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A NOTE ON THE HÖLDER INEQUALITY

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ABSTRACT. In the present paper the authors present some new results concerning the Hölder inequality.

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In the following, $(\Omega,\ \mathcal{F})$ is a measure space and μ is a positive measure on Ω. Let $f, g : \Omega \to [0, \infty)$ be two measurable functions. For $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, the classical Hölder's integral inequality is the following one ([2], [3])

$$\int_{\Omega} f(x)g(x)d\mu(x) \leq \left(\int_{\Omega} f^p(x)d\mu(x)\right)^{\frac{1}{p}} \left(\int_{\Omega} g^q(x)d\mu(x)\right)^{\frac{1}{q}}. \tag{1}$$

Inequality (1) may be written equivalently as

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q, \tag{2}$$

where

$$\|f\|_p = \left(\int_{\Omega} f^p(x)d\mu(x)\right)^{\frac{1}{p}}$$

The classical proof of (1) is based on Young's inequality

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}, \tag{3}$$

where $u, v \geq 0$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Moveover, recently the following result about (3) were obtained in ([1]):

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Lemma 1. *Let* $u,\ v \geq 0$ *and* $p, q \geq 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for* $p \geq 2$

$$P(u,v) \leq \frac{1}{2} u^{2-p} (v - u^{p-1})^2, \tag{4}$$

where

$$P(u,v) = \frac{u^p}{p} + \frac{v^q}{q} - uv.$$

If $p \in (1,\ 2]$, *then the reverse inequality in (4) is valid. For* $p = 2$ *we have the identity in (4).*

First, we shall give a new proof of Lemma 1.

Proof. Inequality (4) is equivalent to the following

$$\frac{1}{2} u^{2-p} v^2 + \left(\frac{1}{2} - \frac{1}{p} \right) u^p - \frac{v^q}{q} \geq 0,$$

i.e.

$$\frac{u^{2-p}}{q} \left(\frac{q}{2} v^2 + \frac{q(p-2)}{2p} u^{2(p-1)} - v^q u^{p-2} \right) \geq 0. \tag{5}$$

Let us denote by $Q(u,v)$ the left-hand side of (5). Observe that

$$\frac{q}{2} + \frac{q(p-2)}{2p} = 1.$$

Suppose that $p \geq 2$, that is $q \leq 2$. Using the known arithmetic-geometric inequality ([2], [3]) we obtain

$$\frac{q}{2} v^2 + \frac{q(p-2)}{2p} u^{2(p-1)} \geq (v^2)^{\frac{q}{2}} (u^{2(p-1)})^{\frac{q(p-2)}{2p}} \equiv v^q u^{p-2}.$$

Thus $Q(u,v) \geq 0$ and (5) is valid. For $p \in (1,\ 2]$ applying the reverse arithmetic-geometric inequality we have the reverse inequality in (5). □

We will prove the next theorem.

Theorem 2. *Suppose that* $\frac{1}{p} + \frac{1}{q} = 1$ *for* $1 < q \leq 2 \leq p < \infty$. *Then the following inequalities are valid*

$$\frac{1}{2} \frac{\left\| g^{2-q} \left(f \|g\|_q^{\frac{q}{p}} - g^{q-1} \|f\|_p \right)^2 \right\|_1}{\|f\|_p \, \|g\|_q^{\frac{q}{p}}} \leq \|f\|_p \, \|g\|_q - \|fg\|_1 \leq \frac{1}{2} \frac{\left\| f^{2-p} \left(g \|f\|_p^{\frac{p}{q}} - f^{p-1} \|g\|_q \right)^2 \right\|_1}{\|f\|_p^{\frac{p}{q}} \, \|g\|_q}. \tag{6}$$

Proof. If we set in (4)

$$u = \frac{f(x)}{\|f\|_p}, \qquad v = \frac{g(x)}{\|g\|_q}, \tag{7}$$

we obtain

$$\frac{1}{p} \frac{f^p(x)}{\|f\|_p^p} - \frac{f(x)\, g(x)}{\|f\|_p \, \|g\|_q} + \frac{1}{q} \frac{g^q(x)}{\|g\|_q^q} \leq \frac{1}{2} \frac{f^{2-p}(x)}{\|f\|_p^{2-p}} \left(\frac{g(x)}{\|g\|_q} - \frac{f^{p-1}(x)}{\|f\|_p^{p-1}} \right)^2.$$

Integrating the last inequality, we obtain

$$1-\frac{\|fg\|_1}{\|f\|_p\,\|g\|_q}\le\frac{1}{2\|f\|_p^{2-p}}\left\|f^{2-p}\ \left(\frac{g}{\|g\|_q}-\frac{f^{p-1}}{\|f\|_p^{\frac{p}{q}}}\right)^2\right\|_1,$$

i.e.,

$$\|f\|_p\,\|g\|_q-\|fg\|_1\le\frac{1}{2}\frac{\left\|f^{2-p}\left(g\|f\|_p^{\frac{p}{q}}-f^{p-1}\|g\|_q\right)^2\right\|_1}{\|f\|_p^{\frac{p}{q}}\|g\|_q},$$

which proves the right-hand side of (6).

For the left-hand side of (6) we use the reverse of the inequality in (4). After the substitutions $u\to v,\ v\to u,\ p\to q$ and $q\to p$ we have

$$P(u,v)\ge\frac{1}{2}v^{2-q}(u-v^{q-1})^2.$$

For u and v from (7) we can similarly obtain the first inequality in (6). □

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