

“NON-STRICT” L’HOSPITAL-TYPE RULES FOR MONOTONICITY: INTERVALS OF CONSTANCY

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ABSTRACT. Assuming that a “derivative” ratio $\rho := f'/g'$ of the ratio $r := f/g$ of differentiable functions f and g is strictly monotonic (that is, ρ is increasing or decreasing), it was shown in previous papers that then r can switch at most once, from decrease to increase or vice versa. In the present paper, it is shown that, if ρ is non-strictly monotonic (that is, non-increasing or non-decreasing), then r can have at most one maximal interval of constancy (m.i.c.); on the other hand, any one m.i.c. of a given derivative ratio ρ is the m.i.c. of an appropriately constructed original ratio r .

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1. INTRODUCTION

Let f and g be differentiable functions defined on the interval (a, b) , where $-\infty \leq a < b \leq \infty$, and let

$$r := \frac{f}{g} \quad \text{and} \quad \rho := \frac{f'}{g'}.$$

It is assumed throughout that g and g' do not take on the value zero anywhere on (a, b) . The function ρ may be referred to as a *derivative ratio* for the “original” ratio r . In [16], general “rules” for monotonicity patterns, resembling the usual l’Hospital rules for limits, were given. In particular, according to [16, Proposition 1.9 and Remark 1.14], one has the dependence of the monotonicity pattern of r (on (a, b)) on that of ρ (and also on the sign of gg') as given by Table 1.1. The vertical double line in the table separates the conditions (on the left) from the corresponding conclusions (on the right).

Here, for instance, $r \searrow \swarrow$ means that there is some $c \in [a, b]$ such that $r \searrow$ (that is, r is non-increasing) on (a, c) and $r \swarrow$ (r is non-decreasing) on (c, b) ; in particular, if $c = a$ then $r \searrow \swarrow$ simply means that $r \swarrow$ on the entire interval (a, b) ; and if $c = b$ then $r \searrow \swarrow$ means that $r \searrow$ on

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ρ	gg'	r
\nearrow	> 0	$\searrow \nearrow$
\searrow	> 0	$\nearrow \searrow$
\nearrow	< 0	$\nearrow \searrow$
\searrow	< 0	$\searrow \nearrow$

Table 1.1: "Non-strict" general rules for monotonicity.

(a, b) . Thus, if one also knows whether $r \nearrow$ or $r \searrow$ in a right neighborhood of a and in a left neighborhood of b , then Table 1.1 uniquely determines the "non-strict" monotonicity pattern of r . (The "strict" counterparts of these rules, with terms "increasing" and "decreasing" in place of "non-decreasing" and "non-increasing" respectively, also hold, according to the same Proposition 1.9 of [16].)

Clearly, the stated l'Hospital-type rules for monotonicity patterns are helpful wherever the l'Hospital rules for limits are so, and even beyond that, because these monotonicity rules do not require that both f and g (or either of them) tend to 0 or ∞ at any point. (Special rules for monotonicity, which do require that both f and g vanish at an endpoint of (a, b) , were given, in different forms and with different proofs, in [9, 14, 3, 15, 25].)

Thus, it should not be surprising that a wide variety of applications of the l'Hospital-type rules for monotonicity patterns were given: in areas of analytic inequalities [10, 6, 15, 16, 22, 1, 13, 31, 32, 33, 34, 35]; approximation theory [17]; differential geometry [8, 9, 11, 24], information theory [15, 16]; (quasi)conformal mappings [2, 3, 4, 5]; probability and statistics [14, 16, 17, 20, 26, 27, 28, 29], including the very recent papers [26, 27, 28, 29], where these monotonicity rules have become a standard tool. (For the references to [13, 31, 32, 33, 34, 35] I thank a referee.)

The mentioned rules for monotonicity, both general and special, are potentially helpful when f' or g' can be expressed simpler than or similarly to f or g , respectively. Such functions f and g are essentially the same as the functions that could be taken to play the role of u in the integration-by-parts formula $\int u dv = uv - \int v du$; this class of functions includes algebraic, exponential, trigonometric, logarithmic, inverse trigonometric and inverse hyperbolic functions, and as well as non-elementary "anti-derivative" functions of the form $x \mapsto c + \int_a^x h(u) du$ or $x \mapsto c + \int_x^b h(u) du$.

"Discrete" analogues, for f and g defined on \mathbb{Z} , of the l'Hospital-type rules for monotonicity are available as well [23].

In this paper, we describe different facets of the relation of the (maximal) interval(s) of constancy of the original ratio r with those of the derivative ratio ρ . In particular, it may seem quite surprising that, as it turns out, the assumption of only non-strict monotonicity of ρ results necessarily in a significant degree of strictness on the monotonicity pattern of r ; namely, r can then have at most one maximal interval of constancy. Thus, new insight into the nature of the general rules for monotonicity is provided, which complements the previously made observation (see e.g. [25, paragraph around (5.1)]) that "the relation between the monotonicity patterns of r and ρ is not reversible in any reasonable sense".

The question of strictness of inequalities plays a prominent role in the fundamental monograph by Hardy, Littlewood and Pólya [12]. This question has been of significant interest in various problems; e.g. see [30, 7, 19, 18]; in particular, the investigation of the problem of

strictness in [19] led to an extension [18] of the well-known theorem by P. Hall in combinatorial theory; other extensions of P. Hall's theorem were subsequently given in [21].

2. RESULTS

In what follows, let us always assume that ρ is (not necessarily strictly) monotonic (that is, \nearrow or \searrow) on (a, b) .

Let us say that an interval $I \subseteq (a, b)$ is an *interval of constancy (i.c.)* of a function $h: (a, b) \rightarrow \mathbb{R}$ if I is of nonzero length and h is constant on I . If an i.c. I is not contained in any other i.c., let us say that I is a *maximal i.c. (m.i.c.)* It is easy to see that any i.c. is contained in a unique m.i.c. (which is simply the union of all i.c.'s containing the given i.c.).

It is easy to see that every i.c. of r is an i.c. of ρ . One might think that, if ρ has more than one m.i.c., then this can also be the case for the original ratio r . It may therefore be unexpected that the opposite is true, and even in the following strong sense.

Proposition 2.1. *The rules given by Table 1.1 can be strengthened as shown in Table 2.1.*

ρ	gg'	r
\nearrow	> 0	\searrow
\searrow	> 0	\nearrow
\nearrow	< 0	\nearrow
\searrow	< 0	\searrow

Table 2.1: Improved “non-strict” general rules for monotonicity.

Here, for instance, $r \searrow$ means that there is a subinterval $[c, d] \subseteq [a, b]$ (possibly of length 0) such that $r' < 0$ on (a, c) , r is constant on (c, d) , and $r' > 0$ on (d, b) .

Why is this proposition true? The key notion here is that of the function

$$\tilde{\rho} := r' \frac{g^2}{|g'|},$$

introduced in [16] and further studied in [25]. The key lemma concerning $\tilde{\rho}$ [25, Lemma 1 and Remark 4] states, as presented in Table 2.2 here, that the monotonicity pattern of $\tilde{\rho}$ is the same as that of ρ if $gg' > 0$, and opposite to the pattern of ρ if $gg' < 0$.

ρ	gg'	$\tilde{\rho}$
\nearrow	> 0	\nearrow
\searrow	> 0	\searrow
\nearrow	< 0	\searrow
\searrow	< 0	\nearrow

Table 2.2: The monotonicity patterns of ρ and $\tilde{\rho}$ mirror each other

From this relation between ρ and $\tilde{\rho}$, the rules given by Table 1.1 can be easily deduced, since

$$\text{sign}(r') = \text{sign } \tilde{\rho}.$$

A simple but important observation is that the derivative ratio ρ and its counterpart $\tilde{\rho}$ are continuous functions [25, Remark 4]. Since the ratio r is differentiable, it is continuous as well. Therefore, any m.i.c. I of r or ρ or $\tilde{\rho}$ is closed (as a set) in (a, b) ; that is, I has the form $[c, d]$ or $(a, c]$ or $[d, b)$ or (a, b) , for some c and d such that $a < c < d < b$. Moreover, it is seen from Table 2.2 that the m.i.c.'s of $\tilde{\rho}$ are the same as those of ρ , because any real function h is constant on an interval I if and only if h is both non-decreasing and non-increasing on I .

Proposition 2.1 now follows easily.

Proof of Proposition 2.1. It suffices to consider only the first line of Table 2.1 (since the other three lines can then be obtained by “vertical” reflection $f \leftrightarrow -f$ and/or “horizontal” reflection $x \leftrightarrow -x$). So, assume that $\rho \nearrow$ on (a, b) and $gg' > 0$. Then $\tilde{\rho} \nearrow$ on (a, b) . If $\tilde{\rho} > 0$ and hence $r' > 0$ on the entire interval (a, b) , let $c := d := a$, to obtain the conclusion that $r \searrow$ on (a, b) . If $\tilde{\rho} < 0$ and hence $r' < 0$ on (a, b) , let $c := d := b$. It remains to consider the case when the sign of $\tilde{\rho}$ takes on at least two different values (of the set $\{-1, 0, 1\}$ of all sign values). Then, since the function $\tilde{\rho}$ is non-decreasing and continuous on (a, b) , the level-0 set $\ell_0(\tilde{\rho}) := \{u \in (a, b) : \tilde{\rho}(u) = 0\}$ of $\tilde{\rho}$ must be a non-empty interval (which in fact must be an m.i.c. of $\tilde{\rho}$ and hence a set closed in (a, b)); in this case, take c and d to be the left and right endpoints, respectively, of the interval $\ell_0(\tilde{\rho})$ (at that, it is possible that $c = a$ and/or $d = b$). Then $\tilde{\rho} < 0$ and hence $r' < 0$ on (a, c) ; $\tilde{\rho} = 0$ and hence $r' = 0$ and $r = \text{const}$ on (c, d) ; and $\tilde{\rho} > 0$ and hence $r' > 0$ on (d, b) . \square

By Proposition 2.1, r can have no more than one m.i.c. On the other hand, one has

Proposition 2.2. *If r has an m.i.c. I , then I must be an m.i.c. of ρ and $\tilde{\rho}$ as well.*

Proof. Suppose that I is the (necessarily unique) m.i.c. of r , so that $\frac{f}{g} = r = K$ on I for some constant K . Then obviously $\rho = K$ and $\tilde{\rho} = 0$ on I , so that I is an i.c. of ρ and $\tilde{\rho}$. Let then J be the unique m.i.c. of ρ such that $J \supseteq I$, whence $\frac{f'}{g'} = \rho = K_1$ on J for some constant K_1 , and so, $f = K_1g + C$ and $r = K_1 + \frac{C}{g}$ on J , and hence on I , for some constant C . But r is constant on the nonzero-length interval I , while g is not constant on I (because $g'(x) \neq 0$ for any $x \in (a, b)$). It follows that $C = 0$ and thus $r = K_1$ on J . Finally, since I is an m.i.c. of r and $J \supseteq I$, one concludes that $J = I$, and so, I is an m.i.c. of ρ and hence of $\tilde{\rho}$. \square

We complete the description of the relation between the m.i.c.'s of r and ρ by observing that any one m.i.c. I of a given derivative ratio ρ is the m.i.c. of an appropriately constructed original ratio r (which must, in view of Proposition 2.1, depend on the choice of I):

Proposition 2.3. *For*

- any differentiable function $g: (a, b) \rightarrow \mathbb{R}$ such that $gg'(x) \neq 0$ for each $x \in (a, b)$,
- any (not necessarily strictly) monotonic continuous function $\rho: (a, b) \rightarrow \mathbb{R}$, and
- any m.i.c. I of ρ

there exists a differentiable function $f: (a, b) \rightarrow \mathbb{R}$ such that $\frac{f'}{g'} = \rho$ and the only m.i.c. of $r := \frac{f}{g}$ is I .

Proof. Let g , ρ , and I satisfy the conditions listed in Proposition 2.3, so that $\rho = K$ on I for some constant K . Note that the condition on g implies that either $g' > 0$ on the entire interval (a, b) or $g' < 0$ on (a, b) (see e.g. [25, Remark 3]), so that g is monotonic and hence of locally bounded variation on (a, b) . Take any point z in the interval I (which is an i.c. and hence non-empty) and define f by the formula

$$f(x) := Kg(z) + \int_z^x \rho(u) dg(u)$$

for all x in (a, b) , where the integral may be understood in the Riemann-Stieltjes sense, with the convention that $\int_z^x := -\int_x^z$ if $x < z$. Because ρ is continuous and g is differentiable, it follows that for the so defined function f one has $\frac{f'}{g'} = \rho$; moreover, $f = Kg$ on I , so that I is an i.c. of r . But any i.c. of r is also an i.c. of ρ , and I was assumed to be an m.i.c. of ρ . It follows that I is an m.i.c. (and hence *the only* m.i.c.) of r . \square

Let us summarize our findings as

Theorem 2.4.

- The set of all m.i.c.'s of $\tilde{\rho}$ is the same as that of ρ .
- The “original” ratio r can have at most one m.i.c., and its m.i.c. must also be an m.i.c. of ρ and thus of $\tilde{\rho}$; moreover, then the m.i.c. of r is the level-0 set of $\tilde{\rho}$.
- Any one m.i.c. of a given derivative ratio ρ is the m.i.c. of an appropriately constructed original ratio r .

This result can be illustrated by

Example 2.1. Let the derivative ratio $\rho: (0, \infty) \rightarrow \mathbb{R}$ for a ratio $r = f/g$ be given by the formula

$$\rho(x) := \min(x - k, k + 1) = \begin{cases} x - k & \text{if } 2k \leq x < 2k + 1, \\ k + 1 & \text{if } 2k + 1 \leq x < 2k + 2, \end{cases}$$

where $k := \lfloor \frac{x}{2} \rfloor$. Let $g(x) := x + 1$ for all $x \in (0, \infty)$. Then, for the formula $\rho = f'/g'$ to hold, the corresponding function $f: (0, \infty) \rightarrow \mathbb{R}$ must be given by

$$\begin{aligned} f(x) = f_c(x) &:= c + \int_0^x \rho(u) g'(u) du \\ &= \begin{cases} c + \frac{1}{2}((x - m)^2 + m(m + 1)) & \text{if } 2m \leq x < 2m + 1, \\ c + (m + 1)(x - m - \frac{1}{2}) & \text{if } 2m + 1 \leq x < 2m + 2, \end{cases} \end{aligned}$$

where $m := \lfloor \frac{x}{2} \rfloor$ and c is any real number (which obviously equals $f_c(0)$). Letting now $r_c := f_c/g$, one sees that the derivative ratio f'_c/g' of r_c equals the given function ρ , for every value of c . This derivative ratio, ρ , is nondecreasing (and continuous) on $(0, \infty)$, with infinitely many m.i.c.'s: $[1, 2], [3, 4], \dots$. In contrast, r_c may have at most one m.i.c. More specifically, r_c has exactly one m.i.c., $[2m + 1, 2m + 2]$ (which is also one of the infinitely many m.i.c.'s of ρ), if $c = c_{2m+1} := \frac{1}{2}(m + 1)(2m + 3) -$ the root of the equation $r_c(2m + 1) = \rho(2m + 1)$, for each $m = 0, 1, \dots$; and r_c has no m.i.c. for any real value of c different from all the c_m 's. Figure 2.1 shows the graphs of ρ and the “original” ratios r_c , for the values $c = -1, 0, \frac{3}{4}, \frac{3}{2}, 3$, and 5 . For these selected values of c , the ratio r_c has exactly one m.i.c. $- [1, 2]$ or $[3, 4]$ – only if $c = \frac{3}{2} = c_1$ or $c = 5 = c_3$, respectively; and r_c has no m.i.c. if $c = -1, 0, \frac{3}{4}$, or 3 .

To visualize this example in particular and the monotonicity rules in general, one can imagine a tank with the solution of a liquid in water. Initially, at time $x = 0$, the amounts in the tank of the liquid and water (not necessarily measured in the same units) are $f_c(0) = c$ and $g(0) = 1$, respectively, so that the relative concentration $r_c = f_c/g$ of the liquid (with respect to water) is initially $r_c(0) = c$. The liquid and water are added to the tank continuously through a pipe so that water is added at a constant rate 1. The relative concentration ρ of the liquid in the pipe is initially 0 (that is, $\rho(0) = 0$); moreover, ρ increases at a constant rate 1 in each of the “odd” unit time intervals $[0, 1], [2, 3], \dots$ and remains constant in each of the “even” unit time intervals $[1, 2], [3, 4], \dots$.

Then for any strictly positive value $c = r_c(0)$, the relative concentration ρ of the liquid in the pipe is initially less than the relative concentration r_c of the liquid in the tank, so that r_c

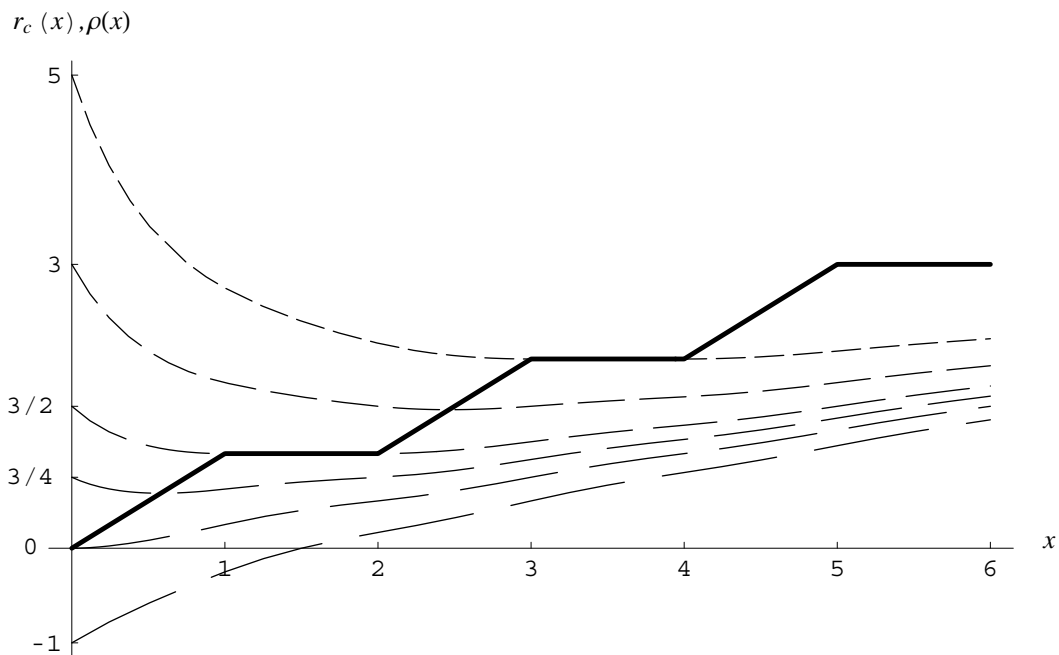


Figure 2.1: Graphs of ρ and r_c : ρ , thick, solid; r_c , dashed, with dash length decreasing in c .

will initially be decreasing. However, since ρ is non-decreasing to ∞ in time, ρ will eventually overtake r_c , and the latter will then be forever strictly increasing (after possibly staying constant, together with ρ , over the unit time interval $[2m + 1, 2m + 2]$ for some $m = 0, 1, \dots$, provided that $c = \frac{1}{2}(m + 1)(2m + 3)$). However, if the initial relative concentration c in the tank is 0 (or, somehow, negative), then the relative concentration r_c in the tank will be always strictly increasing, yet never reaching the relative concentration ρ in the pipe (cf. [16, Proposition 1.18] or identity [25, (1.1)]).

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