_3 = 2\theta$. By the properties of f, A_3 has two image options. But the above argument says that $f(A_1) \neq f(A_3)$ and hence f preserves the angle 2θ. Consequently f preserves an angle of $m\theta$ for any positive integer m. And thus for θ irrational we will get a dense set of angles in $(0, \pi)$ that is preserved by f. Thus by continuity, f preserves all angles.

Next consider $n = 2$. Let A, B, C be three points on the sphere such that $\angle AOB = \angle BOC = \theta$ and $\angle AOC = 2\theta$. Also assume that A, B, C lies in a great circle.

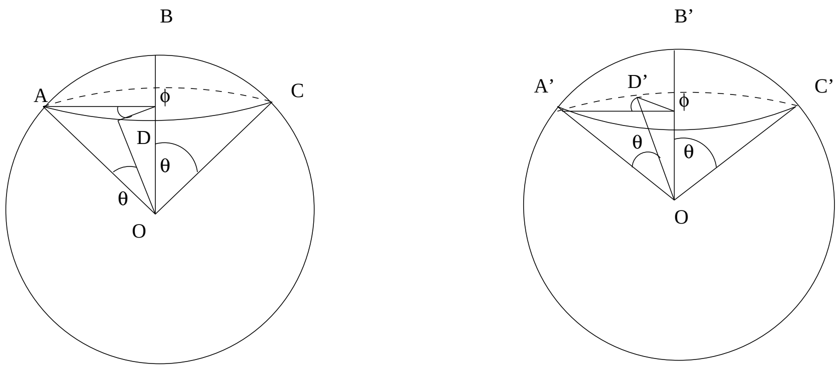

Figure 2.6:

The points that make an angle of θ with B lie on the circle ADC of radius $\sin\theta$. Under f this circle will go to a circle of the same radius, say $A'D'C'$, with $f(A) = A'$ and $f(B) = B'$. Note that $f(C)$ will lie on this circle. Let A, D be points on the circle ADC that make an angle of θ with the center of a sphere and angle, say ϕ, with center of the circle ADC. Then

$$2(1 - \cos\theta) = \bar{AD}^2 = 2\sin^2\theta(1 - \cos\phi)$$
$$\implies \phi = \cos^{-1}\left(\frac{1}{1 + \sec\theta}\right)$$

Therefore any two points on the circle ADC that make an angle of θ with the center of the sphere would make an angle of $\phi = \cos^{-1}(\frac{1}{1+\sec\theta})$ with the centre of the circle and vice versa. Since f preserves an angle of θ on the sphere it preserves an angle of $\phi = \cos^{-1}(\frac{1}{1+\sec\theta})$ when it is considered as a map from circle ADC to $A'D'C'$. Therefore, using the same argument as above, we have that f preserves all the angles w.r.t. the center of this circle. Hence the anti-podal points on the circle are mapped to the anti-podal points. This proves $f(C) = C'$ and $\angle A'OC' = 2\theta$. This proves that f preserves an angle of 2θ on a great circle. Therefore using Theorem 2.2 above, the proof is completed.

We can use similar arguments as above to prove the case for any $n \geq 3$. $\square$

Remark 2. If there exists a angle θ such that $\cos^{-1}(\frac{1}{n+\sec\theta})$ is irrational for all $n \geq 0$ then any continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ that preserves the angle θ is an isometry.

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