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BOUNDEDNESS OF THE WAVELET TRANSFORM IN CERTAIN FUNCTION SPACES

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ABSTRACT. Using convolution transform theory boundedness results for the wavelet transform are obtained in the Triebel space-$L_p^{\Omega,k}$, Hörmander space-$B_{p,q}(\mathbb{R}^n)$ and general function space-$L_{\infty,k}$, where k denotes a weight function possessing specific properties in each case.

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1. INTRODUCTION

The wavelet transform W of a function f with respect to the wavelet ψ is defined by

$$\tilde{f}(a,b) = (W_\psi f)(a,b) = \int_{\mathbb{R}^n} f(t)\overline{\psi_{a,b}(t)}dt = (f * h_{a,0})(b), \tag{1.1}$$

where $\psi_{a,b} = a^{-\frac{n}{2}}\psi(\frac{x-b}{a})$, $h(x) = \overline{\psi(-x)}$, $b \in \mathbb{R}^n$ and $a > 0$, provided the integral exists. In view of (1.1) the wavelet transform $(W_\psi f)(a,b)$ can be regarded as the convolution of f and $h_{a,0}$. The existence of convolution $f * g$ has been investigated by many authors. For this purpose Triebel [6] defined the space $L_p^{\Omega,k}$ and showed that for certain weight functions k, $f * g \in L_p^{\Omega,k}$, where $f, g \in L_p^{\Omega,k}$, $0 < p \leq 1$. Convolution theory has also been developed by Hörmander in the generalized Sobolev space $B_{p,q}(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

In Section 2 of the paper, a definition and properties of the space $L_p^{\Omega,k}$ are given and a boundedness result for the wavelet transform $W_\psi f$ is obtained. In Section 3 we recall the definition and properties of the generalized Sobolev space $B_{p,q}(\mathbb{R}^n)$ due to Hörmander [1] and obtain a certain boundedness result for $W_\psi f$. Finally, using Young's inequality a third boundedness result is also obtained.

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2. BOUNDEDNESS OF W IN $L_p^{\Omega,k}$

Let us recall the definition of the space $L_p^{\Omega,k}$ by Triebel [6].

Definition 2.1. Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$. If $k(x)$ is a non-negative weight function in $\mathbb{R}^n$ and $0 < p \leq \infty$, then

$$L_p^{\Omega,k} = \Bigg\{ f | f \in \mathscr{S}', \operatorname{supp} Ff \subset \overline{\Omega}; \quad \| f \|_{L_p^{\Omega,k}} = \| kf \|_{L_p} = \left(\int_{\mathbb{R}^n} k^p(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \Bigg\}. \tag{2.1}$$

If $k(x) = 1$ then $L_p^{\Omega,k} = L_p^{\Omega}$.

We need the following theorem [6, p. 369] in the proof of our boundedness result.

Theorem 2.1 (Hans Triebel). *If k is one of the following weight functions:*

$$k(x) = |x|^\alpha, \qquad \alpha \geq 0 \tag{2.2}$$

$$k(x) = \prod_{j=1}^n |x_j|^{\alpha_j}, \qquad \alpha_j \geq 0 \tag{2.3}$$

$$k(x) = k_{\beta,\gamma}(x) = e^{\beta |x|^\gamma}, \qquad \beta \geq 0, 0 \leq \gamma \leq 1 \tag{2.4}$$

and $0 < p \leq 1$, then

$$L_p^{\Omega,k} * L_p^{\Omega,k} \subset L_p^{\Omega,k} \tag{2.5}$$

and there exists a positive number C such that for all $f, g \in L_p^{\Omega,k}$,

$$\|f * g\|_{L_p^k} \leq C \, \|f\|_{L_p^k} \, \|g\|_{L_p^k} . \tag{2.6}$$

Using the above theorem we obtain the following boundedness result for the wavelet transform $W_\psi f$.

Theorem 2.2. *Let $f \in L_p^{\Omega,k}$ and $\psi \in L_p^{\Omega,k}$, $0 < p \leq 1$, then for the wavelet transform $W_\psi f$ we have the estimates:*

$$\| (W_\psi f)(a,b) \|_{L_p^k} \leq C a^{\alpha + \frac{n}{2}} \| f \|_{L_p^k} \| \psi \|_{L_p^k} \quad \textit{for (2.2);} \tag{2.7}$$

$$\| (W_\psi f)(a,b) \|_{L_p^k} \leq C a^{|\alpha| + \frac{n}{2}} \| f \|_{L_p^k} \| \psi \|_{L_p^k} \quad \textit{for (2.3);} \tag{2.8}$$

$$\| (W_\psi f)(a,b) \|_{L_p^{k_{\beta,\gamma}}} \leq C a^{\frac{n}{2}} e^{\frac{1}{2}\beta a^{2\gamma}} \| f \|_{L_p^{k_{\beta,\gamma}}} \| \psi \|_{L_p^{k_{\beta,2\gamma}}} \quad \textit{for (2.4),} \tag{2.9}$$

where $b \in \mathbb{R}^n$ and $a > 0$.

Proof. For $k(x) = |x|^{\alpha}$, $\alpha > 0$,we have $k(az) = a^{\alpha}k(z)$ and

$$\begin{aligned}
\|h_{a,0}\|_{L_p^k} &= \left(\int_{\mathbb{R}^n} k^p(x)(a^{-\frac{n}{2}}|h(\frac{x}{a})|)^p dx\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}}\left(\int_{\mathbb{R}^n} k^p(az)|h(z)|^p dz\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}}\left(\int_{\mathbb{R}^n} a^{p\alpha}k^p(z)|h(z)|^p dz\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}+\alpha}\left(\int_{\mathbb{R}^n} k^p(z)|h(z)|^p dz\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}+\alpha} \parallel h \parallel_{L_p^k} \\
&= a^{\frac{n}{2}+\alpha} \parallel \psi \parallel_{L_p^k} .
\end{aligned}$$

For $k(x) = \prod_{j=1}^n |x_j|^{\alpha_j}$, $\alpha_j \geq 0$, we have $k(az) = a^{|\alpha|}k(z)$ and

$$\begin{aligned}
\|h_{a,0}\|_{L_p^k} &= \left(\int_{\mathbb{R}^n} k^p(x)(a^{-\frac{n}{2}}|h(\frac{x}{a})|)^p dx\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}}\left(\int_{\mathbb{R}^n} k^p(az)|h(z)|^p dz\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}}\left(\int_{\mathbb{R}^n} a^{p|\alpha|}k^p(z)|h(z)|^p dz\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}+|\alpha|}\left(\int_{\mathbb{R}^n} k^p(z)|h(z)|^p dz\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}+|\alpha|} \parallel h \parallel_{L_p^k} \\
&= a^{\frac{n}{2}+|\alpha|} \parallel \psi \parallel_{L_p^k} .
\end{aligned}$$

Next, for $k(x) = k_{\beta,\gamma}(x) = e^{\beta|x|^{\gamma}}$, $\beta \geq 0$, $0 \leq \gamma \leq 1$, we have

$$k_{\beta,\gamma}(az) = e^{\beta|az|^{\gamma}} = e^{\beta a^{\gamma}|z|^{\gamma}} \leq e^{\beta\frac{a^{2\gamma}+|z|^{2\gamma}}{2}} = e^{\frac{1}{2}\beta a^{2\gamma}}e^{\frac{1}{2}\beta|z|^{2\gamma}} = e^{\frac{1}{2}\beta a^{2\gamma}}k_{\beta,2\gamma}(z),$$

and

$$\begin{aligned}
\|h_{a,0}\|_{L_p^{k_{\beta,\gamma}}} &= \left(\int_{\mathbb{R}^n} k^p_{\beta,\gamma}(x)\left(a^{-\frac{n}{2}}\left|h\left(\frac{x}{a}\right)\right|\right)^p dz\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}}\left(\int_{\mathbb{R}^n} k^p_{\beta,\gamma}(az)|h(z)|^p dz\right)^{\frac{1}{p}} \\
&\leq a^{\frac{n}{2}}\left(\int_{\mathbb{R}^n} e^{\frac{1}{2}p\beta a^{2\gamma}}k^p_{\beta,2\gamma}(z)|h(z)|^p dz\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}}e^{\frac{1}{2}\beta a^{2\gamma}}\left(\int_{\mathbb{R}^n} k^p_{\beta,2\gamma}(z)|h(z)|^p dz\right)^{\frac{1}{p}} \\
&= a^{\frac{n}{2}}e^{\frac{1}{2}\beta a^{2\gamma}} \parallel h \parallel_{L_p^{k_{\beta,2\gamma}}} \\
&= a^{\frac{n}{2}}e^{\frac{1}{2}\beta a^{2\gamma}} \parallel \psi \parallel_{L_p^{k_{\beta,2\gamma}}} .
\end{aligned}$$

The proofs of (2.7), (2.8) and (2.9) follow from (2.6). □

3. BOUNDEDNESS OF W IN $B_{p,k}$

The space $B_{p,k}(\mathbb{R}^n)$ was introduced by Hörmander [1], as a generalization of the Sobolev space $H^s(\mathbb{R}^n)$, in his study of the theory of partial differential equations. We recall its definition.

Definition 3.1. A positive function k defined in $\mathbb{R}^n$ will be called a temperate weight function if there exist positive constants C and N such that

$$k(\xi+\eta) \leq (1+C|\xi|)^N k(\eta); \quad \xi,\eta \in \mathbb{R}^n, \tag{3.1}$$

the set of all such functions k will be denoted by $\mathscr{K}$. Certain properties of the weight function k are contained in the following theorem whose proof can be found in [1].

Theorem 3.1. *If k_1 and k_2 belong to $\mathscr{K}$,then $k_1+k_2, k_1k_2, \sup(k_1,k_2), \inf(k_1,k_2)$, are also in $\mathscr{K}$. If $k \in \mathscr{K}$ we have $k^s \in \mathscr{K}$ for every real s,and if μ is a positive measure we have either $\mu * k \equiv \infty$ or else $\mu * k \in \mathscr{K}$.*

Definition 3.2. If $k \in \mathscr{K}$ and $1 \leq p \leq \infty$, we denote by $B_{p,k}$ the set of all distributions $u \in \mathscr{S}'$ such that $\hat{u}$ is a function and

$$\| u \|_{p,k} = (2\pi)^{-n} \left(\int |k(\xi)\hat{u}|^p d\xi \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty; \tag{3.2}$$

$$\| u \|_{\infty,k} = ess\sup |k(\xi)\hat{u}(\xi)|. \tag{3.3}$$

We need the following theorem [1, p.10] in the proof of our boundedness result.

Theorem 3.2 (Lars Hörmander)**.** *If $u_1 \in B_{p,k_1} \bigcap \mathscr{E}'$and $u_2 \in B_{\infty,k_2}$ then $u_1 * u_2 \in B_{\infty,k_1k_2}$, and we have the estimate*

$$\| u_1 * u_2 \|_{p,k_1k_2} \leq \| u_1 \|_{p,k_1} \| u_2 \|_{\infty,k_2}, \quad 1 \leq p < \infty. \tag{3.4}$$

Using the above theorem we obtain the following boundedness result.

Theorem 3.3. *Let k_1 and k_2 belong to $\mathscr{K}$. Assume that $f \in B_{p,k_1} \bigcap \mathscr{E}'$ and $\psi \in B_{\infty,k_2}$ then the wavelet transform $(W_\psi f)(a,b) = (f * h_{a,0})(b)$, defined by (1.1) is in B_{p,k_1k_2}, and*

$$\| W_\psi f(a,b) \|_{p,k_1k_2} \leq a^{\frac{n}{2}} k_2\left(\frac{1}{2a^2}\right) \| f \|_{p,k_1} \left\| \left(1+\frac{C}{2}t^2\right)^N \hat{\psi}(t) \right\|_\infty . \tag{3.5}$$

Proof. Since

$$\begin{aligned} \|h_{a,0}\|_{\infty,k_2} &= ess\sup \left| k_2(\xi)\hat{h}_{a,0}(\xi) \right| \\ &= ess\sup \left| k_2(\xi) a^{\frac{n}{2}} \hat{\psi}(a\xi) \right| \\ &\leq a^{\frac{n}{2}} ess\sup \left| k_2(\tfrac{t}{a})\hat{\psi}(t) \right| \\ &\leq a^{\frac{n}{2}} k_2\left(\frac{1}{2a^2}\right) ess\sup \left| \left(1+\frac{C}{2}t^2\right)^N \hat{\psi}(t) \right| \end{aligned}$$

on using (3.1). Hence by Theorem 3.2 we have

$$\begin{aligned} \|W_\psi f(a,b)\|_{p,k_1k_2} &= \| (f * h_{a,0}(b) \|_{p,k_1k_2} \\ &\leq a^{\frac{n}{2}} k_2\left(\frac{1}{2a^2}\right) \| f \|_{p,k_1} \left\| \left(1+\frac{C}{2}t^2\right)^N \hat{\psi}(t) \right\|_\infty . \end{aligned}$$

This proves the theorem. □

4. A GENERAL BOUNDEDNESS RESULT

Using Young's inequality for convolution we obtained a general boundedness result for the wavelet transform. In the proof of our result the following theorem will be used [3, p. 90].

Theorem 4.1. *Let* $p, q, r \geq 1$ *and* $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$. *Let* $k \in L^p(\mathbb{R}^n)$, $f \in L^q(\mathbb{R}^n)$ *and* $g \in L^r(\mathbb{R}^n)$, *then*

$$\begin{aligned}\|f * g\|_{\infty,k} &= \left|\int_{\mathbb{R}^n} k(x)(f*g)(x)dx\right| \\ &= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(x)f(x-y)g(y)dxdy\right| \\ &\leq C_{p,q,r;n} \parallel k \parallel_p \parallel f \parallel_q \parallel g \parallel_r .\end{aligned}$$

The sharp constant $C_{p,q,r;n} = (C_p C_q C_r)^n$, where $C_p^2 = \frac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}$ with $(\frac{1}{p}+\frac{1}{p'}=1)$. Using Theorem 4.1 and following the same method of proof as for Theorem 3.3 we obtain the following boundedness result.

Theorem 4.2. *Let* $p, q, r \geq 1$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$ *and* $k \in L^p(\mathbb{R}^n)$. *Let* $f \in L^q(\mathbb{R}^n)$ *and* $\psi \in L^r(\mathbb{R}^n)$, *then*

$$\parallel W_\psi f \parallel_{\infty,k} \quad\leq\quad C_{p,q,r;n} a^{\frac{n}{r}-\frac{n}{2}} \parallel k \parallel_p \parallel f \parallel_q \parallel \psi \parallel_r$$

where $C_{p,q,r;n} = (C_p C_q C_r)^n$, $C_p^2 = \frac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}$ *with* $\frac{1}{p}+\frac{1}{p'}=1$.

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