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HYERS-ULAM STABILITY OF THE GENERALIZED TRIGONOMETRIC FORMULAS

AHMED REDOUANI[1], ELHOUCIEN ELQORACHI[1], AND BELAID BOUIKHALENE[2]

[1]LABORATORY LAMA
HARMONIC ANALYSIS AND FUNCTIONAL EQUATIONS TEAM
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES, UNIVERSITY OF IBN ZOHR
AGADIR, MOROCCO
redouani_ahmed@yahoo.fr

elqorachi@hotmail.com

[2]LABORATORY LAMA, DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES, UNIVERSITY OF IBN TOFAIL
KENITRA, MOROCCO
bbouikhalene@yahoo.fr

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ABSTRACT. In this paper, we will investigate the Hyers-Ulam stability of the following functional equations

$$\int_G\int_K f(xtk\cdot y)dkd\mu(t) = f(x)g(y)+g(x)f(y),\quad x,y\in G$$

and

$$\int_G\int_K f(xtk\cdot y)dkd\mu(t) = f(x)f(y)-g(x)g(y),\quad x,y\in G,$$

where K is a compact subgroup of morphisms of G, dk is a normalized Haar measure of K, μ is a complex K-invariant measure with compact support, the functions f,g are continuous on G and f is assumed to satisfies the Kannappan type condition $K(\mu)$

$$\int_G\int_G f(ztxsy)d\mu(t)d\mu(s) = \int_G\int_G f(ztysx)d\mu(t)d\mu(s),\ x,y,z\in G.$$

The paper of Székelyhidi [30] is the essential motivation for the present work and the methods used here are closely related to and inspired by those in [30].
The concept of the generalized Hyers-Ulam stability of mappings was introduced in the subject of functional equations by Th. M. Rassias in [20].

Key words and phrases: Locally compact group, Functional equation, Hyers-Ulam stability, Superstability.

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1. INTRODUCTION

The Hyers-Ulam stability problem for functional equation has its origin in the following question posed by S. Ulam [41] in 1940.

Given a group G and a metric group (G', d) and given a number $\varepsilon > 0$, does there exist a $\delta > 0$ such that, if $f : G \longrightarrow G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta, \text{ for all } x, y \in G,$$

then a homomorphism $a : G \longrightarrow G'$ exists such that

$$d(f(x), a(x)) < \varepsilon, \quad \text{for all } x \in G?$$

The first affirmative answer to Ulam's question for linear mappings came within a year when D. H. Hyers [8] proved the following result.

Theorem 1.1 ([8]). *Let B and B' be Banach spaces and let $f : B \longrightarrow B'$ be a function such that for some $\delta > 0$*

$$\| f(x+y) - f(x) - f(y) \| \leq \delta, \text{ for all } x, y \in B.$$

Then there exists a unique additive function $\varphi : B \longrightarrow B'$ such that $\| f(x) - \varphi(x) \| \leq \delta$, for all $x \in G$.
Furthermore, the continuity of f at a point $y \in B$ implies the continuity of φ on B. The continuity, for each $x \in B$, of the function $t \longrightarrow f(tx)$, $t \in \mathbb{R}$, implies the homogeneity of φ.

After Hyers's result a great number of papers on the subject have been published, generalizing Ulam's problem and Hyers's theorem in various directions. In 1951 D.G. Bourgin [3] treated this problem for additive mappings. In 1978, Th. M. Rassias [20] provided a remarkable generalization of Hyers's theorem, a fact which rekindled interest in the field of functional equations.

Theorem 1.2 ([20]). *Let $f : V \longrightarrow X$ be a mapping between Banach spaces and let $p < 1$ be fixed. If f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \geq 0$ and for all $x, y \in V$ ($x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique additive mapping $T : V \longrightarrow X$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{|2 - 2^p|} \|x\|^p$$

for all $x \in V$ ($x \in V \setminus \{0\}$ if $p < 0$).
If, in addition, $f(tx)$ is continuous in t for each fixed x, then T is linear.

This theorem of Th. M. Rassias stimulated several mathematicians working in the theory of functional equations to investigate this kind of stability for a variety of significant functional equations. By taking into consideration the influence of S. M. Ulam, D. H. Hyers and Th. M. Rassias on the study of stability problems of functional equations in mathematical analysis, the stability phenomenon that was proved by Th. M. Rassias is called the Hyers-Ulam-Rassias stability.

The Hyers-Ulam-Rassias stability was taken up by a number of mathematicians and the study of this area has the grown to be one of the central subjects in the mathematical analysis area. For more information, we can see for examples ([3], [7], [8], [10], [12], ..., [40]) and the monographs [4], [9], [11] by D. H. Hyers, G. Isac and Th. M. Rassias, by S.-M. Jung and by S. Czerwik (ed.).

L. Székelyhidi in [30], studied the stability property of two well known functional equations: The sine and cosine functional equations

$$f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in G \tag{1.1}$$

and

$$f(xy) = f(x)f(y) - g(x)g(y), \quad x, y \in G, \tag{1.2}$$

where f, g are complex-valued functions on an amenable group G. More precisely, he proved that if $f, g : G \longrightarrow \mathbb{C}$ are given functions, G is an amenable group, and the function $(x, y) \longrightarrow f(xy) - f(x)g(y) - f(y)g(x)$ is bounded, then there exists a solution (f_0, g_0) of (1.1) such that $f - f_0$ and $g - g_0$ are bounded. An analogous result holds for equation (1.2).

The aim of the present paper is to extend the Székelyhidi's results [30] to the functional equations

$$\int_G \int_K f(xtk \cdot y) dk d\mu(t) = f(x)g(y) + g(x)f(y), \quad x, y \in G \tag{1.3}$$

and

$$\int_G \int_K f(xtk \cdot y) dk d\mu(t) = f(x)f(y) + g(x)g(y), \quad x, y \in G, \tag{1.4}$$

where K is a compact subgroup of $Mor(G)$, μ is a complex K-invariant measure with compact support, f, g are continuous functions on G and f is assumed to satisfy the Kannappan type condition $K(\mu)$

$$\int_G \int_G f(ztxsy) d\mu(t) d\mu(s) = \int_G \int_G f(ztysx) d\mu(t) d\mu(s), \quad x, y, z \in G.$$

Furthermore, in the last subsection we study a superstability result of the generalized quadratical functional equation

$$\int_G \int_K f(xtk \cdot y) dk d\mu(t) = f(x) + f(y), \quad x, y \in G. \tag{1.5}$$

The result can be viewed as a generalization of the ones obtained by G. Maksa and Z. Páles in [12].

2. NOTATION AND PRELIMINARY RESULTS

Our notation is described in the following Set Up and it will be used throughout the paper.

Set-Up. We let G be a locally compact group, $C(G)$ (resp. $C_b(G)$) the complex algebra of all continuous (resp. continuous and bounded) complex valued functions on G. $M(G)$ denotes the topological dual of $C_0(G)$: the Banach space of continuous functions vanishing at infinity. We let K be a compact subgroup of the group $\mathrm{Mor}(G)$ of all mappings k of G onto itself that are either automorphisms and homeomorphisms (i.e. $k \in K^+$), or anti-automorphisms and homeomorphisms (i.e. $k \in K^-$). The action of $k \in K$ on $x \in G$ will be denoted by $k \cdot x$ and the normalized Haar measure on K by dk.

For any function f on G, we put $(k \cdot f)(x) = f(k^{-1} \cdot x)$. For any $\mu \in M(G)$, $k \in K$ and any $f \in C_b(G)$, we put $\langle k \cdot \mu, f \rangle = \langle \mu, k \cdot f \rangle$, and we say that μ is K-invariant if $k \cdot \mu = \mu$, for all $k \in K$.

A non-zero function $\phi \in C_b(G)$ is said to be a solution of Badora's functional equation if it satisfies

$$\int_K \int_G \phi(xtk \cdot y) d\mu(t) dk = \phi(x)\phi(y), \quad x, y \in G. \tag{2.1}$$

Recently the functional equation (2.1) was completely solved in abelian groups by Badora [2] and by E. Elqorachi, M. Akkouchi, A. Bakali, and B. Bouikhalene [6] in non-abelian groups and the Hyers-Ulam-Rassias stability of this equation was investigated in [1] and [5].

In the following, we prove some lemmas that we need later.

Lemma 2.1. *Let K be a compact subgroup of $\mathrm{Mor}(G)$. Let μ be a K-invariant bounded measure on G. If $f \in C_b(G)$ satisfies the Kannappan condition $K(\mu)$:*

$$\int_G \int_G f(ztxsy)d\mu(t)d\mu(s) = \int_G \int_G f(ztysx)d\mu(t)d\mu(s), \quad x, y, z \in G,$$

then we have

$$\int_K \int_K \int_G \int_G f(zsk \cdot (xtk' \cdot y))dkdk'd\mu(s)d\mu(t) = \int_K \int_K \int_G \int_G f(zsk \cdot xtk' \cdot y)dkdk'd\mu(s)d\mu(t),$$

for all $x, y, z \in G$.

Proof. Let $x, y, z \in G$. Let $f \in C_b(G)$ be a complex function such that f satisfies $K(\mu)$. Then

$$\begin{aligned}\int_K \int_K \int_G \int_G & f(zsk \cdot (xtk' \cdot y))dkdk'd\mu(s)d\mu(t) \\ &= \int_{K^+} \int_K \int_G \int_G f(zsk \cdot xk \cdot t(kk') \cdot y)dkdk'd\mu(s)d\mu(t) \\ &\quad + \int_{K^-} \int_K \int_G \int_G f(zs(kk') \cdot yk \cdot tk \cdot x)dkdk'd\mu(s)d\mu(t).\end{aligned}$$

Since μ is K-invariant and dk' is invariant by translation, then we get

$$\int_{K^+} \int_K \int_G \int_G f(zsk \cdot xk \cdot t(kk') \cdot y)dkdk'd\mu(s)d\mu(t) = \int_{K^+} \int_K \int_G \int_G f(zsk \cdot xtk' \cdot y)dkdk'd\mu(s)d\mu(t),$$

$$\begin{aligned}\int_{K^-} \int_K \int_G \int_G & f(zs(kk') \cdot yk \cdot tk \cdot x)dkdk'd\mu(s)d\mu(t) \\ &= \int_{K^-} \int_K \int_G \int_G f(zsk' \cdot ytk \cdot x)dkdk'd\mu(s)d\mu(t) \\ &= \int_{K^-} \int_K \int_G \int_G f(zsk \cdot xtk' \cdot y)dkdk'd\mu(s)d\mu(t),\end{aligned}$$

because f satisfies $K(\mu)$.

Consequently,

$$\begin{aligned}\int_K\int_K\int_G\int_G & f(zsk\cdot(xtk'\cdot y))dkdk'd\mu(s)d\mu(t)\\ &= \int_{K^+}\int_K\int_G\int_G f(zsk\cdot xtk'\cdot y)dkdk'd\mu(s)d\mu(t)\\ &\quad + \int_{K^-}\int_K\int_G\int_G f(zsk\cdot xtk'\cdot y)dkdk'd\mu(s)d\mu(t)\\ &= \int_K\int_K\int_G\int_G f(zsk\cdot xtk'\cdot y)dkdk'd\mu(s)d\mu(t).\end{aligned}$$

This completes the proof. □

The following result is a generalization of the lemma obtained by G. Maksa and Z. Páles in [12].

Lemma 2.2. *Let* K *be a compact subgroup of* $\mathrm{Mor}(G)$*. Let* μ *be a* K*-invariant bounded measure on* G *such that* $\langle\mu, 1_G\rangle = 1$*. Let* $f \in C_b(G)$ *be a complex function which satisfies* $K(\mu)$*, then the continuous and bounded function*

$$L(x,y) = f(x) + f(y) - \int_G\int_K f(xtk\cdot y)dkd\mu(t), \quad x,y\in G \tag{2.2}$$

satisfies the functional equation

$$\begin{aligned}L(x,y) + \int_G\int_K & L((xtk\cdot y),z)dkd\mu(t)\\ &= L(y,z) + \int_G\int_K L(x,(ytk\cdot z))dkd\mu(t), \quad x,y,z\in G.\end{aligned} \tag{2.3}$$

Proof. The proof is closely related to the computation in ([12, Section 2, Lemma]), where K is a finite subgroup of $Aut(G)$ and $\mu = \delta_e$). Let f be a bounded and continuous function on G which satisfies the Kannappan condition $K(\mu)$ and let $L(x,y)$ be the function defined by (2.2), then we have

$$\begin{aligned}L(x,y)+\int_K\int_G & L((xtk\cdot y),z)dkd\mu(t)\\ &= f(x)+f(y)-\int_K\int_G f(xtk\cdot y)dkd\mu(t)\\ &\quad + \int_K\int_G f(xtk\cdot y)dkd\mu(t) + \langle\mu,1_G\rangle\langle dk,1_K\rangle f(z)\\ &\quad - \int_K\int_G\int_K\int_G f(xtk\cdot ysk'\cdot z)dkdk'd\mu(s)d\mu(t)\\ &= f(x)+f(y)+f(z)-\int_K\int_G\int_K\int_G f(xtk\cdot ysk'\cdot z)dkdk'd\mu(s)d\mu(t).\end{aligned}$$

On the other hand, we have

$$\begin{aligned} L(y,z)+&\int_K\int_G L(x,(ytk\cdot z))dkd\mu(t)\\ &= f(y)+f(z)-\int_K\int_G f(ytk\cdot z)dkd\mu(t)\\ &\qquad + \langle \mu, 1_G\rangle \langle dk, 1_K\rangle f(x) + \int_K\int_G f(ytk\cdot z)dkd\mu(t)\\ &\qquad - \int_K\int_G\int_K\int_G f(xsk'\cdot(ytk\cdot z))dkdk'd\mu(s)d\mu(t)\\ &= f(y)+f(z)+f(x)-\int_K\int_G\int_K\int_G f(xsk'\cdot ytk\cdot z)dkdk'd\mu(s)d\mu(t). \end{aligned}$$

This ends the proof of Lemma 2.2. □

3. STABILITY OF EQUATION (1.3)

In this section, we investigate the stability properties of the functional equation (1.3), it's a generalization of the stability of equation (1.1) proved by Székelyhidi in [30].

Theorem 3.1. *Let K be a compact subgroup of $\mathrm{Mor}(G)$, and let μ be a K-invariant measure with compact support. Let f, g be continuous complex-valued functions such that f satisfies $K(\mu)$ and the following function*

$$(G,G)\ni(x,y)\longrightarrow\int_K\int_G f(xtk\cdot y)dkd\mu(t)-f(x)g(y)-f(y)g(x) \tag{3.1}$$

is bounded. Then

i) *$f=0$, g arbitrary in $C(G)$ or*
ii) *f, g are bounded or*
iii) *f is unbounded, g is a bounded solution of Badora's equation or*
iv) *There exists φ a solution of Badora's equation, there exists b a continuous bounded function on G and $\gamma\in\mathbb{C}$ such that $f=\gamma(\varphi-b)$ and $g=\frac{\varphi+b}{2}$ or*
v) *f, g are solutions of (1.3).*

Proof. If $f=0$, then g can be chosen arbitrarily in $C(G)$. This is case (i). If $f\neq 0$ is bounded, then the function $G\ni x\longmapsto f(x)g(y)+f(y)g(x)$ is bounded for all $y\in G$, so g is bounded. This is case (ii). If f is unbounded and g is bounded, the function

$$G\ni x\longmapsto\int_K\int_G f(xtk\cdot y)dkd\mu(t)-f(x)g(y)$$

is bounded, for all $y\in G$. In view of [5, Theorem 3.1], we get that g is a solution of Badora's equation. This is case (iii). If f, g are unbounded functions, we distinguish two cases:
First case. We assume that there exist $\alpha,\beta\in\mathbb{C}\backslash\{0\}$ such that $\alpha f+\beta g$ is bounded, then g can be written as $g=\frac{f}{2\gamma}+b$, where b is a bounded function and $\gamma\in\mathbb{C}\backslash\{0\}$. Consequently, the function

$$G\ni x\longmapsto\int_K\int_G f(xtk\cdot y)dkd\mu(t)-\left(\frac{f(y)}{\gamma}+b(y)\right)f(x)$$

is bounded, for all $y\in G$. Hence by [5, Theorem 3.1], it follows that $\varphi(y)=\frac{f(y)}{\gamma}+b(y)$ is a solution of Badora's equation. This is case (iv).
Second case. For all $\alpha,\beta\in\mathbb{C}\setminus\{0\}$, $\alpha f+\beta g$ is an unbounded function on G. In this case

we shall prove that f, g are solutions of (1.3). The idea of the proof is closely inspired by some good computations used in [30, Lemma 2.2]. We define the mapping

$$F(x,y) = \int_K \int_G f(xtk \cdot y) dk d\mu(t) - f(x)g(y) - f(y)g(x), \quad x, y \in G$$

and we will prove that $F(x,y) = 0$, for all $x, y \in G$. By assumption, there exist $\gamma, \delta, \lambda \in \mathbb{C}$ and $a \in G$ such that

$$g(x) = \gamma f(x) + \delta \int_K \int_G f(xtk \cdot a) dk d\mu(t) + \lambda F(x,a), \quad x \in G. \tag{3.2}$$

For all $x, y, z \in G$, we have

$$\begin{aligned}
&\int_K \int_K \int_G \int_G f((xtk \cdot y)sk' \cdot z) dk dk' d\mu(t) d\mu(s) \\
&\quad = g(z) \int_K \int_G f(xtk \cdot y) dk d\mu(t) + f(z) \int_K \int_G g(xtk \cdot y) dk d\mu(t) \\
&\qquad + \int_K \int_G F((xtk \cdot y), z) dk d\mu(t) \\
&\quad = g(z)f(x)g(y) + g(z)f(y)g(x) + g(z)F(x,y) + \gamma f(z) \int_K \int_G f(xtk \cdot y) dk d\mu(t) \\
&\qquad + \delta f(z) \int_K \int_K \int_G \int_G f(xtk \cdot ysk' \cdot a) dk dk' d\mu(s) d\mu(t) \\
&\qquad + \lambda f(z) \int_K \int_G F((xtk \cdot y), a) dk d\mu(t) + \int_K \int_G F((xtk \cdot y), z) dk d\mu(t).
\end{aligned}$$

In view of Lemma 2.1, we get

$$\begin{aligned}
&\int_K \int_K \int_G \int_G f((xtk \cdot y)sk' \cdot z) dk dk' d\mu(t) d\mu(s) \\
&\quad = g(z)f(x)g(y) + g(z)f(y)g(x) \\
&\qquad + g(z)F(x,y) + \gamma f(z) \int_K \int_G f(xtk \cdot y) dk d\mu(t) \\
&\qquad + \delta f(z) \int_K \int_K \int_G \int_G f(xtk \cdot ysk' \cdot a) dk dk' d\mu(s) d\mu(t) \\
&\qquad + \lambda f(z) \int_K \int_G F((xtk \cdot y), a) dk d\mu(t) + \int_K \int_G F((xtk \cdot y), z) dk d\mu(t) \\
&\quad = g(z)f(x)g(y) + g(z)f(y)g(x) + g(z)F(x,y) + \gamma f(z)f(x)g(y) \\
&\qquad + \gamma f(z)f(y)g(x) + \gamma f(z)F(x,y) + \delta f(z)f(x) \int_K \int_G g(ysk' \cdot a) dk' d\mu(s) \\
&\qquad + \delta f(z)g(x) \int_K \int_G f(ysk' \cdot a) dk' d\mu(s) + \delta f(z) \int_K \int_G F(x, ysk' \cdot a) dk' d\mu(s) \\
&\qquad + \lambda f(z) \int_K \int_G F((xtk \cdot y), a) dk d\mu(t) + \int_K \int_G F((xtk \cdot y), z) dk d\mu(t).
\end{aligned}$$

By using again Lemma 2.1, we obtain

$$\begin{aligned}
&\int_K\int_K\int_G\int_G f((xtk\cdot y)sk'\cdot z)dkdk'd\mu(t)d\mu(s)\\
&\qquad= \int_K\int_K\int_G\int_G f(xtk\cdot(ysk'\cdot z))dkdk'd\mu(t)d\mu(s)\\
&\qquad= f(x)\int_K\int_G g(ysk'\cdot z)dk'd\mu(s) + g(x)\int_K\int_G f(ysk'\cdot z)dk'd\mu(s)\\
&\qquad\quad + \int_K\int_G F(x,(ysk'\cdot z))dk'd\mu(s).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&f(x)\Big[g(y)g(z)+\gamma g(y)f(z)+\delta f(z)\int_K\int_G g(ysk'\cdot a)dk'd\mu(s)\\
&- \int_K\int_G g(ysk'\cdot z)dk'd\mu(s)\Big] + g(x)\Big[f(y)g(z)+\gamma f(y)f(z)\\
&+\delta f(z)\int_K\int_G f(ysk'\cdot a)dk'd\mu(s) - \int_K\int_G f(ysk'\cdot z)dk'd\mu(s)\Big]\\
&= \int_K\int_G F(x,(ysk'\cdot z))dk'd\mu(s) - g(z)F(x,y) - \gamma f(z)F(x,y)\\
&\quad - \delta f(z)\int_K\int_G F(x,(ysk'\cdot a))dk'd\mu(s) - \lambda f(z)\int_K\int_G F((xtk\cdot y),a)dkd\mu(t)\\
&\quad - \int_K\int_G F((xtk\cdot y),z)dk'd\mu(s).
\end{aligned}$$

Since the right-hand side is bounded as a function of x for all fixed $y,z\in G$, then we get

$$\begin{aligned}
&g(z)F(x,y)+f(z)\Big[\gamma F(x,y)+\delta\int_K\int_G F(x,ysk'\cdot a))dk'd\mu(s)\\
&\qquad + \lambda\int_K\int_G F((xtk\cdot y),a)dkd\mu(t)\Big]\\
&\qquad = \int_K\int_G F(x,(ysk'\cdot z)))dk'd\mu(s) - \int_K\int_G F((xtk\cdot y),z)dkd\mu(t).
\end{aligned}$$

Since the right-hand side is bounded as a function of z for all fixed $x,y\in G$, then we obtain $F(x,y)=0$, for all $x,y\in G$. This is case (v) and the proof of Theorem 3.1 is completed. □

4. STABILITY OF EQUATION (1.4)

In this section, we study the problem of the Hyers-Ulam stability of equation (1.4). It is a generalization of the stability of equation (1.2) proved by Székelyhidi in [30].

Theorem 4.1. *Let K be a compact subgroup of $\mathrm{Mor}(G)$, let μ be a K-invariant measure with compact support. Let f,g be continuous complex-valued functions such that f satisfies $K(\mu)$ and the function*

$$(G,G)\ni(x,y)\longrightarrow \int_K\int_G f(xtk\cdot y)dkd\mu(t) - f(x)f(y)+g(x)g(y) \tag{4.1}$$

is bounded. Then,

i) *f,g are bounded or*

ii) *f is a solution of Badora's equation, g is bounded, or*
iii) *f, g are unbounded, $f+g$ or $f-g$ are bounded solutions of Badora's equation or*
iv) *There exists φ a solution of Badora's equation, there exists b a continuous bounded function on G and $\gamma \in \mathbb{C} \setminus \{\pm 1\}$ such that*

$$f = \frac{\gamma^2\varphi - b}{\gamma^2 - 1}, \quad g = \frac{\gamma}{\gamma^2 - 1}(\varphi - b)$$

or

v) *f, g are solutions of (1.4).*

Proof. If g is bounded, then we obtain that the function

$$G \times G \ni (x, y) \longmapsto \int_K \int_G f(xtk \cdot y) dk d\mu(t) - f(x)f(y)$$

is bounded. So by [5, Theorem 3.1], we have either f is bounded or f is a solution of Badora's equation. This is cases (i) and (ii). If g is unbounded, then f is unbounded. As in the preceding proof, we distinguish two cases.

First case. Assume that there exist $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that $\alpha f + \beta g$ is a bounded function on G, then there exists a constant $\gamma \in \mathbb{C} \setminus \{0\}$ such that $f = \gamma g + b$, where b is a bounded function on G. Hence the function

$$G \ni x \longmapsto \int_K \int_G g(xtk \cdot y) dk d\mu(t) - \frac{(\gamma^2 - 1)g(y) + \gamma b(y)}{\gamma} g(x)$$

is bounded for all $y \in G$. It follows from [5, Theorem 3.1] that $\varphi = \frac{\gamma^2 - 1}{\gamma} g + b$ is a solution of Badora's equation. Hence, we obtain case (iii) for $\gamma^2 = 1$ and (iv) for $\gamma^2 \neq 1$.

Second case. For all $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\alpha f + \beta g$ is an unbounded function on G. We put

$$H(x, y) = \int_K \int_G f(xtk \cdot y) dk d\mu(t) - f(x)f(y) + g(x)g(y), \quad x, y \in G$$

and follow some computation used by Székelyhidi in [30]. There exists $\gamma, \delta, \lambda \in \mathbb{C}$ and $a \in G$ such that

$$g(x) = \gamma f(x) + \delta \int_K \int_G f(xtk \cdot a) dk d\mu(t) + \lambda H(x, a), \quad x \in G.$$

Now, for all $x, y, z \in G$, we get

$$\begin{aligned}
&\int_K \int_K \int_G \int_G f((xsk \cdot y)tk' \cdot z) dk dk' d\mu(t) d\mu(s) \\
&= f(z) \int_K \int_G f(xsk \cdot y) dk d\mu(s) - g(z) \int_K \int_G g(xsk \cdot y) dk d\mu(s) \\
&\quad + \int_K \int_G H((xsk \cdot y), z) dk d\mu(s) \\
&= f(x)f(y)f(z) - g(x)g(y)f(z) + f(z)H(x, y) - \gamma f(x)f(y)g(z) \\
&\quad + \gamma g(x)g(y)g(z) - \gamma g(z)H(x, y) - \delta g(z)f(x) \int_K \int_G f(ytk' \cdot a) dk' d\mu(t) \\
&\quad + \delta g(x)g(z) \int_K \int_G g(ytk' \cdot a) dk' d\mu(t) - \delta g(z) \int_K \int_G H(x, (ytk' \cdot a)) dk' d\mu(t) \\
&\quad - \lambda g(z) \int_K \int_G H((xsk \cdot y), a) dk d\mu(s) + \int_K \int_G H((xsk \cdot y), z) dk d\mu(s).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_K\int_K\int_G\int_G & f(xsk\cdot ytk'\cdot z)dkdk'd\mu(t)d\mu(s)\\ &= \int_K\int_K\int_G\int_G f(xsk\cdot(ytk'\cdot z))dkdk'd\mu(t)d\mu(s)\\ &= f(x)\int_K\int_G f(ytk'\cdot z)dk'd\mu(t)\\ &\qquad - g(x)\int_K\int_G g(ytk'\cdot z)dk'd\mu(t) + \int_K\int_G H(x,(ytk'\cdot z))dk'd\mu(t).\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}f(x)\Bigg[&f(y)f(z)-\gamma f(y)g(z)-\delta g(z)\int_K\int_G f(ytk'\cdot a))dk'd\mu(t)\\ &-\int_K\int_G f(ytk'\cdot a))dk'd\mu(t)\Bigg]-g(x)\Bigg[g(y)f(z)-\gamma g(y)g(z)\\ &-\delta g(z)\int_K\int_G g(ytk'\cdot a))dk'd\mu(t)-\int_K\int_G g(ytk'\cdot z)dk'd\mu(t)\Bigg]\\ &=\int_K\int_G H(x,(ytk'\cdot z))dk'd\mu(t)-f(z)H(x,y)+\gamma g(z)H(x,y)\\ &\qquad+\delta g(z)\int_K\int_G H(x,(ytk'\cdot a))dk'd\mu(t)\\ &\qquad\quad+\lambda g(z)\int_K\int_G H((xsk\cdot y),a)dkd\mu(s)\\ &\qquad\qquad-\int_K\int_G H((xsk\cdot y),z)dkd\mu(s).\end{aligned}$$

Since the right hand side is bounded as a function of x for all fixed $y,z\in G$, then we get

$$\begin{aligned}f(z)[-H(x,y)]+g(z)\Bigg[&\gamma H(x,y)+\delta\int_K\int_G H(x,(ytk'\cdot a))dk'd\mu(t)\\ &+\lambda\int_K\int_G H((xsk\cdot y),a)dkd\mu(s)\Bigg]\\ &=\int_K\int_G H(xsk\cdot y),z)dkd\mu(s)-\int_K\int_G H(x,(ytk'\cdot z))dk'd\mu(t).\end{aligned}$$

Since the right-hand side is bounded as a function of z for all fixed $x,y\in G$, we conclude that $H(x,y)=0,$ for all $x,y\in G$, which is case (v). This ends the proof of the theorem. □

5. SUPERSTABILITY OF EQUATION (1.5)

In this subsection, we study a superstability of the functional equation

$$\int_K\int_G f(xtk\cdot y))dkd\mu(t)=f(x)+f(y)\ x,y\in G. \tag{5.1}$$

Theorem 5.1. *Let μ be a $K-$invariant measure with compact support. Let $\delta: G\times G\longmapsto\mathbb{R}^+$ be an arbitrary function and assume that there exists a sequence $(u_n)\in G$ such that*

$$\lim_{n\longrightarrow+\infty}\delta(u_nx,y)=0,\ \textit{for all}\ \ x,y\in G\ \textit{(uniform convergence).}$$

Let $f : G \longmapsto \mathbb{C}$ *be a continuous function, which satisfies the Kannappan type condition* $K(\mu)$*. If* f *satisfies the inequality*

$$\left|\int_K \int_G f(xtk \cdot y))dkd\mu(t) - f(x) - f(y)\right| \leq \delta(x,y), \quad \textit{for all} \;\; x, y \in G, \tag{5.2}$$

then f *is a solution of equation (5.1).*

Proof. Assume that $f \in C(G)$ is such that f satisfies $K(\mu)$ and inequality (5.2). It follows that there exists a sequence u_n such that $\lim_{n\longrightarrow+\infty} L(u_n x, y) = 0$ (uniformly). Now, by Lemma 2.2, we get

$$\begin{aligned} L(u_n x, y) + \int_G \int_K L((u_n xtk \cdot y), z)dkd\mu(t) \\ = L(y, z) + \int_G \int_K L(u_n x, (yt \cdot z))dkd\mu(t), \end{aligned} \tag{5.3}$$

for all $x, y, z \in G$ and $n \in \mathbb{N}$. By letting $n \longrightarrow +\infty$, we deduce the desired result and the proof of the theorem is complete. □

Remark 5.2. If K is a compact subgroup of $Aut(G)$, the condition $K(\mu)$ is not necessary.

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