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AN INEQUALITY FOR THE CLASS NUMBER

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Abstract

We prove in an elementary way a new inequality for the average order of the Piltz divisor function with application to class number of number fields.

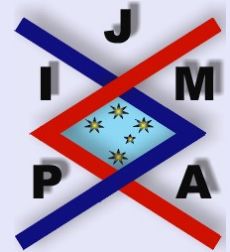
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I would like to thank my wife Véronique for her help.

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1. Introduction

It could be interesting to use tools from analytic number theory to solve problems of algebraic number theory. For example, let \mathbb{K} be a number field of degree n , signature (r_1, r_2) , class number $h_{\mathbb{K}}$, regulator $\mathcal{R}_{\mathbb{K}}$, and $w_{\mathbb{K}}$ is the number of roots of unity in \mathbb{K} , $\zeta_{\mathbb{K}}$ the Dedekind zeta function, $A_{\mathbb{K}} := 2^{-r_2} \pi^{-n/2} d_{\mathbb{K}}^{1/2}$ where $d_{\mathbb{K}}$ is the absolute value of the discriminant of \mathbb{K} . The following formula, valid for any real number $\sigma > 1$,

$$(1.1) \quad A_{\mathbb{K}}^{\sigma} \Gamma^{r_1} \left(\frac{\sigma}{2} \right) \Gamma^{r_2} (\sigma) \zeta_{\mathbb{K}} (\sigma) = \frac{2^{r_1} h_{\mathbb{K}} \mathcal{R}_{\mathbb{K}}}{\sigma (\sigma - 1) w_{\mathbb{K}}} + \sum_{\mathfrak{a} \neq 0} \int_{\|y\| \geq 1} \left\{ \|y\|^{\sigma/2} + \|y\|^{\frac{1-\sigma}{2}} \right\} e^{-g(\mathfrak{a}, y)} \frac{dy}{y},$$

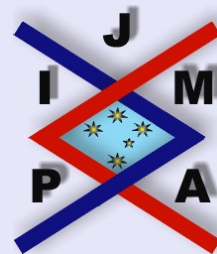
where $g(\mathfrak{a}, y)$ is a certain function depending on a nonzero integral ideal \mathfrak{a} and vector $y := (y_1, \dots, y_{r_1+r_2}) \in (\mathbb{R}_+)^{r_1+r_2}$ (here $\|y\| := \max |y_i|$), is the generalization of the well-known formula

$$\pi^{-\sigma/2} \Gamma \left(\frac{\sigma}{2} \right) \zeta (\sigma) = \frac{1}{\sigma (\sigma - 1)} + \sum_{n=1}^{\infty} \int_1^{\infty} \left\{ y^{\sigma/2} + y^{\frac{1-\sigma}{2}} \right\} e^{-\pi n^2 y} \frac{dy}{y}$$

for the classical Riemann zeta function. Since the integrand in (1.1) is positive, we get

$$(1.2) \quad h_{\mathbb{K}} \mathcal{R}_{\mathbb{K}} \leq \sigma (\sigma - 1) w_{\mathbb{K}} 2^{-r_1} A_{\mathbb{K}}^{\sigma} \Gamma^{r_1} \left(\frac{\sigma}{2} \right) \Gamma^{r_2} (\sigma) \zeta_{\mathbb{K}} (\sigma)$$

for any real number $\sigma > 1$. The study of the function on the right-hand side of (1.2) provides upper bounds for $h_{\mathbb{K}} \mathcal{R}_{\mathbb{K}}$ (see [3] for example).



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In a more elementary way, one can connect the class number $h_{\mathbb{K}}$ with the Piltz divisor function τ_n by using the following result ([1]):

Lemma 1.1. *Let $b_{\mathbb{K}} > 0$ be a real number such that every class of ideals of \mathbb{K} contains a nonzero integral ideal with norm $\leq b_{\mathbb{K}}$. If τ_n is the Piltz divisor function, then:*

$$h_{\mathbb{K}} \leq \sum_{m \leq b_{\mathbb{K}}} \tau_n(m).$$

Recall that τ_n is defined by the relations $\tau_1(m) = m$ and $\tau_n(m) = \sum_{d|m} \tau_{n-1}(d)$ ($n \geq 2$). This function has been studied by many authors (see [6] for a good survey of its properties). A standard argument from analytic number theory gives if $n \geq 4$

$$\sum_{m \leq x} \tau_n(m) = x\mathcal{P}_{n-1}(\log x) + O_{\varepsilon}\left(x^{\frac{n-1}{n+2}+\varepsilon}\right),$$

where \mathcal{P}_{n-1} is a polynomial of degree $n - 1$ and leading coefficient $\frac{1}{(n-1)!}$. For some improvements of the error term and related results, see [4]. Note that the Lindelöf Hypothesis is equivalent to $\alpha_n = (n - 1) / (2n)$ for any $n = 2, 3, \dots$ where α_n is the least number such that

$$\sum_{m \leq x} \tau_n(m) - x\mathcal{P}_{n-1}(\log x) = O_{\varepsilon}\left(x^{\alpha_n+\varepsilon}\right).$$

If we are interested in finding upper bounds of the form

$$\sum_{m \leq x} \tau_n(m) \ll_n x (\log x)^{n-1},$$

one mostly uses arguments based upon induction and the following inequality:



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Lemma 1.2. We set $S_n(x) := \sum_{m \leq x} \tau_n(m)$. Then:

$$S_{n+1}(x) \leq S_n(x) + x \int_1^x t^{-2} S_n(t) dt.$$

Proof. It suffices to use the definition above, interchange the summations and integrate by parts. \square

Using this lemma, it is easy to show by induction the following bound:

$$\sum_{m \leq x} \tau_n(m) \leq \frac{x}{(n-1)!} (\log x + n - 1)^{n-1}$$

which enables us to obtain Lenstra's bound again (see [2]), namely:

$$(1.3) \quad h_{\mathbb{K}} \leq \frac{b_{\mathbb{K}}}{(n-1)!} (\log b_{\mathbb{K}} + n - 1)^{n-1}.$$

In what follows, n is a positive integer and we set

$$S_n(x) := \sum_{m \leq x} \tau_n(m)$$

for any real number $x \geq 1$. $b_{\mathbb{K}}$ is a positive real number always satisfying the hypothesis of Lemma 1.1. \mathbb{K} is a number field of degree n and class number $h_{\mathbb{K}}$. $d_{\mathbb{K}}$ is the absolute value of the discriminant of \mathbb{K} . For some tables giving values of $b_{\mathbb{K}}$, see [7]. The functions ψ and ψ_2 are defined by

$$\psi(t) = t - [t] - \frac{1}{2},$$

$$\psi_2(t) = \int_0^t \psi(u) du + \frac{1}{8} = \frac{\psi^2(t)}{2},$$



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where $[t]$ denotes the integral part of t . Recall that we have for all real numbers t :

$$|\psi(t)| \leq \frac{1}{2},$$

$$0 \leq \psi_2(t) \leq \frac{1}{8}.$$

We denote by γ and γ_1 the Euler-Mascheroni constant and the first Stieltjes constant, defined respectively by:

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right),$$

$$\gamma_1 = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{\log k}{k} - \frac{(\log n)^2}{2} \right).$$

The following results are well-known (see [5] for example):

$$0.577215 < \gamma < 0.577216,$$

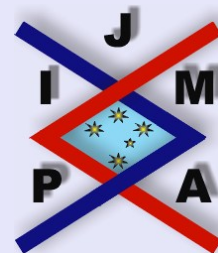
$$-0.072816 < \gamma_1 < -0.072815,$$

and

$$(1.4) \quad \gamma = \frac{1}{2} - 2 \int_1^{\infty} \frac{\psi_2(t)}{t^3} dt$$

and

$$(1.5) \quad \gamma_1 = - \int_1^{\infty} \frac{2 \log t - 3}{t^3} \psi_2(t) dt.$$



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2. Results

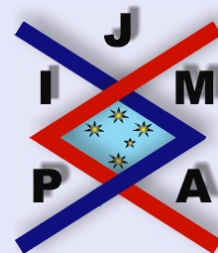
Theorem 2.1. *Let $n \geq 3$ be an integer. For any real number $x \geq 13$, we have:*

$$\sum_{m \leq x} \tau_n(m) \leq \frac{x}{(n-1)!} (\log x + n - 2)^{n-1}.$$

Applying this result with Lemma 1.1 allows us to improve upon (1.3) :

Theorem 2.2. *Let \mathbb{K} be a number field of degree $n \geq 3$. If $b_{\mathbb{K}} \geq 13$ satisfies the hypothesis of Lemma 1.1, then:*

$$h_{\mathbb{K}} \leq \frac{b_{\mathbb{K}}}{(n-1)!} (\log b_{\mathbb{K}} + n - 2)^{n-1}.$$



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3. The Case $n = 3$

The aim of this section is to show that the result of Theorem 2.1 is true for $n = 3$. Hence we will prove the following inequality for S_3 :

Lemma 3.1. *For any real number $x \geq 13$, we have:*

$$S_3(x) \leq \frac{x}{2} (\log x + 1)^2.$$

We first check this result for $13 \leq x \leq 670$ with the PARI/GP system [8], and then suppose $x > 670$. The lemma will be a direct consequence of the following estimation:

Lemma 3.2. *For any real number $x > 670$, we have:*

$$S_3(x) = x \left\{ \frac{(\log x)^2}{2} + (3\gamma - 1) \log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1 \right\} + R(x)$$

where:

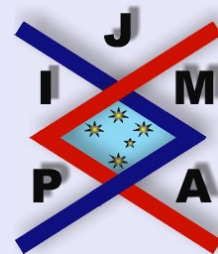
$$|R(x)| \leq 2.36 x^{2/3} \log x.$$

The proof of this lemma needs some technical results:

Lemma 3.3. *Let $x, y \geq 1$ be real numbers.*

(i) *If $e^{3/2} \leq y \leq x$, then we have:*

$$\sum_{k \leq y} \frac{1}{k} \log \left(\frac{x}{k} \right) = \log x \log y - \frac{(\log y)^2}{2} + \gamma \log x - \gamma_1 + R_1(x, y)$$



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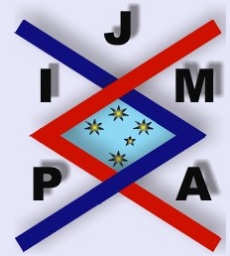


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with:

$$|R_1(x, y)| \leq \frac{\log(x/y)}{2y} + \frac{\log x}{4y^2}.$$

(ii)

$$S_2(y) = y \log y + (2\gamma - 1)y + R_2(y)$$

with:

$$|R_2(y)| \leq y^{1/2} + \frac{1}{2}.$$

(iii)

$$\sum_{n \leq y} \frac{\tau(n)}{n} = \frac{(\log y)^2}{2} + 2\gamma \log y + \gamma^2 - 2\gamma_1 + R_3(y)$$

with:

$$|R_3(y)| \leq \frac{1}{y^{1/2}} + \frac{1}{y}.$$

Proof. (i) By the Euler-MacLaurin summation formula, we get:

$$\begin{aligned} & \sum_{k \leq y} \frac{1}{k} \log \left(\frac{x}{k} \right) \\ &= \frac{\log x}{2} + \int_1^y \frac{1}{t} \log \left(\frac{x}{t} \right) dt - \frac{\psi(y)}{y} \log \left(\frac{x}{y} \right) \\ & \quad - \frac{\psi_2(y)}{y^2} \left(\log \left(\frac{x}{y} \right) + 1 \right) - \int_1^y \frac{2 \log(x/t) + 3}{t^3} \psi_2(t) dt \end{aligned}$$

$$\begin{aligned}
&= \log x \log y - \frac{(\log y)^2}{2} + \left(\frac{1}{2} - 2 \int_1^\infty \frac{\psi_2(t)}{t^3} dt \right) \log x \\
&\quad + \int_1^\infty \frac{2 \log t - 3}{t^3} \psi_2(t) dt - \frac{\psi(y)}{y} \log \left(\frac{x}{y} \right) - \frac{\psi_2(y)}{y^2} \left(\log \left(\frac{x}{y} \right) + 1 \right) \\
&\quad + 2 \log x \int_y^\infty \frac{\psi_2(t)}{t^3} dt - \int_y^\infty \frac{2 \log t - 3}{t^3} \psi_2(t) dt
\end{aligned}$$

and using (1.4) and (1.5) we get:

$$\sum_{k \leq y} \frac{1}{k} \log \left(\frac{x}{k} \right) = \log x \log y - \frac{(\log y)^2}{2} + \gamma \log x - \gamma_1 + R_1(x, y)$$

and since $e^{3/2} \leq y \leq x$, we have:

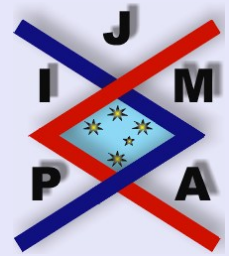
$$\begin{aligned}
|R_1(x, y)| &\leq \frac{\log(x/y)}{2y} + \frac{\log(x/y) + 1}{8y^2} + \frac{\log x}{8y^2} + \frac{\log y - 1}{8y^2} \\
&= \frac{\log(x/y)}{2y} + \frac{\log x}{4y^2}.
\end{aligned}$$

(ii) This result is well-known (see [1] for example).

(iii) Using a result from [5], we have for any real number $y \geq 1$:

$$-y^{-1/2} - \left(\frac{3}{4} + \frac{1}{8e^3} \right) y^{-1} - \frac{y^{-3/2}}{8} - \frac{y^{-2}}{64} \leq R_3(y) \leq y^{-1/2} + \left(\frac{1}{2} + \frac{1}{8e^3} \right) y^{-1}$$

which concludes the proof of Lemma 3.3. \square



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Proof of Lemmas 3.1 and 3.2. The Dirichlet hyperbola principle and the estimations of Lemma 3.3 give, for any real number $e^{3/2} \leq T < x$:

$$\begin{aligned}
 S_3(x) &= \sum_{n \leq T} S_2\left(\frac{x}{n}\right) + \sum_{n \leq x/T} \tau(n) \left[\frac{x}{n}\right] - [T] S_2\left(\frac{x}{T}\right) \\
 &= \sum_{n \leq T} \left(\frac{x}{n} \log\left(\frac{x}{n}\right) + (2\gamma - 1) \frac{x}{n} + R_4(x, n)\right) + x \sum_{n \leq x/T} \frac{\tau(n)}{n} - \frac{1}{2} S_2\left(\frac{x}{T}\right) \\
 &\quad - \sum_{n \leq x/T} \tau(n) \psi\left(\frac{x}{n}\right) - T S_2\left(\frac{x}{T}\right) + \frac{1}{2} S_2\left(\frac{x}{T}\right) + \psi(T) S_2\left(\frac{x}{T}\right) \\
 &= \sum_{n \leq T} \left(\frac{x}{n} \log\left(\frac{x}{n}\right) + (2\gamma - 1) \frac{x}{n} + R_4(x, n)\right) \\
 &\quad + x \sum_{n \leq x/T} \frac{\tau(n)}{n} - T S_2\left(\frac{x}{T}\right) + R_5(x, T)
 \end{aligned}$$

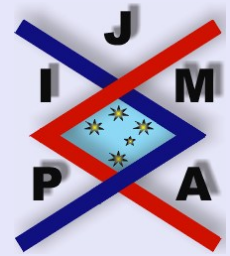
with

$$|R_4(x, n)| \leq \sqrt{\frac{x}{n}} + \frac{1}{2}$$

$$|R_5(x, T)| \leq S_2\left(\frac{x}{T}\right) \leq \frac{x}{T} \log\left(\frac{x}{T}\right) + (2\gamma - 1) \frac{x}{T} + \sqrt{\frac{x}{T}} + \frac{1}{2}$$

and hence:

$$S_3(x) = x \left\{ \log x \log T - \frac{(\log T)^2}{2} + \gamma \log x \right.$$



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$$\begin{aligned}
& -\gamma_1 + R_6(x, T) + (2\gamma - 1)(\log T + \gamma + R_7(T)) \Big\} + \sum_{n \leq T} R_4(x, n) \\
& + x \left\{ \frac{(\log(x/T))^2}{2} + 2\gamma \log\left(\frac{x}{T}\right) + \gamma^2 - 2\gamma_1 + R_8(x, T) \right\} \\
& + R_5(x, T) - x \log\left(\frac{x}{T}\right) - (2\gamma - 1)x - TR_9(x, T)
\end{aligned}$$

with, if $e^{3/2} \leq T < x$:

$$|R_6(x, T)| \leq \frac{\log(x/T)}{2T} + \frac{\log x}{4T^2}$$

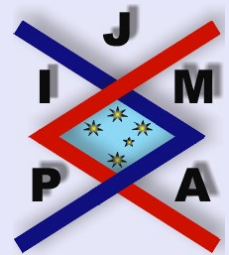
$$|R_7(T)| \leq \frac{1}{T}$$

$$|R_8(x, T)| \leq \sqrt{\frac{T}{x}} + \frac{T}{x}$$

$$|R_9(x, T)| \leq \sqrt{\frac{x}{T}} + \frac{1}{2}$$

and thus:

$$\begin{aligned}
S_3(x) = x \left\{ \frac{(\log x)^2}{2} + (3\gamma - 1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1 \right\} \\
+ xR_6(x, T) + (2\gamma - 1)xR_7(T) + R_{10}(x, T) \\
+ xR_8(x, T) + R_5(x, T) - TR_9(x, T)
\end{aligned}$$



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with

$$\begin{aligned}
 |R_{10}(x, T)| &\leq \sum_{n \leq T} |R_4(x, n)| \\
 &\leq \sqrt{x} \sum_{n \leq T} \frac{1}{\sqrt{n}} + \frac{T}{2} \\
 &\leq 2\sqrt{xT} - \sqrt{x} + \frac{T}{2}
 \end{aligned}$$

and therefore:

$$S_3(x) = x \left\{ \frac{(\log x)^2}{2} + (3\gamma - 1) \log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1 \right\} + R_{11}(x, T)$$

with:

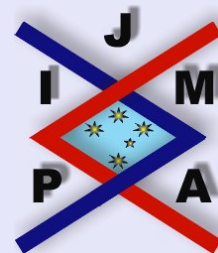
$$\begin{aligned}
 |R_{11}(x, T)| &\leq \frac{x \log(x/T)}{2T} + \frac{x \log x}{4T^2} + 4\sqrt{xT} - \sqrt{x} \\
 &\quad + \frac{2x}{T} \log\left(\frac{x}{T}\right) + 2(2\gamma - 1) \frac{x}{T} + \sqrt{\frac{x}{T}} + 2T + \frac{1}{2}.
 \end{aligned}$$

We choose:

$$T = x^{1/3},$$

which gives:

$$S_3(x) = x \left\{ \frac{(\log x)^2}{2} + (3\gamma - 1) \log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1 \right\} + R_{12}(x),$$



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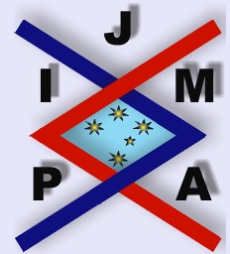
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where:

$$\begin{aligned} |R_{12}(x)| &\leq \frac{5}{3}x^{2/3} \log x + 2(2\gamma + 1)x^{2/3} - x^{1/2} + \frac{1}{4}x^{1/3} \log x + 3x^{1/3} + \frac{1}{2} \\ &\leq 2.36x^{2/3} \log x \end{aligned}$$

since $x > 670$. This concludes the proof of Lemma 3.2, and then of Lemma 3.1. \square



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4. Proof of Theorem 2.1

We first need the following simple bounds:

Lemma 4.1. For any integer $n \geq 3$, we have:

$$\int_1^{13} t^{-2} S_n(t) dt < \frac{n^3}{4} \leq \frac{1}{n!} \left(n + \frac{1}{2}\right)^n.$$

Proof. This follows from straightforward computations which give:

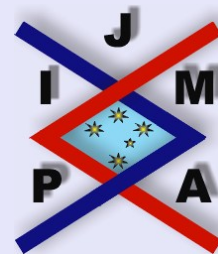
$$\begin{aligned} \int_1^{13} t^{-2} S_n(t) dt &= \frac{7}{624} n^3 + \frac{2281}{9360} n^2 + \frac{90283}{90090} n + 1 - \frac{1}{13} \\ &< \frac{n^3}{4} \end{aligned}$$

since $n \geq 3$. The second inequality follows from studying the sequence (u_n) defined by

$$u_n = \frac{n^3 \times n!}{4(n + 1/2)^n}.$$

We get:

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{2(n+1)^4}{n^3(2n+3)} \left(\frac{2n+1}{2n+3}\right)^n \\ &\leq \frac{512}{243} \left(1 - \frac{2}{2n+3}\right)^n \leq \frac{512e^{-1}}{243} < 1 \end{aligned}$$



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and hence (u_n) is decreasing, and thus:

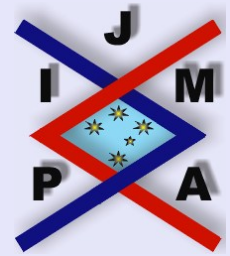
$$u_n \leq u_3 = \frac{324}{343} \leq 1,$$

which concludes the proof of Lemma 4.1. \square

Proof of Theorem 2.1. We use induction, the result being true for $n = 3$ by Lemma 3.1. Now suppose the inequality is true for some integer $n \geq 3$. By Lemmas 1.2, 4.1 and the induction hypothesis, we get:

$$\begin{aligned} S_{n+1}(x) &\leq S_n(x) + x \int_1^{13} t^{-2} S_n(t) dt + x \int_{13}^x t^{-2} S_n(t) dt \\ &\leq x \left\{ \frac{(\log x + n - 2)^{n-1}}{(n-1)!} + \frac{1}{n!} \left(n + \frac{1}{2}\right)^n + \frac{1}{(n-1)!} \int_{13}^x \frac{(\log t + n - 2)^{n-1}}{t} dt \right\} \\ &= x \left\{ \frac{(\log x + n - 2)^n}{n!} + \frac{(\log x + n - 2)^{n-1}}{(n-1)!} \right. \\ &\quad \left. + \frac{1}{n!} \left(\left(n + \frac{1}{2}\right)^n - (n + \log(13e^{-2}))^n \right) \right\} \\ &\leq \frac{x}{n!} \{ (\log x + n - 2)^n + (n-1)(\log x + n - 2)^{n-1} \} \\ &\leq \frac{x}{n!} (\log x + n - 1)^n. \end{aligned}$$

The proof of Theorem 2.1 is now complete. \square



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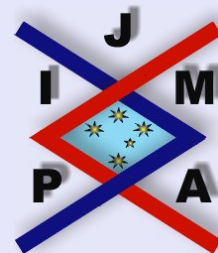
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