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ON THE q -ANALOGUE OF GAMMA FUNCTIONS AND RELATED INEQUALITIES

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Abstract

In this paper, we obtain a q -analogue of a double inequality involving the Euler gamma function which was first proved geometrically by Alsina and Tomás [1] and then analytically by Sándor [6].

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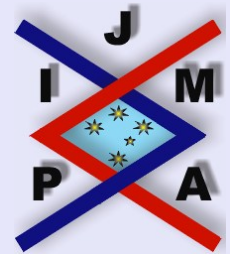
Key words: Euler gamma function, q -gamma function.

The authors express their sincere gratitude to Professor J. Sándor for his valuable comments and suggestions.

Dedicated to H. M. Srivastava on his 65th birthday.

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1. Introduction

F. H. Jackson defined the q -analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1, \text{ cf. [2, 4, 5, 7],}$$

and

$$\Gamma_q(x) = \frac{(q^{-1}; q^{-1})_\infty}{(q^{-x}; q^{-1})_\infty} (q - 1)^{1-x} q^{\binom{x}{2}}, \quad q > 1,$$

where

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

It is well known that $\Gamma_q(x) \rightarrow \Gamma(x)$ as $q \rightarrow 1^-$, where $\Gamma(x)$ is the ordinary Euler gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0.$$

Recently Alsina and Tomás [1] have proved the following double inequality on employing a geometrical method:

Theorem 1.1. *For all $x \in [0, 1]$, and for all nonnegative integers n , one has*

$$(1.1) \quad \frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$



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Sándor [6] has obtained a generalization of (1.1) by using certain simple analytical arguments. In fact, he proved that for all real numbers $a \geq 1$, and all $x \in [0, 1]$,

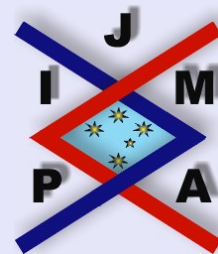
$$(1.2) \quad \frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1.$$

But to prove (1.2), Sándor used the following result:

Theorem 1.2. For all $x > 0$,

$$(1.3) \quad \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + (x-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(x+k)}.$$

In an e-mail message, Professor Sándor has informed the authors that, relation (1.2) follows also from the log-convexity of the Gamma function (i.e. in fact, the monotonous increasing property of the ψ -function). However, (1.3) implies many other facts in the theory of gamma functions. For example, the function $\psi(x)$ is strictly increasing for $x > 0$, having as a consequence that, inequality (1.2) holds true with strict inequality (in both sides) for $a > 1$. The main purpose of this paper is to obtain a q -analogue of (1.2). Our proof is simple and straightforward.



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2. Main Result

In this section, we prove our main result.

Theorem 2.1. *If $0 < q < 1$, $a \geq 1$ and $x \in [0, 1]$, then*

$$\frac{1}{\Gamma_q(1+a)} \leq \frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)} \leq 1.$$

Proof. We have

$$(2.1) \quad \Gamma_q(1+x) = \frac{(q; q)_\infty}{(q^{1+x}; q)_\infty} (1-q)^{-x}$$

and

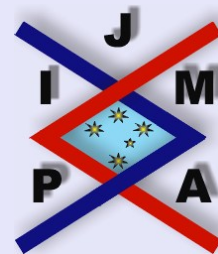
$$(2.2) \quad \Gamma_q(1+ax) = \frac{(q; q)_\infty}{(q^{1+ax}; q)_\infty} (1-q)^{-ax}.$$

Taking the logarithmic derivatives of (2.1) and (2.2), we obtain

$$(2.3) \quad \frac{d}{dx} (\log \Gamma_q(1+x)) = -\log(1-q) + \log q \sum_{n=0}^{\infty} \frac{q^{1+x+n}}{1-q^{1+x+n}}, \text{ cf. [3, 4, 5],}$$

and

$$(2.4) \quad \frac{d}{dx} (\log \Gamma_q(1+ax)) = -a \log(1-q) + a \log q \sum_{n=0}^{\infty} \frac{q^{1+ax+n}}{1-q^{1+ax+n}}.$$



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Since $x \geq 0, a \geq 1, \log q < 0$ and

$$\frac{q^{1+ax+n}}{1 - q^{1+ax+n}} - \frac{q^{1+x+n}}{1 - q^{1+x+n}} = \frac{q^{1+ax+n} - q^{1+x+n}}{(1 - q^{1+ax+n})(1 - q^{1+x+n})} \leq 0,$$

we have

$$(2.5) \quad \frac{d}{dx} (\log \Gamma_q(1 + ax)) \geq a \frac{d}{dx} (\log \Gamma_q(1 + x)).$$

Let

$$g(x) = \log \frac{\Gamma_q(1 + x)^a}{\Gamma_q(1 + ax)}, \quad a \geq 1, x \geq 0.$$

Then

$$g(x) = a \log \Gamma_q(1 + x) - \log \Gamma_q(1 + ax)$$

and

$$g'(x) = a \frac{d}{dx} (\log \Gamma_q(1 + x)) - \frac{d}{dx} (\log \Gamma_q(1 + ax)).$$

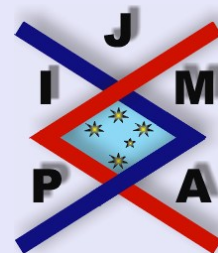
By (2.5), we get $g'(x) \leq 0$, so g is decreasing. Hence the function

$$f(x) = \frac{\Gamma_q(1 + x)^a}{\Gamma_q(1 + ax)}, \quad a \geq 1$$

is a decreasing function of $x \geq 0$. Thus for $x \in [0, 1]$ and $a \geq 1$, we have

$$\frac{\Gamma_q(2)^a}{\Gamma_q(1 + a)} \leq \frac{\Gamma_q(1 + x)^a}{\Gamma_q(1 + ax)} \leq \frac{\Gamma_q(1)^a}{\Gamma_q(1)}.$$

We complete the proof by noting that $\Gamma_q(1) = \Gamma_q(2) = 1$. □



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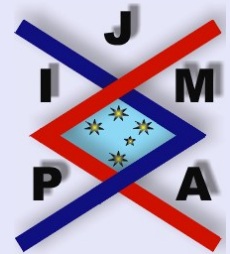
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Remark 1. Letting q to 1 in the above theorem. we obtain (1.2).

Remark 2. Letting q to 1 and then putting $a = n$ in the above theorem, we get (1.1).



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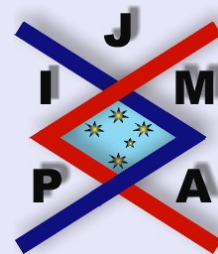
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