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ON MAXIMAL INEQUALITIES ARISING IN BEST APPROXIMATION

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ABSTRACT. Let f be a function in an Orlicz space L^{Φ} and $\mu(f,\mathcal{L})$ be the set of all the best Φ-approximants to f, given a $\sigma-$lattice $\mathcal{L}$. Weak type inequalities are proved for the maximal operator $f^* = \sup_n |f_n|$, where f_n is any selection of functions in $\mu(f,\mathcal{L}_n)$, and $\mathcal{L}_n$ is an increasing sequence of σ-lattices. Strong inequalities are proved in an abstract set up which can be used for an operator as f^*.

Key words and phrases: Best approximants, Φ-approximants, σ-lattices, Maximal inequalities.

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1. INTRODUCTION AND MAIN RESULT

Let $(\Omega,\mathcal{A},\mu)$ be a finite measure space and $\mathcal{M}=\mathcal{M}(\Omega,\mathcal{A},\mu)$ the set of all $\mathcal{A}$-measurable real valued functions. Let Φ be a Young function, that is an even and convex function $\Phi:\mathbb{R}\to\mathbb{R}^+$ such that $\Phi(a)=0$ iff $a=0$. We denote by L^{Φ} the space of all the functions $f\in\mathcal{M}$ such that

$$\int_\Omega \Phi(tf)\,d\mu<\infty, \tag{1.1}$$

for some $t>0$.

We say that the function Φ satisfies the Δ_2 condition ($\Phi\in\Delta_2$) if there exists a positive constant $\Lambda=\Lambda_\Phi$ such that for all $a\in\mathbb{R}$

$$\Phi(2a)\le\Lambda\Phi(a).$$

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Under this condition, it is easy to check that $f \in L^{\Phi}$ iff inequality (1.1) holds for every positive number t.

The function Φ satisfies the ∇_2 condition ($\Phi \in \nabla_2$) if there exists a constant $\lambda = \lambda_{\Phi} > 2$ such that

$$\Phi(2a) \geq \lambda \Phi(a).$$

A subset $\mathcal{L} \subset \mathcal{A}$ is a σ-lattice iff $\emptyset, \Omega \in \mathcal{L}$ and $\mathcal{L}$ is closed under countable unions and intersections. Set $L^{\Phi}(\mathcal{L})$ for the set of $\mathcal{L}$-measurable functions in $L^{\Phi}(\Omega)$. Here, $\mathcal{L}$-measurable function means the class of functions $f : \Omega \to \mathbb{R}$ such that $\{f > a\} \in \mathcal{L}$, for all $a \in \mathbb{R}$.

A function $g \in L^{\Phi}(\mathcal{L})$ is called a best Φ-approximation to $f \in L^{\Phi}$ iff

$$\int_{\Omega} \Phi(f-g)d\mu = \min_{h \in L^{\Phi}(\mathcal{L})} \int_{\Omega} \Phi(f-h)d\mu.$$

We denote by $\mu(f, \mathcal{L})$ the set of all the best Φ-approximants to f. It is well known that $\mu(f, \mathcal{L}) \neq \emptyset$, for every $f \in L^{\Phi}$, see [9].

When $\mathcal{L}$ is a σ-field $\mathcal{B} \subset \mathcal{A}$ and $\Phi(t) = t^2$, the set $\mu(f, \mathcal{B})$ has exactly one element, namely the conditional expectation $E_{\mathcal{B}}(f)$ relative to $\mathcal{B}$, which is a linear operator in L^2 and can be continuously extended to all L^1. For $\Phi(t) = t^p$, $1 < p < \infty$ we obtain the p-predictor $P_{\mathcal{B}}(f)$ in the sense of Ando and Amemiya [1], which coincides with the conditional expectation for $p = 2$. The p-predictor operator $P_{\mathcal{B}}(f)$ is generally non-linear, and it is possible to extend it to L^{p-1} as a unique operator preserving a property of monotone continuity, see [10], where $P_{\mathcal{L}}$ is studied for the σ-lattice $\mathcal{L}$. The operator $P_{\mathcal{L}}(f)$, when $\mathcal{L}$ is a σ-lattice and $p = 2$, falls within what is called the theory of isotonic regression, first introduced by Brunk [4] (for applications and further development, see [2, 14]). When $\Phi(x) = x$ and $\mathcal{B}$ is a σ-field, a function g in the set $\mu(f, \mathcal{B})$ is a conditional median, see [15] and [11] for more recent results. All the situations described above are dealt with by considering minimization problems using convex functions and Orlicz Spaces L^{Φ}. For other and more detailed applications, see [2, 14] and chapter 7 of [13].

We adjust a Young function Φ to the origin by $\hat{\Phi}(x) = \int_0^x \hat{\varphi}(t)dt$ with $\hat{\varphi}(x) = \varphi_+(x) - \varphi_+(0)\text{sign}(x)$, where φ_+ denotes the right continuous derivative of Φ. Now we can state our principal result.

Theorem 1.1. *Let Φ be a Young function such that $\hat{\Phi} \in \Delta_2 \cap \nabla_2$. Suppose that $\mathcal{L}_n$ is an increasing sequence of σ-lattices, i.e. $\mathcal{L}_n \subset \mathcal{L}_{n+1}$ for every $n \in \mathbb{N}$. Let f be a nonnegative function in L^{Φ}, let f_n be any selection of functions in $\mu(f, \mathcal{L}_n)$, and consider the maximal function $f^* = \sup_n f_n$. Then there exists constants C and c such that f^* satisfies the following weak type inequality:*

$$\mu(\{f^* > \alpha\}) \leq \frac{C}{\varphi_+(\alpha)} \int_{\{f > c\alpha\}} \varphi_+(f)d\mu, \tag{1.2}$$

for every $\alpha > 0$.

The constant C only depends on $\Lambda_{\hat{\Phi}}$ and c depends on $\Lambda_{\hat{\Phi}}$ and $\lambda_{\hat{\Phi}}$.

If $\varphi_+(0) = 0$ we can set $c = \frac{1}{2}$ and we also have

$$\mu(\{f^* > \alpha\}) \leq \frac{C}{\varphi_+(\alpha)} \int\limits_{\{f^* > \alpha\}} \varphi_+(f)d\mu, \tag{1.3}$$

for every $\alpha > 0$.

The constants $\Lambda_{\hat{\Phi}}$ and $\lambda_{\hat{\Phi}}$ are those used in the definitions of the conditions Δ_2 and ∇_2 respectively.

Theorem 1.1 (in particular inequality (1.3) with $\varphi_+(t) = t^{p-1}$, $1 < p < \infty$) is an Orlicz version of the "martingale maximal theorem", Theorem 5.1 given in [6]. The classical Doob result is given by inequality (1.3) with $\varphi_+(t) = t$ and $f_n = E_{\mathcal{B}_n}[f]$ where $\mathcal{B}_n$ is a increasing sequence of σ-fields in $\mathcal{A}$.

We emphasise that our maximal operator f^* is built up with functions $f_n \in \mu(f, \mathcal{L}_n)$ obtained as a minimization problem in L^{Φ}, though (1.2) and (1.3) can be seen as some sorts of weak type inequalities in L^{φ_+} for functions $f \in L^{\Phi}$, a strictly smaller subset of L^{φ_+}. The extension of the operator $\mu(f, \mathcal{L})$ to all L^{φ_+} is not an easy task for general Φ and $\mathcal{L}$, see [5] for some results in this direction and Theorem 1.1 can be applied to the extension operator given there.

Since operators such as f^* as well as other operators obtained as a best approximation function are not linear or even not sublinear, and in many cases are not positive homogeneous operators, we will assume that the inequalities (1.2) or (1.3) hold for two fixed measurable functions f and f^* and any $a > 0$. From this set up, we interpolate to obtain the so called strong inequalities. Now we state the interpolation problem as follows.

Let φ be a nondecreasing function from $\mathbb{R}^+$ into itself, and we consider two fixed measurable functions $f, g : \Omega \to \mathbb{R}^+$ satisfying the following *weak type inequality*

$$\mu(\{f > a\}) \leq \frac{C_w}{\varphi(a)} \int_{\{f>a\}} \varphi(g) d\mu, \tag{1.4}$$

for any $a > 0$.

We try to find functions Ψ such that the *strong type inequality* below holds:

$$\int_{\Omega} \Psi(f) d\mu \leq C_s \int_{\Omega} \Psi(g) d\mu, \tag{1.5}$$

where $C_s = C_s(\varphi, \Psi, C_w)$. That is, C_s depends only on φ, Ψ and the constant C_w in inequality (1.4).

An inequality closely related to (1.4) is the following one:

$$\mu(\{f > a\}) \leq \frac{\widetilde{C_w}}{\varphi(a)} \int_{\{g>ca\}} \varphi(g) d\mu, \tag{1.6}$$

for every $a > 0$, and c a constant less than one.

It is well known in harmonic analysis and classical differentiation theory that is possible to obtain inequality (1.6) from inequality (1.4) when the functions f, g are related by $f = Tg$ and the function T is a sublinear operator bounded from L^∞ into itself (see [6] or [16], and the last part of the proof of the Theorem 1.1). In this case we need to assume that inequality (1.4) holds for any measurable function g in the domain of T and any $a > 0$. We see that inequality (1.4) implies inequality (1.6) if the function $\Phi(x) = \int_0^x \varphi(t) dt$ is ∇_2 (see Lemma 2.2).

The strong inequality (1.5) will be a consequence of standard arguments in interpolation theory [16]. In Theorem 2.3 we introduce the notion of quasi-increasing functions which implicitly appears in some theorems (see Theorem 1.2.1 in [8]). The notion of quasi-increasing functions is used to define when a function Φ_2 is "bigger" than a function Φ_1 and we will write $\Phi_1 \prec \Phi_2$ (see Definition 2.2). This notation is used to state interpolation results for Orlicz spaces in Corollaries 2.4, 2.5 and 2.6. In [8] a condition related to $x^2 \prec \Phi(x)$ is used to obtain strong inequalities. The relation $x \prec \varphi(x)$ is also named as a Dini condition, i.e.

$$\int_0^x \frac{\varphi(t)}{t} dt \leq C\phi(x),$$

for all $x > 0$ (see Theorem 1 and Proposition 3 in [3]). More on the relation $\Phi_1 \prec \Phi_2$ is given in Section 3 where we extend some results of [7].

The results of Sections 2 and 3 can be used to obtain the strong inequalities (1.5) for the particular operator f^* given in Theorem 1.1.

It was proved in [7], in an abstract set up, that if two functions η and ξ are related by a weak type inequality (1.4) with respect to the function Φ', that is,

$$\mu(\{\eta > a\}) \leq \frac{C_w}{\Phi'(a)} \int_{\{\eta > a\}} \Phi'(\xi) d\mu, \tag{1.7}$$

for any $a > 0$, then η and ξ satisfy the strong inequality

$$\int \Psi(\eta) d\mu \leq C_\Psi \int \Psi(\xi) d\mu,$$

for the functions $\Psi : \Psi = (\Phi'^p$, $1 \leq p$, and $\Psi = (\Phi)^p$, for $1 \leq p$ (also for some p in the range $0 < p < 1$). In proving these results the conjugate function Φ^* was heavily used. We recall that

$$\Phi^*(s) = \sup_t \{st - \Phi(t)\}.$$

As consequence of Sections 2 and 3 we obtain a result more general than those in [7] without appealing to the conjugate function.

2. A Simple Theorem

The following lemma is well known, see [12].

Lemma 2.1. *For every $a \in \mathbb{R}^+$ we have $\Phi(a) \leq a\varphi_+(a)$. Moreover, $\Phi \in \Delta_2$ iff there exists a constant $C > 0$ such that $a\varphi_+(a) \leq C\Phi(a)$.*

Lemma 2.2. *Let φ be a nondecreasing function from $\mathbb{R}^+$ into itself such that $\varphi(rx) \leq \frac{1}{2}\varphi(x)$, for a constant $0 < r < 1$, and every $x > 0$. Suppose that f and g are nonnegative measurable functions defined on Ω satisfying inequality* (1.4)*. Then there exists a positive constant $c = c(r, C_w) < 1$ such that*

$$\mu(\{f > a\}) \leq \frac{2C_w}{\varphi(a)} \int_{\{g > ca\}} \varphi(g) d\mu, \tag{2.1}$$

for every $a > 0$.

Proof. By an inductive argument we get

$$2^n \varphi(r^n a) \leq \varphi(a). \tag{2.2}$$

Let $n \in \mathbb{N}$ be such that $\frac{C_w}{2^n} < \frac{1}{2}$, and set $c = r^n$. Now, we split the integral on the right hand side of (1.4) into the sets $\{g \leq ca\}$ and $\{g > ca\}$. By (2.2) we get

$$\mu(\{f > a\}) \leq \frac{C_w}{\varphi(a)} \int_{\{g > ca\}} \varphi(g) d\mu + \frac{1}{2}\mu(\{f > a\}).$$

Therefore inequality (2.1) follows. $\square$

Remark 1. It is not difficult to see that a Young function Ψ satisfies the ∇_2 condition iff its right derivative ψ_+ fulfills the condition on Lemma 2.2. That is, $\psi_+(rx) \leq \frac{1}{2}\psi_+(x)$, for a constant $0 < r < 1$, and every $x > 0$.

Proof. Since $\Psi(x) = \int_0^x \psi_+(t)dt$, the condition on ψ_+ implies that $\Psi(rx) \leq \frac{1}{2}\Psi(x)$, for every $x > 0$, which is equivalent to the ∇_2 condition given before, see [12]. Now, if we have this condition for Ψ, it is readily seen that $\psi_+(\frac{r}{2}x) \leq \frac{1}{2}\psi_+(x)$. $\square$

We note that if $\Phi \in \nabla_2$ then $\varphi_+(0) = \varphi_-(0) = 0$, see Remark 1.

Definition 2.1. We say that the function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a quasi-increasing function iff there exists a constant $\rho > 0$ such that

$$\frac{1}{x}\int_0^x \eta(t)dt \leq \rho\eta(x), \tag{2.3}$$

for every $x \in \mathbb{R}^+$.

Theorem 2.3. *Let f and g be measurable and positive functions defined on Ω satisfying inequality* (2.1)*. Let Ψ be a $C^1([0,+\infty))$ Young function and let ψ be its derivative. Assume that $\frac{\psi}{\varphi}$ is a quasi-increasing function.Then*

$$\int_\Omega \Psi(f)d\mu \leq 2C_w\rho \int_\Omega \Psi\left(\frac{2}{c}g\right)d\mu. \tag{2.4}$$

Proof. We have that

$$\begin{aligned}\int_\Omega \Psi(f)d\mu &= \int_0^\infty \psi(a)\mu(\{f>a\})d\mu \\ &\leq 2C_w \int_0^\infty \frac{\psi(a)}{\varphi(a)}\left(\int_{\{g>ca\}}\varphi(g)d\mu\right)da \\ &= 2C_w \int_\Omega \varphi(g)\left(\int_0^{c^{-1}g}\frac{\psi(a)}{\varphi(a)}da\right)d\mu.\end{aligned} \tag{2.5}$$

Now, we get

$$\begin{aligned}\int_0^{c^{-1}g}\frac{\psi(a)}{\varphi(a)}da &\leq \rho c^{-1}g\frac{\psi(c^{-1}g)}{\varphi(c^{-1}g)} \\ &\leq \rho\frac{\Psi(2c^{-1}g)}{\varphi(c^{-1}g)}\end{aligned} \tag{2.6}$$

Therefore, from equations (2.5), (2.6) and since $c < 1$ in Lemma 2.2, we obtain Theorem 2.3. □

Definition 2.2. Let φ_1, φ_2 be two functions from $\mathbb{R}_+$ into $\mathbb{R}_+$. We say that $\varphi_1 \prec \varphi_2$ iff $\varphi_2\varphi_1^{-1}$ is a quasi-increasing function.

The notation $\Phi_1 \prec \Phi_2$ is also used if both Φ_1 and Φ_2 are Young functions, in this case Definition 2.2 is applied for the restriction of these functions to $\mathbb{R}_+$.

Remark 2. Let Φ_1 and Φ_2 be two Young functions and let φ_{1+}, φ_{2+} be their right derivatives. If $\Phi_1, \Phi_2 \in \Delta_2$, using Lemma 2.1, we have $\Phi_1 \prec \Phi_2 \Leftrightarrow \varphi_{1+} \prec \varphi_{2+}$.

Remark 3. Despite the symbol used, $\prec$ is not an order relation. We have $x^2 \prec x^{\frac{3}{2}}$ and $x^{\frac{3}{2}} \prec x$, but the relation $x^2 \prec x$ is false. In fact, for two arbitrary powers we have $x^\alpha \prec x^\beta \Leftrightarrow \alpha - 1 < \beta$.

We may define, and it is useful, the relation $\varphi_1 \prec \varphi_2$ only for x near zero, say $0 < x \leq 1$, and only for large values of x, i.e. $1 \leq x$. In the example given below, we will omit the rather straightforward calculations.

Example 2.1. For $0 < x \leq 1$ we have $x^\alpha \prec \ln(1+x)$ if and only if $0 < \alpha < 2$, and for $1 \leq x$ the same relation is true only in the range $0 < \alpha < 1$. On the other hand $\ln(1+x) \prec x^\alpha$ for all x and $0 < \alpha$. All the functions involved here are Δ_2 functions, but $(1+x)\ln(1+x) - x$ is not ∇_2, so its derivative $\ln(1+x)$ does not fulfill the condition on Lemma 2.2 (see Remark 1).

In the following corollaries of Theorem 2.3 the Young function Φ is the one given by $\Phi(x) = \int_0^x \varphi(t)dt$. They are obtained using this theorem, Lemma 2.2, Remark 1 and Remark 2.

Corollary 2.4. *Let f and g be measurable and positive functions defined on Ω satisfying inequality* (1.4)*. Let Ψ be a $C^1([0,+\infty))$ Young function and let ψ be its derivative. Assume that $\varphi \prec \psi$ and the ∇_2 condition for the function Φ holds. Then we have inequality* (2.4)*.*

Corollary 2.5. *Let f and g be measurable and positive functions defined on Ω satisfying inequality* (2.1)*, and assume Φ is a Δ_2 function. Let Ψ be a $C^1([0,+\infty)) \cap \Delta_2$ Young function. If $\Phi \prec \Psi$, then*

$$\int_\Omega \Psi(f)d\mu \leq C \int_\Omega \Psi(g)d\mu, \tag{2.7}$$

where the constant C is independent of the functions f and g.

Corollary 2.6. *Let f and g be measurable and positive functions defined on Ω satisfying inequality* (1.4) *, and assume Φ is a $\Delta_2 \cap \nabla_2$ function. Let Ψ be a $C^1([0,+\infty)) \cap \Delta_2$ Young function. If $\Phi \prec \Psi$, then*

$$\int_\Omega \Psi(f)d\mu \leq C \int_\Omega \Psi(g)d\mu, \tag{2.8}$$

where the constant C is independent of the functions f and g.

Remark 4. By Corollary 2.6 we obtain inequality (1.5) for the following functions Ψ (all the theorems quoted here belong to [7] and see that paper for a proof using conjugate functions). If $\Psi = \Phi$, clearly $\Phi \prec \Phi$, that is Theorem 3.3. The case $p > 1$ of Theorem 3.8 follows by setting $\Psi = \Phi^p$. For Theorem 3.4, set $\Psi = \varphi^p$, $p > 1$ and observe that $\varphi \prec p\varphi^{p-1}\varphi'$.

The operator f^* introduced in [7] is a monotone operator and $(f + c)^* = f^* + c$ for any constant c. We can use Corollary 2.5 to obtain

$$\int_\Omega \Psi(f^*)d\mu \leq C \int_\Omega \Psi(f)d\mu, \tag{2.9}$$

for every function $f \in L^\Psi$, and all Ψ quoted in Remark 4. Now the condition ∇_2 is dropped.

3. THE RELATION $\Phi \prec \Psi$

If $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function then η is clearly a quasi-increasing function. On the other hand, there are decreasing functions which are quasi-increasing functions. We note that if η is a quasi-increasing and nonincreasing function then

$$\rho x \eta(x) \geq \int_0^x \eta(t)dt \geq \int_0^{\frac{x}{2}} \eta(t)dt \geq \frac{x}{2}\eta\left(\frac{x}{2}\right).$$

Therefore, there exists a constant K such that

$$\eta\left(\frac{x}{2}\right) \leq K\eta(x). \tag{3.1}$$

Lemma 3.1. *Let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function. If η satisfies inequality* (3.1) *with $K < 2$, then η is a quasi-increasing function.*

Proof. In addition to the continuous average $A\eta(x) = \frac{1}{x}\int_0^x \eta(t)dt$, is convenient to introduce the discrete averages $A_d\eta(x) = \sum_0^\infty \frac{1}{2^k}\eta(\frac{x}{2^k})$ and $A'_d\eta = A_d\eta - \eta$.

As η is a nonincreasing function we have

$$\frac{1}{2}A_d\eta \leq A\eta \leq A'_d\eta. \tag{3.2}$$

We estimate the discrete average $A'_d\eta$,

$$A'_d\eta(x) = \sum_1^\infty \frac{1}{2^n}\eta\left(\frac{x}{2^n}\right) \le \sum_1^\infty \left(\frac{K}{2}\right)^n \eta(x) = \frac{K}{2-K}\eta(x). \tag{3.3}$$

Now the lemma follows by (3.2) and (3.3). □

Corollary 3.2. *Let $\Psi\Phi^{-1}$ be a nonincreasing function, $\Phi \in \Delta_2$ and $\Psi \in \nabla_2$. Moreover if we assume that $\lambda_\Psi^{-1}\Lambda_\Phi < 2$, then $\Phi \prec \Psi$.*

The next corollary is a version of Theorem 3.8 in [7] for the case $0 < p < 1$.

Corollary 3.3. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function with the Δ_2 condition $2\varphi(x) \le K\varphi(\frac{x}{2})$. Let f and g be measurable nonnegative functions defined on Ω satisfying inequality* (2.1)*. Then*

$$\int_\Omega \Phi^p(f)d\mu \le C\int_\Omega \Phi^p(g)d\mu, \tag{3.4}$$

for any $1 \ge p > \ln(K/2)(\ln K)^{-1}$ and $\Phi(x) = \int_0^x \varphi(t)dt$. Moreover the constant C is $O(1/(2-K^{1-p}))$ as $p \to \ln(K/2)(\ln K)^{-1}$.

Proof. Since $\Phi(x) \le K\Phi(\frac{x}{2})$ we have $\Phi^{p-1}(\frac{x}{2}) \le K^{1-p}\ \Phi^{p-1}(x)$ for $0 < p < 1$. Therefore, by Lemma 3.1, $\Phi \prec \Phi^p$ whenever $K^{1-p} < 2$, and inequality (3.4) follows by Corollary 2.5. □

Remark 5. It is possible to replace (2.1) by (1.4) to again obtain inequality (3.4) for the same range of p if we place on φ the condition $\varphi(rx) \le \frac{1}{2}\varphi(x)$ with a constant $0 < r < 1$, and $2\varphi(x) \le K\varphi(\frac{x}{2})$, that is, if $\Phi \in \Delta_2 \cap \nabla_2$ (see Lemma 2.2).

Proposition 3.4. *Let Φ be in $C^1([0,+\infty)) \cap \Delta_2$ and let Ψ be a quasi increasing function. For the function $\Psi_1(x) = \int_0^x \Psi(t)dt$, suppose that there exists a constant $p > 1$ such that $\frac{\Psi_1}{[\Phi]^p}$ is non-decreasing. Then $\Phi \prec \Psi$.*

Proof. We have that $\log\Psi_1 - p\log\Phi$ is a non-decreasing function in $C^1((0,+\infty))$. Then $\frac{\Psi}{\Psi_1} \ge p\frac{\Phi'}{\Phi}$, or $(q-1)\frac{\Psi}{\Phi} \ge q\frac{\Phi'\Psi_1}{\Phi^2}$, with $q = p/(p-1)$. Therefore

$$q\left(\frac{\Psi\Phi - \Phi'\Psi_1}{\Phi^2}\right) \ge \frac{\Psi}{\Phi}.$$

Integrating the above inequality on $[\epsilon, x]$ we get

$$q\frac{\Psi_1(x)}{\Phi(x)} \ge \int_\epsilon^x \frac{\Psi}{\Phi}dt + \frac{\Psi_1(\epsilon)}{\Phi(\epsilon)}. \tag{3.5}$$

From the hypotheses we have that $\Psi_1(\epsilon)/\Phi(\epsilon) \to 0$, when $\epsilon \to 0$. Therefore inequality (3.5) implies that

$$q\frac{\Psi_1(x)}{\Phi(x)} \ge \int_0^x \frac{\Psi}{\Phi}dt.$$

Taking into account that Ψ is a quasi-increasing function, it follows that $\Phi \prec \Psi$. □

We can use Proposition 3.4 to prove a generalization of Theorem 3.4 of [7] (see the end of Remark 4). Indeed, given functions $\varphi, \theta \in C^1 \cap \Delta_2$ set $\Phi(x) = \int_0^x \varphi(t)dt$ and $\Psi(x) = \theta(\varphi(x))$. Then we have $\Phi \prec \Psi$ if $\theta(x) \div x^p$ is a nondecreasing function for some $p > 1$. In fact, $\Phi' \prec \Psi'$ by Proposition 3.4.

4. PROOF OF THE THEOREM 1.1

We need some additional considerations.

Lemma 4.1. *Let Φ be a convex function satisfying the Δ_2 condition. Then there exists a constant $C > 0$ such that for every $a, x \geq 0$ we have that*

$$\varphi_+(a) + C^2\varphi_+(x-a) \leq (C^2+1)\varphi_+(x).$$

Proof. If $x \geq a$ the assertion in the lemma is trivial. We suppose that $x < a$. Thus $\frac{a}{2} \leq \max\{x, a-x\}$. Then

$$\begin{aligned} \varphi_+(a) &\leq K\varphi_+\left(\frac{a}{2}\right) \\ &\leq K\varphi_+(x) + K\varphi_+(a-x) \\ &\leq K^2\varphi_+(x) + K^2\varphi_+\left(\frac{a-x}{2}\right) \\ &\leq K^2\varphi_+(x) + K^2\varphi_-(a-x). \end{aligned} \tag{4.1}$$

The lemma follows using $\varphi_+(y) = -\varphi_-(-y)$ and (4.1). $\square$

The following theorem was proved in [11]. We denote by $\overline{\mathcal{L}}$ the σ-lattice of the sets D such that $\Omega \setminus D \in \mathcal{L}$.

Theorem 4.2. *Let $f \in L^\Phi$ and $\mathcal{L} \subset \mathcal{A}$ be a σ-lattice. Then $g \in \mu(f, \mathcal{L})$ iff for every $C \in \mathcal{L}$, $D \in \overline{\mathcal{L}}$ and $a \in \mathbb{R}$ the following inequalities hold*

$$\int_{\{g>a\}\cap D} \varphi_\pm(f-a)d\mu \geq 0 \quad \textit{and} \quad \int_{\{g<a\}\cap C} \varphi_\pm(f-a)d\mu \leq 0. \tag{4.2}$$

The set $\mu(f, \mathcal{L})$ admits a minimum and a maximum, i.e. there exist elements $L(f, \mathcal{L}) \in \mu(f, \mathcal{L})$ and $U(f, \mathcal{L}) \in \mu(f, \mathcal{L})$ such that for all $g \in \mu(f, \mathcal{L})$

$$L(f, \mathcal{L}) \leq g \leq U(f, \mathcal{L}).$$

See [9, Theorem 14].

Now we prove Theorem 1.1.

Proof. We define $A_{n,1} = \{f_1 > \alpha\}$ and

$$A_{j,n} := \{f_1 \leq \alpha, \dots, f_{j-1} \leq \alpha, f_j > \alpha\}$$

for $j = 2, \dots, n$.

Then we have that

$$A_n = \left\{\sup_{1\leq j\leq n} f_j > \alpha\right\} = A_{1,n} \cup \cdots \cup A_{n,n}.$$

As a consequence of Theorem 4.2, we obtain

$$\int_{A_{j,n}} \varphi_+(f-\alpha)d\mu \geq 0.$$

Since $A_{j,n} \cap A_{i,n} = \emptyset$ for $i \neq j$, it follows that

$$\int_{\{f^*>\alpha\}} \varphi_+(f-\alpha)d\mu = \lim_{n\to\infty} \int_{A_n} \varphi_+(f-\alpha)d\mu \geq 0.$$

Therefore

$$\begin{aligned}(4.3)\qquad &\varphi_+(0)\mu(\{f<\alpha\}\cap\{f^*>\alpha\})\\ &\qquad\le \varphi_+(0)\mu(\{f\ge\alpha\}\cap\{f^*>\alpha\})+\int_{\{f^*>\alpha\}}\hat\varphi(f-\alpha)d\mu.\end{aligned}$$

Now, using Lemma 4.1 we have

$$(4.4)\qquad \int_{\{f^*>\alpha\}}\hat\varphi(f-\alpha)d\mu \le C_1\int_{\{f^*>\alpha\}}\hat\varphi(f)d\mu - C_2\hat\varphi(\alpha)\mu(\{f^*>\alpha\})$$

with C_i, $i=1,2$, constants depending only on $\Lambda_{\hat\varphi}$. Taking into account (4.3) and (4.4), we get

$$(4.5)\qquad \varphi_+(\alpha)\mu(\{f^*>\alpha\})\le C\varphi_+(0)\mu(\{f\ge\alpha\}\cap\{f^*>\alpha\})+C\int_{\{f^*>\alpha\}}\hat\varphi(f)d\mu,$$

where $C=C(\Lambda_{\hat\varphi})$. Thus we have proved inequality (1.3) of Theorem 1.1.

In order to prove inequality (1.2) of Theorem 1.1, we consider two cases.

Let us begin by assuming that $\varphi_+(0)>0$. We then split the set $\{f^*>\alpha\}$ in the integral of (4.5) in the two regions $\{f^*>\alpha\}\cap\{f>c\alpha\}$ and $\{f\le c\alpha\}\cap\{f^*>\alpha\}$. Now we use the fact that $\hat\Phi\in\nabla_2$ and by Remark 1 there exist constants $0<c<1$ and $0<r$ small such that $\hat\varphi(cx)\le r\hat\varphi(x)$. Then we have:

$$\begin{aligned}(4.6)\qquad &\varphi_+(\alpha)\mu(\{f^*>\alpha\})\le C\varphi_+(0)\mu(\{f\ge\alpha\})\\ &\qquad\qquad + C\int_{\{f>c\alpha\}}\hat\varphi(f)d\mu + rC\varphi_+(\alpha)\mu(\{f^*>\alpha\}).\end{aligned}$$

We now use the Chebyshev inequality, $rC<\frac12$ and $\varphi_+(0)\le\varphi_+(\alpha)$ to obtain inequality (1.2) with constant $4C$.

The second case is $\varphi_+(0)=0$. Now we have

$$\mu(\{f^*>\alpha\})\le\frac{C}{\varphi_+(\alpha)}\int_{\{f^*>\alpha\}}\varphi_+(f)d\mu$$

for every $f\in L^\Phi$ and $\alpha>0$.

Let $f\in L^\Phi$ and define $f_1=f\chi_{\{f\ge\frac{\alpha}{2}\}}$. Thus $f\le f_1+\alpha/2$. Then $f_n\le U(f_1,\mathcal{L}_n)+\alpha/2$ and

$$\{f^*>\alpha\}\subset\left\{\sup_n U(f_1,\mathcal{L}_n)>\frac{\alpha}{2}\right\}.$$

Therefore

$$\begin{aligned}\mu(\{f^*>\alpha\})&\le\mu\left(\left\{\sup_n U(f_1,\mathcal{L}_n)>\frac{\alpha}{2}\right\}\right)\\ &\le\frac{C}{\varphi_+(\alpha)}\int_\Omega\varphi_+(f_1)d\mu\\ &=\frac{C}{\varphi_+(\alpha)}\int_{\{f>\frac{\alpha}{2}\}}\varphi_+(f)d\mu.\end{aligned}$$

□

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