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ON GENERALIZED INVARIANT MEANS AND SEPARATION THEOREMS

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ABSTRACT. We prove the existence of generalized invariant means on some functions spaces which are larger then the space of all bounded functions. Our results are applied to the study of functional inequalities.

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1. INTRODUCTION

Let $\mathcal{F}$ be a non-void subset of the space of all real functions defined on a semigroup $(S, +)$. We say that $\mathcal{F}$ is a *left* (*right*) *invariant* if and only if

$$f \in \mathcal{F} \text{ and } a \in S \text{ implies that } {}_af \in \mathcal{F} \ \ (f_a \in \mathcal{F}), \tag{1.1}$$

where ${}_af$ and f_a denote the *left* and *right translations* of $f \in \mathcal{F}$ by $a \in S$ defined by

$${}_af(x) = f(a + x) \text{ and } f_a(x) = f(x + a), \ \ x \in S.$$

Definition 1.1. Let $\mathcal{F}$ be a left (right) invariant linear space of real functions defined on a semigroup S and let $F : \mathcal{F} \to \mathbb{R}$. A linear functional $\mathcal{M} : \mathcal{F} \to \mathbb{R}$ is termed a *left* (*right*) *invariant F-mean* if and only if it satisfies the following two conditions:

$$\mathcal{M}(f) \le F(f), \ \ f \in \mathcal{F}; \tag{1.2}$$

$$\mathcal{M}({}_af) = \mathcal{M}(f) \ \ (\mathcal{M}(f_a) = \mathcal{M}(f)), \ \ f \in \mathcal{F}, \ a \in S. \tag{1.3}$$

In the case where $\mathcal{F} = B(S, \mathbb{R})$, the space of all real bounded functions on a semigroup S and $F(f) = \sup_{x \in S} f(x)$, for $f \in B(S, \mathbb{R})$, we infer that our definition reduces to the classical definition of an invariant mean.

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In argument with the traditional terminology, if there exists at least one left (right) invariant mean on the space $B(S,\mathbb{R})$ then the underlying semigroup S is said to be *left* (*right*) *amenable*. For the theory of amenability of semigroups and groups see e.g. Greenleaf [7] and Hewitt, Ross [8]. Here we only stress that every Abelian semigroup is (two-sided) amenable.

The concept of invariant means in connection with functional inequalities was invented by L. Székelyhidi (see [12]). In the present paper we are going to extend the concept of an invariant mean to some functions spaces which are essentially larger then the space $B(S,\mathbb{R})$. Next, we present applications of these results to the study of functional inequalities.

2. Generalized Invariant Means

Let us start with the following existence theorem.

Theorem 2.1. *Let $(S,+)$ be a left (right) amenable semigroup and let $\mathcal{F}$ be a left (right) invariant linear space of real functions defined on S. Assume that functionals $\Phi, F : \mathcal{F} \to \mathbb{R}$ satisfy the following conditions:*

$$\Phi(f+g) \le \Phi(f) + \Phi(g), \quad f, g \in \mathcal{F}; \tag{2.1}$$

$$\Phi(\alpha f) = \alpha\Phi(f), \quad f \in \mathcal{F},\ \alpha > 0; \tag{2.2}$$

$$\Phi(f) \le F(f), \quad f \in \mathcal{F} \tag{2.3}$$

and

$$\Phi({}_a f) \le F(f) \quad \left(\Phi(f_a) \le F(f)\right), \quad f \in \mathcal{F},\ a \in S. \tag{2.4}$$

Then there exists a left (right) invariant F-mean on the space $\mathcal{F}$.

Proof. We shall restrict ourselves to the proof of the "left - hand side version" of this theorem.

To start with, note that by condition (2.1)

$$0 \le \Phi(0_S), \tag{2.5}$$

where 0_S denotes the function equal zero on the whole semigroup S. The Hahn - Banach theorem, for the space $X = \mathcal{F}$ and the subspace X_0 degenerated to zero, implies that there exists a linear operator $L : \mathcal{F} \to \mathbb{R}$ such that

$$L(f) \le \Phi(f), \quad f \in \mathcal{F}.$$

Then, by (2.3), we get

$$L(f) \le \Phi(f) \le F(f), \quad f \in \mathcal{F}. \tag{2.6}$$

Let $f \in \mathcal{F}$ be fixed. Condition (2.4) implies

$$L({}_x f) \le \Phi({}_x f) \le F(f), \quad x \in S.$$

Using the linearity of L we have

$$-F(-f) \le L({}_x f) \le F(f), \quad x \in S \tag{2.7}$$

which means that the function

$$S \ni x \longrightarrow L({}_x f) \in \mathbb{R}$$

belongs to the space $B(S,\mathbb{R})$.

Let M be a left invariant mean on $B(S,\mathbb{R})$ which exists by our assumption. We define the map $\mathcal{M} : \mathcal{F} \to \mathbb{R}$ by the formula:

$$\mathcal{M}(f) = M_x(L({}_x f)), \quad f \in \mathcal{F},$$

where the subscript x next to M indicates that the mean M is applied to a function of the variable x.

From the linearity of L and M we obtain that $\mathcal{M}$ is a linear functional. Moreover, condition (2.7) implies

$$\mathcal{M}(f) = M_x(L({}_xf)) \leq \sup_{x\in S} L({}_xf) \leq F(f),$$

for $f \in \mathcal{F}$.

To prove the left invariance of $\mathcal{M}$ we observe that

$${}_y({}_xf) = {}_{x+y} f, \;\; f \in \mathcal{F},\; x, y \in S.$$

Indeed, for every $z \in S$ we get

$${}_y({}_xf)(z) = {}_x f(y+z) = f(x+y+z) = {}_{x+y} f(z), \;\; x, y \in S,$$

which means that our identity holds.

This fact combined with the left invariance of M yields

$$\mathcal{M}({}_af) = M_x(L({}_x({}_af))) = M_x(L({}_{a+x}f)) = M_x(L({}_xf)) = \mathcal{M}(f),$$

for all $f \in \mathcal{F}$ and $a \in S$. Thus, the map $\mathcal{M}$ has all the desired properties for a left invariant F-mean and the proof is completed. □

Remark 2.2. If $\mathcal{M}$ is a left (right) invariant F-mean on the space $\mathcal{F}$, then the linearity of $\mathcal{M}$ jointly with condition (1.2) yields

$$-F(-f) \leq \mathcal{M}(f) \leq F(f), \;\; f \in \mathcal{F}. \tag{2.8}$$

Remark 2.3. If the space $\mathcal{F}$ contains the space C_S of all constant functions on S, then in the proof of Theorem 2.1 we can start with the space $X_0 = C_S$ and with the functional $L_0 : C_S \to \mathbb{R}$ defined by $L_0(c_S) = c\Phi(1_S)$, for $c \in \mathbb{R}$ and we obtain the existence of the F-mean $\mathcal{M}$ such that

$$\mathcal{M}(c_S) = c\Phi(1_S), \;\; c \in \mathbb{R}. \tag{2.9}$$

Now, we will give examples of situations in which all assumptions of Theorem 2.1 are satisfied.

Definition 2.1. A non-empty family $\mathcal{I}$ of subsets of a semigroup S will be called a *proper set ideal* if:

$$S \notin \mathcal{I};$$
$$A, B \in \mathcal{I} \text{ implies } A \cup B \in \mathcal{I};$$
$$A \in \mathcal{J} \text{ and } B \subset A \text{ imply } B \in \mathcal{I}.$$

Moreover, if the set ${}_aA = \{x \in S : a + x \in A\}$ belongs to the family $\mathcal{I}$ whenever $A \in \mathcal{I}$ and $a \in S$, then the set ideal $\mathcal{I}$ is said to be *proper left quasi-invariant* (in short: *p.l.q.i.*). Analogously, the set ideal $\mathcal{I}$ is said to be *proper right quasi-invariant* (in short: *p.r.q.i.*) if the set $A_a = \{x \in S : x + a \in A\}$ belongs to the family $\mathcal{I}$ whenever $A \in \mathcal{I}$ and $a \in S$. In the case where the set ideal satisfies both these conditions we shall call it *proper quasi-invariant* (*p.q.i.*).

The sets belonging to the ideal are intuitively regarded as small sets. For example, if S is a second category subsemigroup of a topological group G then the family of all first category subsets of S is a p.q.i. ideal. If G is a locally compact topological group equipped with the left or right Haar measure μ and if S is a subsemigroup of G with positive measure μ then the family of all subsets of S which have zero measure μ is a p.q.i. ideal. Also, if S is a normed space ($S \neq \{0\}$) then the family of all bounded subsets of S is p.q.i. ideal (see also Gajda [5] and Kuczma [9]).

Let $\mathcal{I}$ be a set ideal of subsets of a semigroup S. For a real function f on S we define $\mathcal{I}_f$ to be the family of all sets $A \in \mathcal{I}$ such that f is bounded on the complement of A. A real function f on S is called *$\mathcal{I}$-essentially bounded* if and only if the family $\mathcal{I}_f$ is non-empty. The space of all $\mathcal{I}$-essentially bounded functions on S will be denoted by $B^{\mathcal{I}}(S, \mathbb{R})$.

It is obvious that, in general, the space $B^{\mathcal{I}}(S, \mathbb{R})$ is essentially larger then the space $B(S, \mathbb{R})$.

For every element f of the space $B^{\mathcal{I}}(S, \mathbb{R})$ the real numbers

$$\mathcal{I} - \operatorname{ess}\inf_{x \in S} f(x) = \sup_{A \in \mathcal{I}_f} \inf_{x \in S \setminus A} f(x),$$

$$\mathcal{I} - \operatorname{ess}\sup_{x \in S} f(x) = \inf_{A \in \mathcal{I}_f} \sup_{x \in S \setminus A} f(x)$$

are correctly defined and are referred to as the *$\mathcal{I}$-essential infimum* and the *$\mathcal{I}$-essential supremum* of the function f, respectively.

Now, we define a map $F^{\mathcal{I}} : B^{\mathcal{I}}(S, \mathbb{R}) \to \mathbb{R}$ by the following formula:

$$F^{\mathcal{I}}(f) = \mathcal{I} - \operatorname{ess}\sup_{x \in S} f(x), \quad f \in B^{\mathcal{I}}(S, \mathbb{R}).$$

If $\mathcal{I}$ is a p.l.q.i. (p.r.q.i.) ideal of a subset of S, then $\mathcal{F} = B^{\mathcal{I}}(S, \mathbb{R})$ is a left right invariant linear space and functions $\Phi = F^{\mathcal{I}}$, $F = F^{\mathcal{I}}$ satisfy conditions (2.1), (2.2), (2.3) and (2.4). So, as a consequence of Theorem 2.1 we obtain the following result which was proved using Silverman's extension theorem by Gajda in [5] (see also [1]).

Corollary 2.4. *If $(S, +)$ is a left (right) amenable semigroup and $\mathcal{I}$ is a p.l.q.i. (p.r.q.i.) ideal of subsets of S, then there exists a real linear functional $M^{\mathcal{I}}$ on the space $B^{\mathcal{I}}(S, \mathbb{R})$ such that*

$$\mathcal{I} - \mathit{ess}\inf_{x \in S} f(x) \le M^{\mathcal{I}}(f) \le \mathcal{I} - \mathit{ess}\sup_{x \in S} f(x)$$

and

$$M^{\mathcal{I}}({}_a f) = M^{\mathcal{I}}(f) \ \ (M^{\mathcal{I}}(f_a) = M^{\mathcal{I}}(f)),$$

for all $f \in B^{\mathcal{I}}(S, \mathbb{R})$ and all $a \in S$.

The next example is a generalization of Gajda's example (see [6]). Here we assume that $p : S \times S \to [0, +\infty)$ is a given function fulfilling the following condition:

$$\inf \left\{ \sum_{i=1}^{n} p(x_i, a_i + s) : s \in S \right\} = 0 \quad \left(\inf \left\{ \sum_{i=1}^{n} p(x_i, s + a_i) : s \in S \right\} = 0 \right), \tag{2.10}$$

for all $a_1, a_2, \dots, a_n \in S$, $x_1, x_2, \dots, x_n \in S$ and $n \in \mathbb{N}$. We say that the function $f : S \to \mathbb{R}$ is *p-bounded* if there exist constants $c_f, C_f \in \mathbb{R}$, $k_f, K_f \ge 0$, $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_n \in S$, $x_1, x_2, \dots, x_n \in S$ such that

$$c_f - k_f \sum_{i=1}^{n} p(x_i, a_i + s) \le f(s) \le C_f + K_f \sum_{i=1}^{n} p(x_i, a_i + s)$$

$$(c_f - k_f \sum_{i=1}^{n} p(x_i, s + a_i) \le f(s) \le C_f + K_f \sum_{i=1}^{n} p(x_i, s + a_i)),$$

for all $s \in S$. The space of all p-bounded functions will be denoted by $B^p(S, \mathbb{R})$. This space is a left (right) invariant linear space.

Let $f \in B^p(S, \mathbb{R})$ be fixed. Then, using the fact that

$$\inf \left\{ K_f \sum_{i=1}^{n} p(x_i, a_i + s) + k_f \sum_{i=1}^{n} p(x_i, a_i + s) : s \in S \right\} = 0$$

$$\left(\inf\left\{K_f\sum_{i=1}^{n}p(x_i,s+a_i)+k_f\sum_{i=1}^{n}p(x_i,s+a_i):s\in S\right\}=0\right)$$

we get $c_f - C_f \leq 0$. So,

$$c_f \leq C_f$$

which means that the set $\mathcal{C}_f$ of all $C_f \in \mathbb{R}$ such that there exist $K_f \geq 0$, $n \in \mathbb{N}$, $a_1, a_2, \ldots, a_n \in S$ and $x_1, x_2, \ldots, x_n \in S$ fulfilling

$$f(s) \leq C_f + K_f\sum_{i=1}^{n}p(x_i,a_i+s) \quad \left(f(s) \leq C_f + K_f\sum_{i=1}^{n}p(x_i,s+a_i)\right), \quad s\in S$$

is bounded from below. Therefore, we can define the map $F^p : B^p(S,\mathbb{R}) \to \mathbb{R}$ by the following formula:

$$F^p(f) = \inf \mathcal{C}_f, \quad f \in B^p(S,\mathbb{R}). \tag{2.11}$$

It is easy to show that functions $\Phi = F^p$ and $F = F^p$ satisfy conditions (2.1), (2.2), (2.3) and (2.4). In this case Theorem 2.1 reduces to the following.

Corollary 2.5. *If $p : S\times S \to [0,+\infty)$ satisfies condition (2.10) and S is a left (right) amenable semigroup, then there exists a real linear functional M^p on the space $B^p(S,\mathbb{R})$ such that*

$$M^p(f) \leq F^p(f), \quad f \in B^p(S,\mathbb{R}); \tag{2.12}$$

and

$$M^p({}_af) = M^p(f) \quad \left(M^p(f_a) = M^p(f)\right), \quad f \in B^p(S,\mathbb{R}), \ a \in S. \tag{2.13}$$

3. SEPARATION THEOREMS

We shall formulate all results of this section in the case corresponding to the left invariant mean only. It will be quite obvious how to rephrase the results so as to obtain its right - handed versions. The proofs of these alternative theorems require only minor changes and, therefore, will be omitted.

Theorem 3.1. *Let S be a left amenable semigroup and let $f, g : S \to \mathbb{R}$. Then there exists an additive function $a : S \to \mathbb{R}$ such that*

$$f(x) \leq a(x) \leq g(x), \quad x \in S \tag{3.1}$$

if and only if there exists a left invariant linear space $\mathcal{F}$ of real functions on S which contains the space of all constant functions on S, the map $F : \mathcal{F} \to \mathbb{R}$ fulfilling

$$F(f+g) \leq F(f) + F(g), \quad f,g \in \mathcal{F}; \tag{3.2}$$

$$F(\alpha f) = \alpha F(f), \quad f \in \mathcal{F}, \ \alpha > 0; \tag{3.3}$$

$$F({}_af) \leq F(f), \quad f \in \mathcal{F}, \ a \in S \tag{3.4}$$

and the following condition:

$$F(f) \leq 0, \ \textit{for } f \leq 0_S, \ f \in \mathcal{F} \textit{ and } F(1_S) > 0, \tag{3.5}$$

functions $\zeta, \eta : S \to [0,+\infty)$, such that $\zeta, \eta \in \mathcal{F}$ and $F(\zeta) = F(\eta) = 0$ and a function $\varphi : S \to \mathbb{R}$ such that, for every $x \in S$, the map:

$$S \ni y \longrightarrow \varphi(x+y) - \varphi(y) \in \mathbb{R} \tag{3.6}$$

belongs to the space $\mathcal{F}$ and

$$f(x) - \zeta(y) \le \varphi(x+y) - \varphi(y) \le g(x) + \eta(y), \quad x, y \in S. \tag{3.7}$$

Proof. Let $f, g : S \to \mathbb{R}$. Assume that there exists an additive function $a : S \to \mathbb{R}$ satisfying (3.1). Then the space $\mathcal{F} = C_S = \{c_S : c \in \mathbb{R}\}$ is a left invariant linear space and the map $F : \mathcal{F} \to \mathbb{R}$ defined by

$$F(c_S) = c, \quad c \in \mathbb{R}$$

fulfills (3.2), (3.3), (3.4) and (3.5). Moreover, taking $\varphi = a$, the additivity of a implies that the function (3.6) is constant (equal $a(x)$, for $x \in S$) - belongs to $\mathcal{F}$ and from condition (3.1) we infer that φ satisfies (3.7) with $\zeta, \eta = 0_S$.

Now, we assume that $\mathcal{F}$ is a left invariant linear space of real functions on S containing the space of all constant functions on S, the map $F : \mathcal{F} \to \mathbb{R}$ satisfies (3.2), (3.3), (3.4) and (3.5), functions $\zeta, \eta : S \to [0, +\infty)$ belong to the space $\mathcal{F}$, $F(\zeta) = F(\eta) = 0$ and that there exists a function $\varphi : S \to \mathbb{R}$ fulfilling (3.6) and (3.7).

Let $\mathcal{M}$ be a left invariant F-mean on the space $\mathcal{F}$ whose existence results from Theorem 2.1 for $\Phi = F$. By Remark 2.3 we can assume that

$$\mathcal{M}(c_S) = cF(1_S), \quad c \in \mathbb{R}. \tag{3.8}$$

Moreover, condition (3.5) implies the monotonicity of $\mathcal{M}$:

$$f, g \in \mathcal{F}, \; f \le g \Longrightarrow \mathcal{M}(f) \le \mathcal{M}(g). \tag{3.9}$$

Indeed, if $f, g \in \mathcal{F}$ satisfy $f \le g$, then using conditions (1.2) and (3.5) we get

$$\mathcal{M}(f) - \mathcal{M}(g) = \mathcal{M}(f-g) \le F(f-g) \le 0.$$

Next, by our assumptions $-\zeta, -\eta \le 0_S$ and $F(\zeta) = F(\eta) = 0$. Applying (3.5) and (2.8) we have

$$0 \le -F(-\zeta) \le \mathcal{M}(\zeta) \le F(\zeta) = 0$$

and

$$0 \le -F(-\eta) \le \mathcal{M}(\eta) \le F(\eta) = 0.$$

Hence,

$$\mathcal{M}(\zeta) = \mathcal{M}(\eta) = 0. \tag{3.10}$$

Now, we put $\alpha(x) = \mathcal{M}_y(\varphi(x+y) - \varphi(y))$, for $x \in S$. Let $x, y \in S$. Then using the linearity and left invariance of $\mathcal{M}$ we get

$$\begin{aligned}
\alpha(x+y) &= \mathcal{M}_z(\varphi(x+y+z) - \varphi(z)) \\
&= \mathcal{M}_z(\varphi(x+y+z) - \varphi(y+z) + \varphi(y+z) - \varphi(z)) \\
&= \mathcal{M}_z(\varphi(x+y+z) - \varphi(y+z)) + \mathcal{M}_z(\varphi(y+z) - \varphi(z)) \\
&= \mathcal{M}_z(\varphi(x+z) - \varphi(z)) + \mathcal{M}_z(\varphi(y+z) - \varphi(z)) \\
&= \alpha(x) + \alpha(y),
\end{aligned}$$

so that α is additive. Moreover, by the definition of α, conditions (3.7), (3.9), (3.10) and (3.8) imply

$$\begin{aligned}
f(x)F(1_S) &= \mathcal{M}_y(f(x)) = \mathcal{M}_y(f(x)) - \mathcal{M}_y(\zeta(y)) = \mathcal{M}_y(f(x) - \zeta(y)) \\
&\le \mathcal{M}_y(\varphi(x+y) - \varphi(y)) = \alpha(x) \\
&\le \mathcal{M}_y(g(x) + \eta(y)) = \mathcal{M}_y(g(x)) + \mathcal{M}_y(\eta(y)) = \mathcal{M}_y(g(x)) \\
&= g(x)F(1_S),
\end{aligned}$$

for all $x \in S$. Consequently, the map $a = F(1_S)^{-1}\alpha$ is an additive function fulfilling (3.1), which ends the proof. □

Applications of Corollary 2.4 can be found in Gajda's paper [5] and in [3]. Applying Corollary 2.5 we have the following result on the separation of two functions by an additive map (see also Páles [11], Nikodem, Páles, Wąsowicz [10] and [4], [3]).

Theorem 3.2. *Let S be a left amenable semigroup with the neutral element, $p : S \times S \to [0, +\infty)$ satisfying condition (2.10) and let $f, g : S \to \mathbb{R}$. Then there exists an additive function $a : S \to \mathbb{R}$ fulfilling (3.1) if and only if there exists a function $\varphi : S \to \mathbb{R}$ such that*

$$f(x) - p(x, y) \le \varphi(x + y) - \varphi(y) \le g(x) + p(x, y), \quad x, y \in S. \tag{3.11}$$

Proof. If a is an additive function fulfilling (3.1), then $\varphi = a$ satisfies (3.11).

Assume that $\varphi : S \to \mathbb{R}$ satisfies (3.11). Then, for every $x \in S$, the map

$$S \ni y \longrightarrow \varphi(x + y) - \varphi(y) \in \mathbb{R}$$

belongs to the space $B^p(S, \mathbb{R})$ and, as in the proof of Theorem 3.1, $a : S \to \mathbb{R}$ defined by the formula:

$$a(x) = \mathcal{M}_y(\varphi(x + y) - \varphi(y)), \quad x \in S$$

is an additive function. Moreover, by the definition of F^p we have

$$\begin{aligned} f(x) &= -(-f(x)) \le -F^p(-(\varphi(x + y) - \varphi(y))) \\ &\le -\mathcal{M}_y(-(\varphi(x + y) - \varphi(y))) = \mathcal{M}_y(\varphi(x + y) - \varphi(y)) \\ &= a(x) \le F^p(\varphi(x + y) - \varphi(y)) \le g(x), \end{aligned}$$

for all $x \in S$ and the proof of Theorem 3.2 is finished. □

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