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A STUDY OF THE REAL HARDY INEQUALITY

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ABSTRACT. We show that some Hardy-type inequalities on the circle can be proved to be true on the real line. Namely, we discuss the idea of getting Hardy inequalities on the real line by the use of corresponding inequalities on the circle. In the last section, we prove the truth of a certain open problem under some restrictions.

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1. INTRODUCTION

When McGehee, Pigno and Smith [4] proved the Littlewood conjecture, many questions regarding the best possible generalization of Hardy's inequality were asked. The longstanding question is: Does there exist a constant $c > 0$ such that

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq c\|f\|_1 + c\sum_{n=1}^{\infty} \frac{|\hat{f}(-n)|}{n} \tag{1.1}$$

for all $f \in L^1(\mathbb{T})$? The truth of this inequality is an open problem. Many attempts were made to answer this question and many partial results were obtained. We refer the reader to [2], [3], [6] and [7] for some partial results.

Almost all articles in the literature treat Hardy-type inequalities on the circle and a very few articles treat them on the real line.

In [8] it was proved that a constant $c > 0$ exists such that for all

$$f \in \left\{ g \in L^1(\mathbb{R}) : \int_{-\infty}^{x} g(t)dt \in L^1(\mathbb{R}) \right\}$$

we have

$$\int_0^{\infty} \frac{|\hat{f}(\xi)|^2}{\xi} d\xi \leq c\|f\|_1^2 + c\int_0^{\infty} \frac{|\hat{f}(-\xi)|^2}{\xi} d\xi.$$

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Although this is not the first proved Hardy inequality on the real line, its proof is the first proof which uses the construction of a bounded function on $\mathbb{R}$ whose Fourier coefficients have some desired decay properties.

We have two main goals in this article. The first is to prove Hardy inequalities on the real line using well known inequalities on the circle and the second is to prove a real Hardy inequality on the real line which is related to the open problem (1.1).

Proving a Hardy inequality usually involves quite a difficult construction and this is because of the way we prove such inequalities. Again we refer the reader to [2], [4], [6], [7] and [8] for more information on how and why we construct bounded functions with desired Fourier coefficients.

2. SETUP

Let L^1 denote the space of all integrable functions (equivalent classes) defined on $\mathbb{R}$. For $f \in L^1$, we define the Fourier transform of f to be

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} d\xi.$$

If $\hat{f} \in L^1$ then the inversion formula for f holds:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} d\xi.$$

If $f \in L^2$ then the Fourier transform of f is defined to be the L^2- limit:

$$\hat{f}(\xi) = \lim_{n\to\infty} \int_{-n}^{n} f(x) e^{-i\xi x} dx.$$

In this case $\hat{f} \in L^2$ and

$$f(x) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} \hat{f}(\xi) e^{ix\xi} d\xi, \qquad \text{in the } L^2 \text{ sense.}$$

The Plancheral theorem then says:

$$\|f\|_2 = \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_2.$$

Lemma 2.1. *For $f, g \in L^2$ we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)} dx = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi.$$

We refer the reader to any standard book in analysis for more on these concepts, see for example [5].

Observe that if $f \in L^1$ is such that $\hat{f}$ is compactly supported, then $\hat{f} \in L^2$ and hence $f \in L^2$.

Now, given $f \in L^1(\mathbb{R})$, define

$$\varphi_N(t) = 2\pi \sum_{j=-\infty}^{\infty} f_N(t + 2\pi j), \qquad \text{where } f_N(x) = Nf(Nx). \tag{2.1}$$

Then we have

$$\varphi_N \in L^1(\mathbb{T}), \hat{\varphi}_N(n) = \hat{f}\left(\frac{n}{N}\right) \quad \text{and} \quad \lim_{N\to\infty} \|\varphi_N\|_{L^1(\mathbb{T})} = \|f\|_{L^1(\mathbb{R})}. \tag{2.2}$$

For a discussion of this idea we refer the reader to [1, p. 160-162].

Thus, the equations in (2.1) and (2.2) enable us to move from a function integrable on the line to a function integrable on the circle. This idea will be used efficiently in the next section to prove some Hardy-type inequalities on the real line using only a corresponding inequality on the circle.

3. From The Circle To The Line

In this section we discuss how one can use a Hardy inequality on the circle to obtain an inequality on the real line.

Recall that Hardy's inequality on the circle states that a constant $C > 0$ exists such that for all $f \in L^1(\mathbb{T})$ with $\hat{f}(n) = 0, n < 0$ we have

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq C\|f\|_1.$$

As a matter of notation, let $H^1(\mathbb{R})$ be defined by

$$H^1(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : \hat{f}(\xi) = 0,\ \forall \xi < 0 \right\}.$$

In the following theorem we use the above stated Hardy's inequality to get the well known Hardy inequality on the line. We should remark that proving such an inequality (on the line) is a difficult task, but transforming the problem from the circle to the line greatly simplifies the proof.

Theorem 3.1. *There exists an absolute constant $C > 0$ such that*

$$\int_0^{\infty} \frac{|\hat{f}(\xi)|}{\xi} \leq C\|f\|_1$$

for all $f \in H^1(\mathbb{R})$.

Proof. Let $f \in H^1(\mathbb{R})$ be arbitrary and let, for $N \in \mathbb{N}$, φ_N (as in (2.1) and (2.2)) be such that:

$$\varphi_N \in L^1(\mathbb{T}), \quad \lim_{N\to\infty} \|\varphi_N\|_{L^1(\mathbb{T})} = \|f\|_{L^1(\mathbb{R})} \quad \text{and} \quad \hat{\varphi}_N(n) = \hat{f}(n/N).$$

Clearly $\hat{\varphi}_N(n) = 0,\ \forall n < 0$, hence Hardy's inequality applies and we have

$$\sum_{n=1}^{\infty} \frac{|\hat{\varphi}_N(n)|}{n} \leq C\|\varphi_N\|_{L^1(\mathbb{T})}.$$

However, this implies that

$$\sum_{n=1}^{N} \frac{1}{N} \frac{|\hat{f}(n/N)|}{n/N} \leq C\|\varphi_N\|_{L^1(\mathbb{T})}.$$

We take the limit as $N \to \infty$ to get

$$\int_0^{1} \frac{|\hat{f}(\xi)|}{\xi} \leq C\|f\|_1. \tag{3.1}$$

Thus, we have shown (3.1) for any function in $H^1(\mathbb{R})$. Now we proceed to prove the inequality stated in the theorem. That is, we would like to replace the upper limit of the above integral by ∞. For this, let $M \in \mathbb{N}$ be arbitrary and put $h(x) = f(x/M)$. Then

$$\|h\|_1 = M\|f\|_1 \quad \text{and} \quad \hat{h}(\xi) = M\hat{f}(M\xi).$$

Now apply (3.1) on h to obtain

$$\int_0^{1} \frac{|\hat{h}(\xi)|}{\xi} \leq C\|h\|_1 \Rightarrow \int_0^{1} \frac{|\hat{f}(M\xi)|}{\xi} d\xi \leq C\|f\|_1.$$

Put $M\xi = t$ to get

$$\int_0^M \frac{|\hat{f}(\xi)|}{\xi} d\xi \le C\|f\|_1. \tag{3.2}$$

Now, letting M tend to $\infty,$ we obtain the result. □

In fact, the idea of this proof can be used to prove many Hardy inequalities on the real line! It is proved that, see for example [3], for all functions $f \in L^1(\mathbb{T})$ we have

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|^2}{n} \le C\|f\|_1^2 + C\sum_{n=1}^{\infty} \frac{|\hat{f}(-n)|^2}{n}.$$

Using an argument similar to that of Theorem 3.1 we can prove:

Theorem 3.2. *There exists a constant $C > 0$ such that for all functions $f \in L^1(\mathbb{R})$ we have*

$$\int_0^{\infty} \frac{|\hat{f}(\xi)|^2}{\xi} d\xi \le C\|f\|_1^2 + C\int_0^{\infty} \frac{|\hat{f}(-\xi)|^2}{\xi} d\xi.$$

This modifies the result proved in [8].

4. ANOTHER HARDY INEQUALITY

In this section we prove (1.1) on the real line under a certain condition on the signs of the Fourier coefficients.

We should remark here that the method used to prove this inequality is standard and all recent articles use this idea; we need a bounded function whose Fourier coefficients obey some desired decay conditions.

One last remark before proceeding, although the given proof is for a real Hardy inequality, we can imitate the given steps to prove a similar inequality on the circle.

For $j \ge 1$ put

$$f_j(x) = \frac{1}{4^j}\int_{4^{j-1}}^{4^j} e^{ix\xi} d\xi, \quad x \in \mathbb{R}.$$

Then we have our first lemma:

Lemma 4.1. *Let f_j be as above, then*

$$\hat{f}_j(\xi) = \begin{cases} \frac{2\pi}{4^j}, & 4^{j-1} < \xi < 4^j, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let

$$g_j(\xi) = \begin{cases} \frac{2\pi}{4^j}, & 4^{j-1} < \xi < 4^j, \\ 0, & \text{otherwise,} \end{cases}$$

then $g_j \in L^2$. Therefore, g_j is the Fourier transform of some function $h_j \in L^2$ and

$$\begin{aligned} h_j(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} g(\xi)e^{ix\xi} d\xi \\ &= f(x). \end{aligned}$$

However, $\hat{h}_j = g_j$, which implies $\hat{f}_j = g_j$ as claimed. □

Observe that the above proof implies that $f_j \in L^2$ and

Lemma 4.2. *For f_j as above, we have*

$$\|f_j\|_2 = \frac{\sqrt{6\pi}}{4^{j/2}}.$$

Proof. Since $f_j \in L^2$ we have

$$\begin{aligned}
\|f_j\|_2 &= \frac{1}{\sqrt{2\pi}}\|\hat{f}_j\|_2 \\
&= \frac{1}{\sqrt{2\pi}}\frac{2\pi}{4^j}\left(\int_{4^{j-1}}^{4^j} d\xi\right)^{\frac{1}{2}} \\
&= \frac{\sqrt{6\pi}}{4^{j/2}}.
\end{aligned}$$

□

Lemma 4.3. *For f_j as above and for $M \in \mathbb{N}$, let*

$$F_M(x) = \sum_{j=1}^{M} \left| f_j(x) - \overline{f_j(x)} \right|,$$

then $\|F_M\|_\infty \leq C$ where C is some absolute constant (independent of M).

Proof. Observe first that

$$\begin{aligned}
\left| f_j(x) - \overline{f_j(x)} \right| &= \frac{2}{4^j}\left| \int_{4^{j-1}}^{4^j} \sin(x\xi) d\xi \right| \\
&= \frac{2}{4^j}\left| \frac{\cos(4^j x) - \cos(4^{j-1}x)}{x} \right| \\
&= \frac{2}{4^j}\left| \frac{2\sin^2(4^{j-1}x/2)\left[-1 + 16\cos^2(4^{j-1}x)\cos^2(4^{j-1}x/2)\right]}{x} \right| \\
&\leq \frac{60}{4^j}\frac{\sin^2(4^{j-1}x/2)}{|x|}.
\end{aligned}$$

Note that F_M is an even function, so it suffices to consider only the case $x > 0$.

Now, fix $x > 0$ and observe that

$$\begin{aligned}
F_M(x) &\leq 60\sum_{j=1}^{\infty}\frac{1}{4^j}\frac{\sin^2(4^{j-1}x/2)}{x} \\
&= 60\left(\sum_{4^{-j}\leq x}\frac{1}{4^j}\frac{\sin^2(4^{j-1}x/2)}{x} + \sum_{4^{-j}> x}\frac{1}{4^j}\frac{\sin^2(4^{j-1}x/2)}{x} \right),
\end{aligned}$$

where, by convention, the second sum is zero if $4^{-j} \leq x$ for all $j \geq 1$.

Now

$$\sum_{4^{-j}\leq x}\frac{1}{4^j}\frac{\sin^2(4^{j-1}x/2)}{x} \leq \frac{1}{x}\sum_{j=k_x}^{\infty}\frac{1}{4^j},$$

where k_x is the smallest positive integer such that $4^{-j} \leq x$ for all $j \geq k_x$.

Consequently

$$\begin{aligned}\sum_{4^{-j}\le x}\frac{1}{4^j}\frac{\sin^2(4^{j-1}x/2)}{x} &\le \frac{1}{x}\frac{4}{3}\frac{1}{4^{k_x}}\\ &\le \frac{1}{x}\frac{4}{3}x=\frac{4}{3}.\end{aligned}$$

On the other hand, if $\sum_{4^{-j}>x}\frac{1}{4^j}\frac{\sin^2(4^{j-1}x/2)}{x}$ is not zero, we get

$$\begin{aligned}\sum_{4^{-j}>x}\frac{1}{4^j}\frac{\sin^2(4^{j-1}x/2)}{x} &= \sum_{j=1}^{\log_4(1/x)}\frac{1}{4^j}\frac{\sin^2(4^{j-1}x/2)}{x}\\ &\le \sum_{j=1}^{\log_4(1/x)}\frac{1}{4^jx}\left(\frac{1}{2}4^{j-1}x\right)^2\\ &= \frac{1}{64}x\sum_{j=1}^{\log_4(1/x)}4^j\\ &\le \frac{1}{64}x\frac{4^{\log_4(1/x)}-4}{3}\\ &\le \frac{1}{192}.\end{aligned}$$

Thus, we have

$$F_M(x)\le 60\left(\frac{4}{3}+\frac{1}{192}\right):=C.$$

$\square$

Now let X be the set of all $f\in L^1$ such that the sign of $\hat f(\xi)$ is constant in the block $(4^{j-1},4^j)$. Then we have our main result:

Theorem 4.4. *There is an absolute constant $K>0$ such that*

$$\int_1^\infty\frac{|\hat f(\xi)|}{\xi}d\xi\le K\|f\|_1+K\int_1^\infty\frac{|\hat f(-\xi)|}{\xi}d\xi \tag{4.1}$$

for all $f\in X$.

Proof. Let $f\in X$ be such that $\hat f$ is compactly supported, let f_j be as above and let $M\in\mathbb{N}$ be such that the support of $\hat f$ is contained in $[-M,M]$. Denote the sign of $\hat f$ in the block $(4^{j-1},4^j)$ by σ_j and put

$$F(x)=\sum_{j=1}^M\sigma_j\left(f_j(x)-\overline{f_j(x)}\right).$$

Then $\|F\|_\infty\le C$, where C is the constant of Lemma 4.3. Moreover, $\hat F(\xi)=\frac{2\pi}{4^j}\sigma_j$, where j is the unique index such that $\xi\in[4^{j-1},4^j]\cup[-4^j,-4^{j-1}]$ if $-4^M\le\xi\le 4^M$ and $\hat F(\xi)=0$ otherwise.

Moreover, since $\hat{f}$ is of compact support, we have $f \in L^2$. Now we apply a standard duality argument:

$$\begin{aligned} C\|f\|_1 &\geq \left| \int_{\mathbb{R}} f(x)\overline{F(x)}dx \right| \\ &= \frac{1}{2\pi} \left| \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi \right| \text{ (see Lemma 2.1)} \\ &\geq \frac{1}{2\pi} \left| \sum_{j=1}^{\infty} \int_{4^{j-1}}^{4^j} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi \right| - \frac{1}{2\pi} \left| \int_{-1}^{1} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi \right| \\ &\qquad - \frac{1}{2\pi} \left| \sum_{j=1}^{\infty} \int_{4^{j-1}}^{4^j} \hat{f}(-\xi)\overline{\hat{F}(-\xi)}d\xi \right|. \end{aligned}$$

That is,

$$\begin{aligned} \frac{1}{2\pi} \left| \sum_{j=1}^{\infty} \int_{4^{j-1}}^{4^j} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi \right| &\leq C\|f\|_1 + \frac{1}{2\pi} \left| \int_{-1}^{1} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi \right| \\ &\qquad + \frac{1}{2\pi} \left| \sum_{j=1}^{\infty} \int_{4^{j-1}}^{4^j} \hat{f}(-\xi)\overline{\hat{F}(-\xi)}d\xi \right|. \end{aligned} \tag{4.2}$$

However, when $\xi \in (4^{j-1}, 4^j) \cup (-4^j, 4^{j-1})$, we have $\hat{F}(\xi) = \frac{2\pi}{4^j}\sigma_j$, where σ_j is the sign of $\hat{f}$ in $(4^j, 4^{j-1})$. Thus, when $\xi \in (4^{j-1}, 4^j)$ we have $\hat{f}(\xi)\overline{\hat{F}(\xi)} = |\hat{f}(\xi)|$. Moreover

$$\begin{aligned} \left| \int_{-1}^{1} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi \right| &\leq \|f\|_1 \int_{-1}^{1} \left|\hat{F}(\xi)\right| d\xi \\ &\leq \|f\|_1 \left(\int_{-1}^{1} d\xi \right)^{\frac{1}{2}} \left(\int_{-1}^{1} |\hat{F}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq \sqrt{2}\|f\|_1 \|\hat{F}\|_2 \\ &= \sqrt{2}\|f\|_1 \sqrt{2\pi}\|F\|_2 \\ &\leq 2\sqrt{\pi}\|f\|_1 \times 2 \sum_{j=1}^{\infty} \|f_j\|_2 \\ &= 4\sqrt{6}\pi\|f\|_1, \end{aligned}$$

where we have used the facts

$$F(x) = \sum_{j=1}^{M} \sigma_j \left(f_j(x) - \overline{f_j(x)} \right) \quad \text{and} \quad \|f_j\|_2 = \sqrt{6\pi}4^{-j/2}.$$

Consequently, (4.2) becomes

$$\sum_{j=1}^{\infty} \int_{4^{j-1}}^{4^j} \frac{|\hat{f}(\xi)|}{4^j} d\xi \leq C\|f\|_1 + 2\sqrt{6}\|f\|_1 + \sum_{j=1}^{\infty} \int_{4^{j-1}}^{4^j} \frac{|\hat{f}(-\xi)|}{4^j} d\xi. \tag{4.3}$$

However, when $\xi \in [4^{j-1}, 4^j]$ we have

$$\frac{1}{4^j} \leq \frac{1}{\xi} \quad \text{and} \quad \frac{1}{4^j} \geq \frac{1}{4\xi}.$$

Therefore, (4.3) boils down to

$$\int_1^\infty \frac{|\hat{f}(\xi)|}{\xi} \leq K\|f\|_1 + K\int_1^\infty \frac{|\hat{f}(-\xi)|}{\xi} d\xi,$$

where $K = 4(C + 2\sqrt{6})$ and where C is the constant of Lemma 4.3.

This completes the proof for $f \in X$ with the property that $\hat{f}$ is compactly supported. Now for general $f \in X$, consider the convolution $f * K_\lambda$ where K_λ is the Fejer Kernel of order λ. A standard limiting process yields the result. $\square$

Remark:

(1) We note that the above proof can be adopted to prove that: There is an absolute constant $C' > 0$ such that for any function $f \in L^1(\mathbb{T})$ whose Fourier coefficients have the same sign on the block $[4^{j-1}, 4^j)$ we have

$$\sum_{n=1}^\infty \frac{|\hat{f}(n)|}{n} \leq C'\|f\|_1 + C'\sum_{n=1}^\infty \frac{|\hat{f}(-n)|}{n}.$$

(2) Observe that the condition that $\hat{f}$ has the same sign on the block $(4^{j-1}, 4^j)$ is flexible. This is because functions in L^p are in fact equivalent classes. Therefore, even if $\hat{f}$ obeys our condition but for a set of measure zero, then we may modify our choice by changing the values of $\hat{f}$ so that the new $\hat{f}$ satisfies our condition.
We should remark that this process does not effect the proof above.

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